LECTURE - 2 7010

Example - Poisson Dist^n

Prob. of certain number of events occuring in a fixed interval:

$$P(Y=y) = \frac{\lambda^y}{e^\lambda y!}$$

- y_i's are "counts"
- y_i's are independent
- $E(y) = \lambda$

Examples

- calls at call-centre
- deaths from horse kicks in Prussian army
- customers
- doctor visits / bank failures / insurance claims / patents

Why not OLS?

- y isn't continuous
- $y \geq 0$

Exercise

$y = \{2, 0, 1, 1, 5, 3\}$. estimate λ.

e.g. might want to know $P_r(Y>6)$ or $E(y)$ or $P_r(Y=0)$.

1) $L(\lambda | y) = \prod_{i=1}^{n} \left(\frac{\lambda^{y_i}}{e^\lambda y_i!} \right)$

2) $\log L = \sum_{i=1}^{n} \left(y_i \log \lambda - \lambda - \log y_i! \right) = \sum y_i \log \lambda - n\lambda - \log \sum y_i!$

3) F.O.C.: $\dfrac{\partial \log L}{\partial \lambda} = \dfrac{\sum y_i}{\lambda} - n = 0$

$$\tilde{\lambda} = \frac{\sum y_i}{n} = \bar{y}$$

4) $\dfrac{\partial^2 \log L}{\partial \lambda^2} = -\dfrac{\sum y_i}{\lambda^2} = (-)\ \forall\ y_i$ and λ. So, $\tilde{\lambda}$ finds global max.

5) $\mathrm{var}(\tilde{\lambda}) = [-E(H)]^{-1} = \dfrac{\lambda^2}{\sum E(y_i)} = \dfrac{\lambda^2}{n\lambda} = \dfrac{\lambda}{n}$

So, given the data, $\tilde{\lambda} = \dfrac{\sum y_i}{n} = \dfrac{12}{6} = 2$,

- $\widetilde{E(y)} = 2$
- $\widetilde{Pr(Y=0)} = \dfrac{\tilde{\lambda}^0}{e^{\tilde{\lambda}}\, 0!} = \dfrac{1}{e^2 \cdot 1} = 0.135$

MLE - Example 1

The geometric distribution may be used to describe the probability of a number of failures occuring before the first success. The p.d.f. for a random variable, y_i, which follows a geometric distribution, is given by:

$$P_r(Y_i = y_i) = (1-p)^{y_i} p \qquad (p = \text{prob. of success})$$

The mean of the distribution (the expected value of y_i) is given by:

$$E(y_i) = \frac{1-p}{p}$$

1) Specify the likelihood function.

$$L(p \mid y_1, \ldots, y_n) = \prod_{i=1}^{n} (1-p)^{y_i} p = (1-p)^{\Sigma y_i} p^n$$

2) Take logs.

$$\log L = \Sigma y_i \log(1-p) + n \log p$$

3) Solve the FOC.

$$\frac{\partial \log L}{\partial p} = -\frac{\Sigma y_i}{1-p} + \frac{n}{p} = 0$$

$\tilde{p} = \dfrac{1}{1+\bar{y}}$. This is the MLE for p.

4) Get the Hessian.

In this case, there is only one parameter to estimate, so the Hessian is scalar:

$$\frac{\partial^2 \log L}{\partial p^2} = \frac{-\Sigma y_i}{(1-p)^2} - \frac{n}{p^2} = H$$

Since the Hessian is negative for all p, the MLE $\tilde{p}$ solves for a global max.

5) Obtain the variance of $\tilde{p}$.

$$\text{var}(\tilde{p}) = -E[H]^{-1}$$

Let's obtain $E[H]$ first.

$$E[H] = E\left[\frac{\partial^2 \log L}{\partial p^2}\right] = -\frac{E[\Sigma y_i]}{(1-p)^2} - \frac{n}{p^2}$$

Using $E(y_i) = \frac{1-p}{p}$, we get:

$$E[H] = \frac{-n\left(\frac{1-p}{p}\right)}{(1-p)^2} - \frac{n}{p} = \frac{-n}{(1-p)p} - \frac{n}{p^2}$$

$$= \frac{-n}{p^2(1-p)}$$

So, $\text{var}(\tilde{p}) = -E[H]^{-1} = \frac{p^2(1-p)}{n}$

Here is some data:

$$y = \{3, 0, 2, 1, 2, 1, 1, 0, 0, 3\}$$

This is real data which I obtained in my office. The y_i's follow the geometric distribution. Test the null hypothesis that this data was generated using a fair coin.

The MLE is: $\tilde{p} = \dfrac{1}{1 + 1.3} = 0.435$

and $\widetilde{var}(\tilde{p}) = \dfrac{0.435^2(0.565)}{10} = 0.011$

$H_0: p = 0.5$ vs. $H_1: p \neq 0.5$

LRT

The LRT test statistic is: $-2 \log (L_0 - L_1)$, or:

$$LRT = 2 \log \left(\frac{L_1}{L_0} \right) = 2 \log \left[\frac{(1 - 0.435)^{13} \, 0.435^{10}}{(1 - 0.5)^{13} \, 0.5^{10}} \right]$$

$$= 2 \log \left[\frac{1.451 \times 10^{-7}}{1.192 \times 10^{-7}} \right] = 0.392$$

Using a 5% critical value of 3.85, we fail to reject the null.

Wald

$$W = (R\tilde{\theta}_1 - r)'[R\, I^*(\tilde{\theta}_1)^{-1} R']^{-1}(R\tilde{\theta}_1 - r)$$

Here, $R\tilde{\theta}_1 - r = 0.435 - 0.5 = -0.065$, and

$I^*(\tilde{\theta}_1)^{-1} = \widetilde{var}(\tilde{p}) = 0.011$

So, $W = \dfrac{(-0.065)^2}{0.011} = 0.384 \sim \chi^2_{(1)}$

Again, we fail to reject.

LM-test

$$LM = [D \log L_1(\tilde{\theta}_0)]' I^*(\tilde{\theta}_0)^{-1} [D \log L_1(\tilde{\theta}_0)]$$

Where $I^*(\tilde{\theta}_0)^{-1} = var(p_0) = 0.0125$. Since θ is scalar, the formula becomes:

$$\left(\frac{\partial \log L_1(p_0)}{\partial p}\right)^2 \times var(p_0) = \left(\frac{-13}{0.5} + \frac{10}{0.5}\right) \times 0.125 = 0.45$$

For a third time, we fail to reject.

(I sat in my office and flipped a coin.)

MLE - Example 2

Exponential distribution describes the time between events in a Poisson process.

P.d.f. : $p(y_i \mid \lambda) = \lambda e^{-y_i \lambda}$

$E(y_i) = 1/\lambda$

1) $L(\lambda \mid y_1, \ldots, y_n) = \prod_{i=1}^{n} p(y_i \mid \lambda) = \lambda^n e^{-\lambda \Sigma y_i}$

2) $\log L = n \log \lambda - \lambda \Sigma y_i$

3) $\dfrac{\partial \log L}{\partial \lambda} = \dfrac{n}{\lambda} - \Sigma y_i = 0$

$\hat{\lambda} = n / \Sigma y_i = 1/\bar{y}$

4) $\dfrac{\partial^2 \log L}{\partial \lambda^2} = \dfrac{-n}{\lambda^2} \quad (-),$

5) $var(\tilde{\lambda}) = \dfrac{1}{-E[H]} = \dfrac{\lambda^2}{n}$

Note that we don't have to worry about the expectation operator above since y_i does not enter the 2^{nd} derivative.

Given a sample of data:

$y = \{0.5, 1, 0.6, 0.2, 3\},$

$\hat{\lambda} = 0.9434, \quad \widetilde{var}(\tilde{\lambda}) = \tilde{\lambda}^2 / n = 0.178$

Test: $H_0: \lambda = 1$ vs. $H_1: \lambda \neq 1$

<u>LRT</u> $= 2[\log \tilde{L}_1 - \log \tilde{L}_0]$

$= 2[n \log \tilde{\lambda}_1 - \tilde{\lambda}_1 \Sigma y_i - n \log \tilde{\lambda}_0 + \tilde{\lambda}_0 \Sigma y_i]$

$= 2[5 \log 0.9434 - 0.9434 \times 5.3 - 0 + 5.3] = 0.0173$

From a $\chi^2_{(1)}$ distribution, with $\alpha = 0.05$, we have a critical value of 3.84. We fail to reject.

<u>Wald</u>

The restriction is $\theta = 1$, so: $R = r = 1$; $\tilde{\theta}_1 = 0.9434$; $\tilde{\theta}_0 = 1$.

$$W = \frac{(0.9434 - 1)^2}{0.178} = 0.0180$$

Fail to reject.

<u>LM</u> $= \left(\frac{n}{\tilde{\lambda}_0} - \Sigma y_i\right)^2 \times \left(\frac{\tilde{\lambda}_0^2}{n}\right) = \left(\frac{5}{1} - 5.3\right)^2 \times 0.2 = 0.0180$

Fail to reject