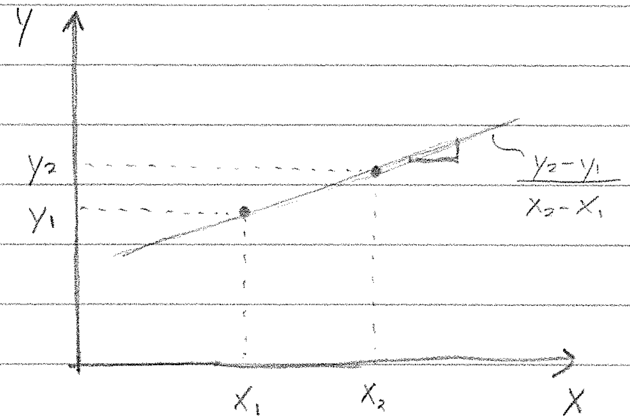
Practice Exercise - Solution



a) Let $j = 1, \ldots, n/2$ and $k = \frac{n}{2}+1, \ldots, n$

$\downarrow$ 1st half $\qquad \downarrow$ 2nd half

$$\hat{\beta}_2 = \frac{\sum y_k - \sum y_j}{\sum x_k - \sum x_j} = \underbrace{\frac{1}{\sum x_k - \sum x_j}}_{c}\left(\sum y_k - \sum y_j\right)$$

$$= c\sum y_k - c\sum y_j$$

b) $$E(\hat{\beta}_2) = c\sum E(y_k) - c\sum E(y_j)$$

$$= c\left(\frac{n}{2}\beta_1 + \beta_2 \sum x_k\right) - c\left(\frac{n}{2}\beta_1 + \beta_2 \sum x_j\right)$$

$$= \frac{\beta_2\left(\sum x_k - \sum x_j\right)}{\sum x_k - \sum x_j} = \beta_2$$

c) $var(\hat{\beta}_2) = var\left(c\left[\Sigma y_K - \Sigma y_j\right]\right)$

$$= c^2\left(\frac{n}{2}\sigma^2 + \frac{n}{2}\sigma^2\right) = \frac{n\sigma^2}{(\Sigma x_K - \Sigma x_j)^2}$$

<u>$var(b_2)$</u>

$$var(b) = \sigma^2 (X'X)^{-1}$$

$$X'X = \begin{bmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix} \qquad (X'X)^{-1} = \frac{1}{n\Sigma x_i^2 - (\Sigma x_i)^2}\begin{bmatrix} \Sigma x_i^2 & -\Sigma x_i \\ -\Sigma x_i & n \end{bmatrix}$$

so, $var(b_2) = \dfrac{n\sigma^2}{n\Sigma x_i^2 - (\Sigma x_i)^2}$

Now, $var(\hat{\beta}_2) > var(b_2)$ if $(\Sigma x_K - \Sigma x_j)^2 < n\Sigma x_i^2 - (\Sigma x_i)^2$

$$(\Sigma x_K - \Sigma x_j)^2 = (\Sigma x_K)^2 + (\Sigma x_j)^2 - 2(\Sigma x_K)(\Sigma x_j)$$

Cauchy-Schwarz : $(\Sigma x_i)^2 \le \Sigma x_i^2$