# Introduction to Econometrics

A textbook for ECON 3040

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# 1

# Introduction

## 1.1 What is Econometrics?

Econometrics is the study of statistical methods applied to economics data. It is a subset of statistics. Similarly, biology has "biometrics", psychology has "psychometrics", etc. Econometrics uses those methods most suited to economics data.

Econometrics can be used to test economics theories. Economics is a social *science*, and economics benefits from the scientific method. Theories are formed and tested using observations from the real world. The testing part mostly relies on econometrics.

Econometrics can be used to estimate *causal effects*, though it should not be used to find them. That is, the theoretical model (e.g. from Micro or Macro) should specify which variable causes which. It is then up to the econometrician to estimate *how much* of an effect one variable has on another. Econometrics may also be used to forecast or predict economic variables, although forecasting is not covered in this course.

Econometrics specializes in dealing with *observational data*. Observational data is in contrast to *experimental data*. In an experiment, there is some element of *control* - a variable can be changed by the researcher, and the effect of the change on another variable can be more easily measured. In observational data the causal variable is changing on its own, and this can be very problematic. Typically there are important *omitted variables* in observational data. An experiment provides a better way to estimate a causal effect, since the missing variables are not a problem in a well constructed experiment.

Economic models often suggest that one variable causes another. This often has *policy implications*. The economic models, however, do not provide quantitative magnitudes of the causal effects. For example:

- How would a change in the *price* of alcohol or cigarettes effect the *quantity* consumed?

- If *income* increases, how much of the increase will be *consumed*?
- If an additional fireplace is added to a house, how much will the price of the house increase?
- How does another year of *education* change *earnings*?

How would you use an experiment to determine the above four causal effects? You will likely conclude that using an experiment would be too costly and/or unethical. Hence, we must rely on observational data, and try to sort out the associated problems.

It is important to be aware of the limitations of statistics. It can never be used to determine *causation*. Causation must be theorized. If two variables are correlated, statistics alone cannot tell which variable causes which, or if there is any causation at all. That is, *correlation does not imply causation*. If, however, we find that two variables are statistically independent from each other, one variable can not cause the other.

**Objectives**

Some objectives of this text are the following:

- Learn a method for estimating causal effects (OLS)
- Understand some theoretical properties of OLS
- Learn about hypothesis testing
- Learn to read regression analyses, so as to understand empirical economics papers in other courses
- Practice OLS using data sets

## 1.2 R Statistical Environment and R Studio

The theory and concepts presented in this course will be illustrated by analysing several data sets. Data analysis will be accomplished through the R Statistical Environment and RStudio. Both are free, and R is fast becoming the best and most widely used statistical software. Download R from `https://cran.r-project.org/bin/windows/base/` (for Windows) or `https://cran.r-project.org/bin/macosx/` (for Mac). Download RStudio from `https://www.rstudio.com/products/rstudio/download/`.

Once you download and install R and R Studio, open R Studio. Figures 1.1, 1.2, and 1.3 give you a basic idea of how to run a command in R.

Figure 1.1: Open up RStudio. It should look like something this:

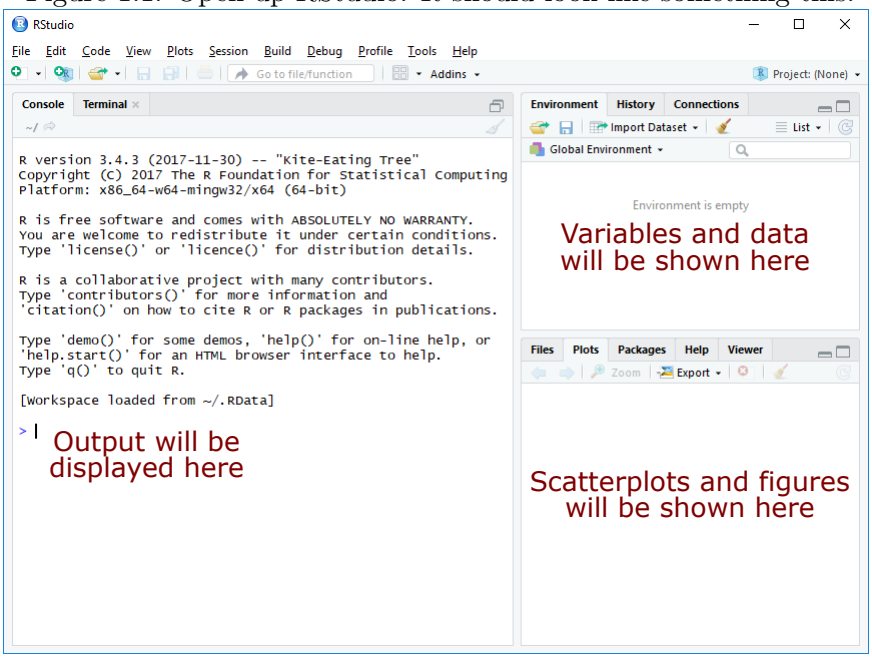



Figure 1.2: Create an R Script. To keep track of your commands, you should use an R Script. Go to "File" →"New File" →"R Script".

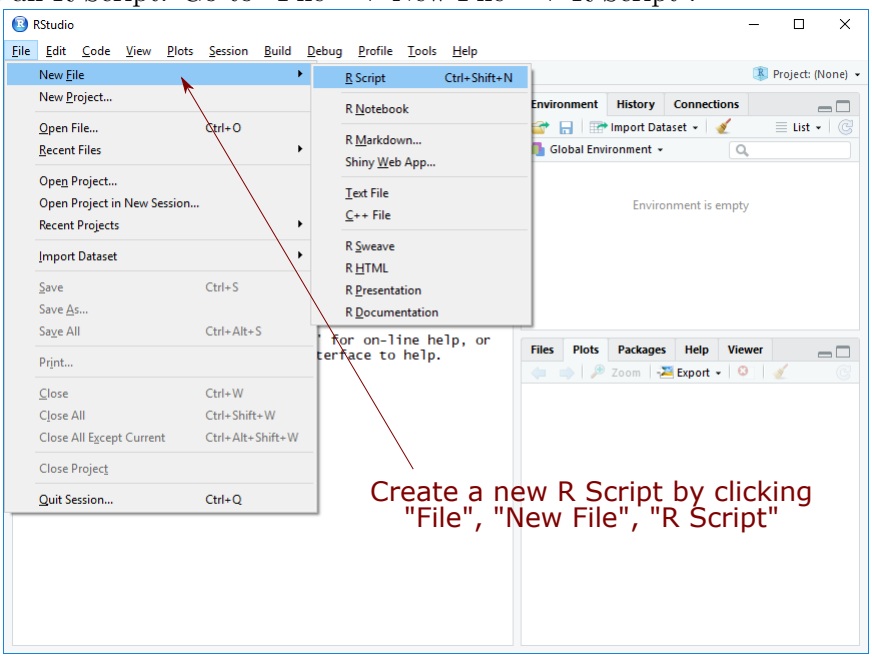
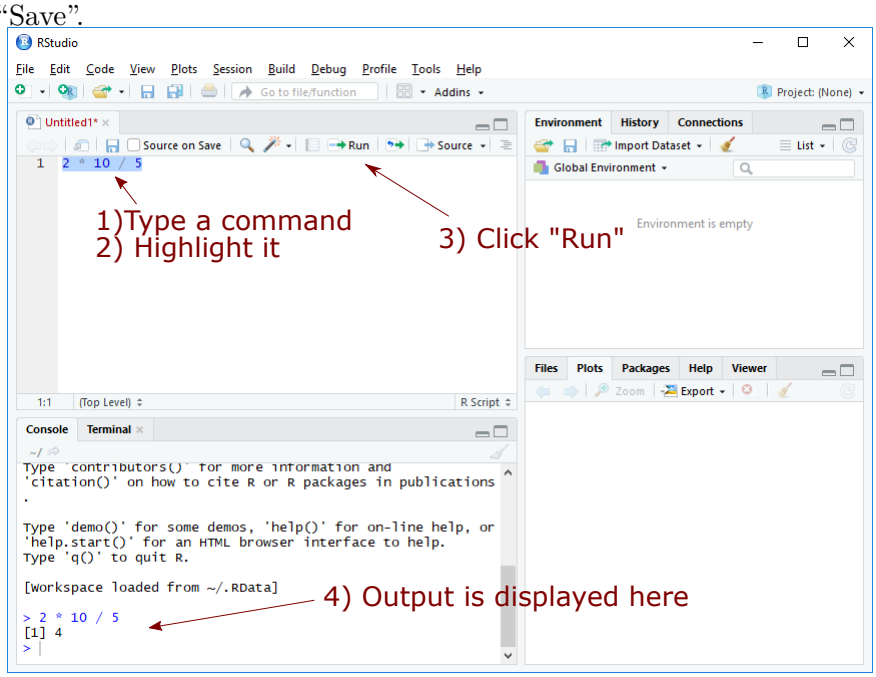


Figure 1.3: To run a command in R Studio: 1) Type a command in the "R Script" window. 2) Highlight the command. 3) Click the "Run" button. 4) The output will be displayed in the "R Console" window. 5) Save your script by making sure the "R Script" window is selected, and click "File" →"Save".



# 2

# Probability Review

This is a brief review. These are concepts that you should know from your previous statistics courses.

## 2.1 Fundamental Concepts

### 2.1.1 Randomness

Randomness is unpredictability. Outcomes that we cannot predict are random. Randomness represents our inability as humans to accurately predict things. For example, if I roll two dice, the outcome is random because I am not smart enough or skilled enough to predict what the roll will be. Things that I cannot, or do not want to predict, are random. We cannot know everything. However, we can attempt to model the randomness mathematically.

> Randomness: the inability to predict an outcome.

This definition of randomness does not oppose a deterministic world view (fate). While many things in our lives *appear* to be random, I still think that at some fundamental level the world is deterministic, and that all events are potentially predictable. In the dice example, it is not far-fetched to believe that a computer could analyze my hand movements and perfectly predict the outcome of the roll.

It is sometimes useful to construct a set, or *sample space* of the possible *outcomes* of interest. In the dice example, the sample space is { ⚀⚀, ⚀⚁, ⚀⚂, ..., ⚅⚅ }. An *event* is a subset of the sample space, and consists of one or more of the possible outcomes. For example, rolling higher than ten is an event consisting of three outcomes { ⚂⚅, ⚅⚂, ⚅⚅ }.

### 2.1.2 Probability

A probability is a number between 0 and 1 that is assigned to an event (sometimes expressed as a percentage). A standard definition is: the probability of an event is the proportion of times it occurs in the long run. This is fine for the dice example, and you may be aware that the probability of rolling a seven is 1/6 or of rolling higher than ten is 1/12. This definition works for this example because we can imagine rolling the dice repeatedly under similar settings and observing that a seven occurs one-sixth of the time.

What about events that occur seldomly or only once? What is the probability that you will obtain an A+ in this course? What is the probability that Donald Trump will be president in 2021? For these events, the former definition of probability is less satisfactory. A more general definition is: probability is a mathematical way of quantifying uncertainty. For the Trump example, the probability of reelection is *subjective*. I may think the probability is 0.1, but someone else may assign a probability of 0.9. Which is right? These problems are better suited to a *Bayesian* framework, which is not discussed in this book. Luckily, the first definition of probability will be sufficient.

> Probability: a number between 0 and 1 representing the portion of times an event will occur, if it could occur repeatedly.

## 2.2 Random variables

A *random variable* translates outcomes into numerical values. For example, a die roll only has numerical meaning because someone has etched numbers onto the sides of a cube. A random variable is a human-made construct, and the choice of numerical values can be arbitrary. Different choices can lead to different properties of the random variable. For example, I could measure temperature in Celsius, Fahrenheit, Kelvin or something new (degrees Ryans). The probability that it will be above 20° tomorrow depends critically on how I have constructed the random variable.

Random variables can be separated into two categories, *discrete* and *continuous*. A discrete random variable takes on a countable number of values, e.g. {0, 1, 2, ...}. The result of the dice roll is a discrete random variable. A continuous random variable takes on a continuum of possible values (an infinite number of possibilities).

Even when the random variable has lower and upper bounds, there are still infinite possibilities. The temperature tomorrow is a continuous random variable. It may be bound between -50°C and 50°C, but there are still infinite possibilities. What is the probability that it is 20°C? What about

20.1°C? What about 20.0001°C? We could keep adding 0s after the decimal. In fact, the probability of the temperature taking on any one value approaches 0. Instead, we must talk about the probability of a *range* of numbers. For example, the probability that the temperature is between 19°C and 21°C.

The continuum of possibilities makes it more difficult to discuss continuous random variables than it does discrete random variables. We will use discrete random variables for examples and try to extend the logic to continuous random variables.

Finally, note the difference between a *random variable* and the *realization of a random variable*. Before I roll the die, the outcome is random. After I roll the die and get a ⚃ (for example), the 4 is just a number - a *realization* of a random variable.

Key Points

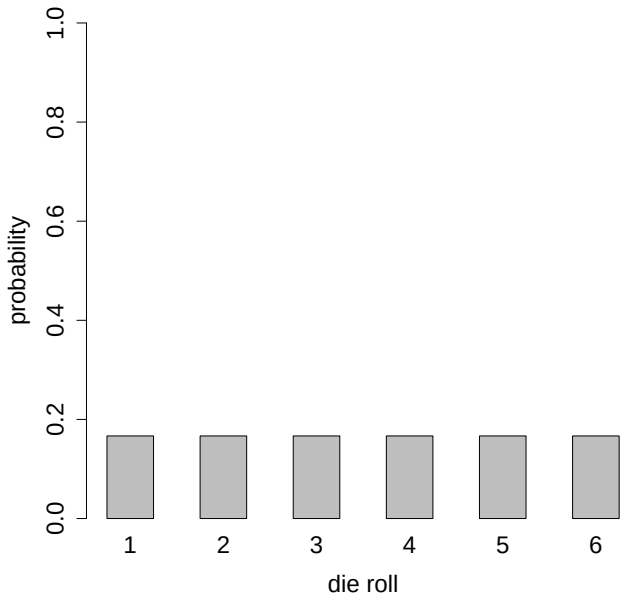
- A random variable can take on different values (or ranges of values), with different probabilities
- There are discrete and continuous random variables
- Continuous random variables can take on an infinite number of possible values, so we can only assign probabilities to *ranges* of values
- We can assign probabilities to all possible values for a discrete random variable
- The *realization* of a random variable is just a number, it used to be random, but now we've seen the outcome

## 2.3 Probability function

A *probability function* is also called a *probability distribution*, or a *probability distribution function* (PDF). Sometimes a distinction is made: *probability mass function* (PMF) for discrete variables instead of PDF for continuous variables. I will use *probability function* for both.

A probability function is an equation (it can also be a graph or table), which contains information about a random variable. The nature and properties of the randomness determines what type of equation is appropriate. A different equation would be used for a dice roll than would be used for the wage of a worker. The probability function is very important. The probability function accomplishes two things: (i) it lists all possible numerical values that the random variable can take, and (ii) assigns a probability

Figure 2.1: Probability function for the result of a die roll




to each value. Note that the probabilities of all outcomes must sum to 1 (something must happen). The probability function contains all possible knowledge that we can have about the random variable (before we observe its realization).

### 2.3.1 Example: probability function for a die roll

Let $Y =$ the result of a die roll. The probability function for $Y$ is:

$$Pr(Y = 1) = \frac{1}{6}, Pr(Y = 2) = \frac{1}{6}, \ldots, Pr(Y = 6) = \frac{1}{6} \tag{2.1}$$

Note how the function lists all possible numerical outcomes and assigns a probability to each. A more compact way of expressing (2.1) is:

$$Pr(Y = y) = \frac{1}{6} \; ; \; y = 1, \ldots, 6 \tag{2.2}$$

The probability function in (2.2) may also be expressed in a graph (see Figure 2.1).

### 2.3.2 Example: probability function for a normally distributed random variable

The normal distribution is an important probability distribution. Later, we will discuss why it is so important and prevalent. For now, I will present the probability function for a random variable (you do not need to memorize this).

$$f(y|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp -\frac{(y-\mu)^2}{2\sigma^2} \; ; \; -\infty < y < \infty \tag{2.3}$$

Do not be scared. $y$ is the random variable, $\mu$ and $\sigma^2$ are the *parameters* that govern the probability of $y$. $\mu$ turns out to be the *mean* or *expected value* of $y$, and $\sigma^2$ turns out to be the *variance* of $y$. If $\mu$ and $\sigma^2$ are known (usually they aren't), then you can determine the probability that $y$ takes on any range of values. However, this requires integration (you won't have to integrate in this course).

### 2.3.3 Probabilities of events

Recall that the probability function contains all possible information about the random variable (all the outcomes, and a probability assigned to each outcome), and that an event is a collection of outcomes. The probability function can be used to calculate the probability of events occurring.

*Example.* Let $Y$ be the result of a die roll. What is the probability of rolling higher than 3?

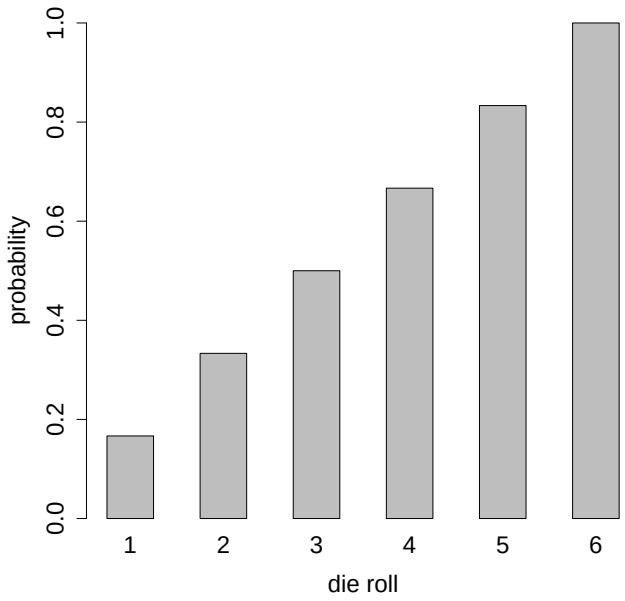
$$Pr(Y > 3) = Pr(Y = 4) + Pr(Y = 5) + Pr(Y = 6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

### 2.3.4 Cumulative distribution function

The *cumulative distribution function* (CDF) is related to the probability function. It is the probability that the random variable is *less than or equal to* a particular value. While every random variable has a probability function, it does not always have a CDF (but usually does). Again, let $Y$ be the result of a die roll, then the CDF for $Y$ is expressed as equation 2.4 or as figure 2.2.

$$\begin{aligned} Pr(Y \leq 1) &= 1/6 \\ Pr(Y \leq 2) &= 2/6 \\ Pr(Y \leq 3) &= 3/6 \\ Pr(Y \leq 4) &= 4/6 \\ Pr(Y \leq 5) &= 5/6 \\ Pr(Y \leq 6) &= 1 \end{aligned} \tag{2.4}$$

Figure 2.2: Cumulative density function for the result of a die roll




## 2.4 Moments of a random variable

The term "moment" is related to a concept in physics. The first moment of a random variable is the mean, the second (central) moment is the variance, the third the skewness, and the fourth the kurtosis. In this book, we will make extensive use of mean and variance, as well as the mixed moment covariance (and correlation).

### 2.4.1 Mean or expected value

The *mean* or *expected value* of a random variable is the value that is expected, or the value that occurs on average through repeated realizations of the random variable. The mean of a random variable can be determined from its probability function. Recall that the probability function contains all possible information we could hope to have about the random variable. So, it should be no surprise that if we want to determine the mean we have to do some math to the probability function. The mean (and variance, etc.) is just summarized information contained in the probability function.

Let $Y$ be the random variable, the result of a die roll for example. Notation for the mean of $Y$ or expectation of $Y$ is $\mu_Y$ or $\mathrm{E}[Y]$. As mentioned above, the mean of $Y$ is determined from its probability function. For such discrete random variables as $Y$, the mean is determined by taking a weighted average of all possible outcomes, where the weights are the probabilities. The

equation for the mean of (Y) is:

$$\mathrm{E}[Y] = \sum_{i=1}^{K} p_i Y_i \tag{2.5}$$

where $p_i$ is the probability of the $\mathrm{i}^{\mathrm{th}}$ event, $Y_i$ is the value of the $\mathrm{i}^{\mathrm{th}}$ outcome, and $K$ is the total number of outcomes ($K$ can be infinite). Study this equation. It is a good way of understanding what the mean is.

Equation 2.5 is valid for any discrete random variable $Y$. For our particular example, using the probability function we have that $K = 6$ and each $p_i = 1/6$, so the mean of $Y$ is:

$$\mathrm{E}(Y) = \frac{1}{6} \times (1) + \frac{1}{6} \times (2) + ... + \frac{1}{6} \times (6) = 3.5$$

Calculating the mean of a continuous random variable is analogous, but more difficult. Again, the mean is determined from the probability function, but instead of *summing* across all possible outcomes we have to *integrate* (since the random variable can take on a continuum of possibilities).

Let $y$ be a continuous random variable. The mean of $y$ is

$$\mathrm{E}[y] = \int y f(y)\, \mathrm{d}y$$

If $y$ is normally distributed, then $f(y)$ is equation (2.3), and the mean of $y$ turns out to by $\mu$. You do not need to integrate for this course, but you should have some idea about how the mean of a continuous random variable is determined from its probability function.

Some properties of the mean are:

- $\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y]$
- $\mathrm{E}[cY] = c\mathrm{E}[Y]$, where $c$ is a constant
- $\mathrm{E}[c + Y] = c + \mathrm{E}[Y]$
- $\mathrm{E}[c] = c$

### 2.4.2 Median and Mode

The *mean* of a random variable is not to be confused with the *median* or *mode* of a random variable, although all three are measures of "central tendency". The *median* is the "middle" value, where 50% of values will be above and below. The *mode* is the value which occurs the most.

For variables that are normally distributed, the *mean*, *median* and *mode* are all the same, but this is not always true. For a die roll, the mean and median are 3.5, but there either is no mode or all of the values are the mode (depending on which statistician you ask).

### 2.4.3 Variance

The variance of a random variable is a measure of its *spread* or *dispersion*. Variance is often denoted by $\sigma^2$. In words, variance is the expected squared difference of the random variable from its mean. In an equation, the variance of $Y$ is

$$\text{Var}(Y) = \text{E}[(Y - \text{E}[Y])^2] \tag{2.6}$$

When $Y$ is a discrete random variable, then equation (2.6) becomes

$$\text{Var}(Y) = \sum_{i=1}^{K} p_i \times (Y_i - \text{E}[Y_i])^2 \tag{2.7}$$

where $p_i$, $Y_i$, and $K$ are defined as before. Note that equation 2.7 is a weighted averaged of squared distances. The variance is measuring how far, on average, the variable is from its mean. The higher the variance, the higher the probability that the random variable will be far away from its expected value.

When the random variable is continuous, equation (2.6) becomes:

$$\text{Var}(y) = \int (y - \text{E}[y])^2 f(y)\, \text{d}y$$

but you don't need to know this for the course.

Some properties of the variance are:

- $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] + 2 \times \text{Cov}[X, Y]$
- $\text{Var}[cY] = c^2\text{Var}[Y]$, where $c$ is a constant
- $\text{Var}[c + Y] = \text{Var}[Y]$
- $\text{Var}[c] = 0$

### 2.4.4 Skewness and Kutosis

Notice in the variance formula (2.6), that there is an expectation of a squared term ($\text{E}[]^2$). This partly explains why the variance is called the *second* (central) moment. Similarly, we could take the expectation of the $Y$ to the third power, or fourth power, etc. Doing so would (almost) give us the third and fourth moments.

The third (central) moment is called *skewness* and the fourth is called *kurtosis*. Much less attention is paid to these moments than is to the mean and the variance. However, it is worth noting that if a random variable is normally distributed, it has a skewness of 0 and a kurtosis of 3.

### 2.4.5 Covariance

Covariance is a measure of the relationship between two random variables. Random variables $Y$ and $X$ are said to have a *joint* probability distribution. The joint probability distribution is like the probability functions we have seen before (equations 2.1 and 2.3), except that it involves two random variables. The joint probability function for $Y$ and $X$ would (i) list all possible combinations that $Y$ and $X$ could take, and (ii) assign a probability to each combination. A useful summary of the information contained in the joint probability function, is the *covariance*.

The covariance between $Y$ and $X$ is the expected difference of $Y$ from its mean, multiplied by the expected value of $X$ from its mean. Covariance tells us something about how two variables *move* together. That is, if the covariance is positive, then when one variable is larger (or smaller) than its mean, the other variable tends to be larger (or smaller) as well. The larger the magnitude of covariance, the more often this statement tends to be true. Covariance tells us about the direction and strength of the relationship between two variables.

The formula for the covariance between $Y$ and $X$ is

$$\text{Cov}(Y, X) = \text{E}[(Y - \mu_Y)(X - \mu_X)] \tag{2.8}$$

The covariance between $Y$ and $X$ is often denoted as $\sigma_{YX}$. Note the following properties of $\sigma_{YX}$:

- $\sigma_{YX}$ is a measure of the *linear* relationship between $Y$ and $X$. Non-linear relationships will be discussed later.
- $\sigma_{YX} = 0$ means that $Y$ and $X$ are linearly independent.
- If $Y$ and $X$ are independent (neither variable causes the other), then $\sigma_{YX} = 0$. The converse is not necessarily true (because of non-linear relationships).
- The $\text{Cov}(Y, Y)$ is the $\text{Var}(Y)$.
- A positive covariance means that the two variables tend to differ from their mean in the *same* direction.
- A negative covariance means that the two variables tend to differ from their mean in the *opposite* direction.

### 2.4.6 Correlation

Correlation is similar to covariance. It is usually denoted with the Greek letter $\rho$. Correlation conveys all the same information that covariance does, but is easier to interpret, and is frequently used instead of covariance when

summarizing the linear relationship between two random variables. The formula for correlation is

$$\rho_{YX} = \frac{\text{Cov}(Y,X)}{\sqrt{\text{Var}(Y)\text{Var}(X)}} = \frac{\sigma_{YX}}{\sigma_Y \sigma_X} \tag{2.9}$$

The difficulty in interpreting the value of covariance is because $-\infty < \sigma_{YX} < \infty$. Correlation transforms covariance so that it is bound between -1 and 1. That is, $-1 \leq \rho_{YX} \leq 1$.

- $\rho_{YX} = 1$ means perfect positive linear association between $Y$ and $X$.
- $\rho_{YX} = -1$ means perfect negative linear association between $Y$ and $X$.
- $\rho_{YX} = 0$ means no linear association between $Y$ and $X$ (linear independence).

### 2.4.7 Conditional distribution and conditional moments

When we introduced covariance, and began to talk about the relationship between two random variable, we introduced the concept of the joint probability distribution function. Recall that the joint probability function lists all combinations of the random variables, assigning a probability to each combination.

Sometimes, however, it is useful to obtain a *conditional* distribution from the joint distribution. The conditional distribution just fixes the value of one of the variables, while providing a probability function for the other. This probability function may change depending on the fixed value.

We need this concept for the *conditional expectation*, which will be important later when we discuss dummy variables. The *conditional expectation* is just the expected or mean value of one variable, conditional on some value for the other variable.

Let $Y$ be a discrete random variable. Then, the conditional mean of $Y$ given some value for $X$ is

$$\text{E}(Y|X=x) = \sum_{i=1}^{K} (p_i|X=x)Y_i \tag{2.10}$$

### 2.4.8 Example: Joint distribution

Suppose that you have a midterm tomorrow, but that there is a possibility of a blizzard. You are wondering if the midterm might be canceled. If there is a blizzard, there is a strong chance of cancellation. If there is no blizzard, then you can only hope that the professor gets severely ill, but that still only gives a small chance of cancellation. The joint probability distribution for

the two random events (occurrence of the blizzard, and occurrence of the midterm) is given in table (2.1). Note how all combinations of events have been described, and a probability assigned to each combination, and that all probabilities in the table sum to 1.

Table 2.1: Joint distribution for snow and a canceled midterm

| | Midterm ($Y = 1$) | No Midterm ($Y = 0$) |
|---|---|---|
| Blizzard ($X = 1$) | 0.05 | 0.20 |
| No Blizzard ($X = 0$) | 0.72 | 0.03 |

What is E[$Y$]? It is 0.77. This means there is a 77% chance you will have a midterm. E[$Y$] is an *unconditional* expectation; it is the mean of $Y$ before you look out the window in the morning and see if there is a blizzard. The conditional expectations, however, are E[$Y|X = 1$] = 0.20 and E[$Y|X = 0$] = 0.96. This means there is only a 20% chance of a midterm if you see a blizzard in the morning, but a 96% chance with no blizzard. Some other review questions using table (2.1) are left to the Review Questions.

## 2.5 Some Special Probability Functions

In this section, we present some common probability functions that we will reference in this course. We start with the normal distribution, and a discussion of the *central limit theorem*.

### 2.5.1 The normal distribution

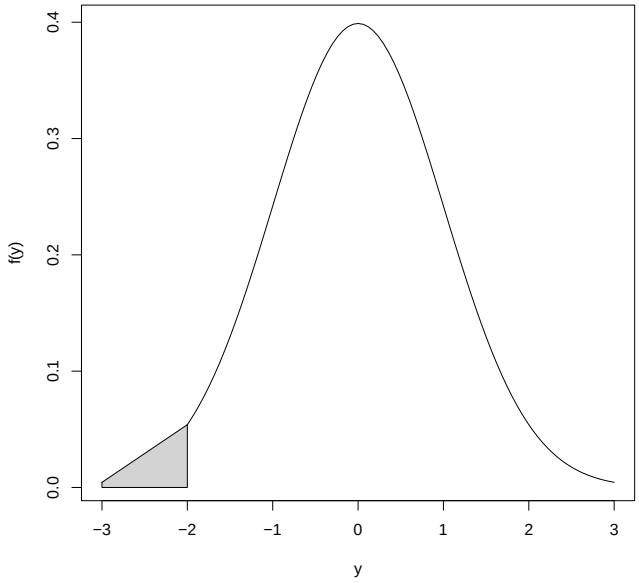
The probability function for a normally distributed random variable, $y$, has already been given in equation (2.3). What is the use of knowing this? If we know that $y$ is normal, and if we knew the parameters $\mu$ and $\sigma^2$ (we will likely have to estimate them) then we know all we can possibly hope to about $y$. That is, we can use equation (2.3) to determine the mean and variance of y. We can draw out equation (2.3), and calculate areas under the curve. These areas would tell us about the probability of events occurring.

Suppose that we knew $y$ had mean 0 and variance 1. What is the probability that $y < -2$? Using equation (2.3), we could draw out the probability function, and calculate the area under the curve, to the left of -2. See figure (2.3). This area, and probability, is 0.023.

### 2.5.2 The standard normal distribution

The probability function drawn out in figure (2.3) is actually the probability function for a standard normal variable. A variable is standard normal when

Figure 2.3: Probability function for a standard normal variable, $p_{y<-2}$ in gray




its mean is 0 and variance is 1. When $\mu = 0$ and $\sigma^2 = 1$, the probability function for a normal variable (equation 2.3) becomes:

$$f(y) = \frac{1}{\sqrt{2\pi}} \exp \frac{-y^2}{2} \tag{2.11}$$

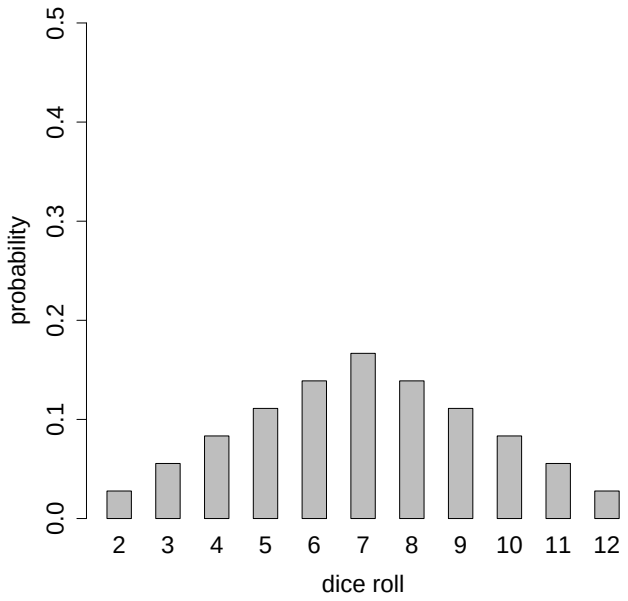
Note that any random normal variable can be "standardized". That is, if we subtract the variable's mean, and divide by it's standard deviation, then we change the mean to 0, and variance to 1. It becomes "standard normal". This practice is useful in hypothesis testing, as we shall see.

### 2.5.3 The central limit theorem

So why do we care so much about the normal distribution? There are hundreds of probability functions, that are appropriate in various situations. The heights of waves might be described by the Nakagami distribution. The probability of successfully drawing a certain number of red balls out of a hat of red and blue balls is described by the binomial distribution. The number of customers that visit a store in an hour might be described by the Poisson distribution. The result of a die roll is uniformly distributed. So why should we pay so much attention to the normal distribution?

The answer is the *central limit theorem* (CLT). Loosely speaking, the

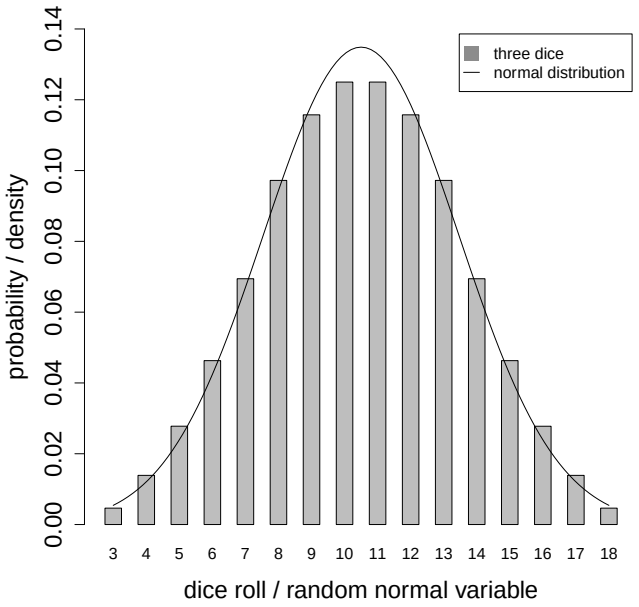
Figure 2.4: Probability function for the sum of two dice




CLT says that if we add up enough random variables, the resulting sum tends to be normal. It doesn't matter if some are Poisson and some are uniform. It only matters that we add up enough. If the random outcomes that we seek to model using probability theory are the results of many random factors all added together, then the central limit theorem applies. This turns out to be plausible for the types of economic models we are going to consider. This has been a very casual explanation of the CLT; you should be aware that there are several conditions required for it to hold, and several versions.

$$\begin{aligned} Pr(Y = 2) &= 1/36 \\ Pr(Y = 3) &= 2/36 \\ Pr(Y = 4) &= 3/36 \\ Pr(Y = 5) &= 4/36 \\ Pr(Y = 6) &= 5/36 \\ Pr(Y = 7) &= 6/36 \\ Pr(Y = 8) &= 5/36 \\ &\vdots \\ Pr(Y = 12) &= 1/36 \end{aligned} \tag{2.12}$$

Figure 2.5: Probability function for three dice, and normal distribution




*Example.* Let $Y$ be the result of summing two die rolls. The probability function for $Y$ is displayed in equation 2.12 and in figure (2.4). Notice how each individual die has a uniform (flat) distribution, but summed together, begins to get a "curve".

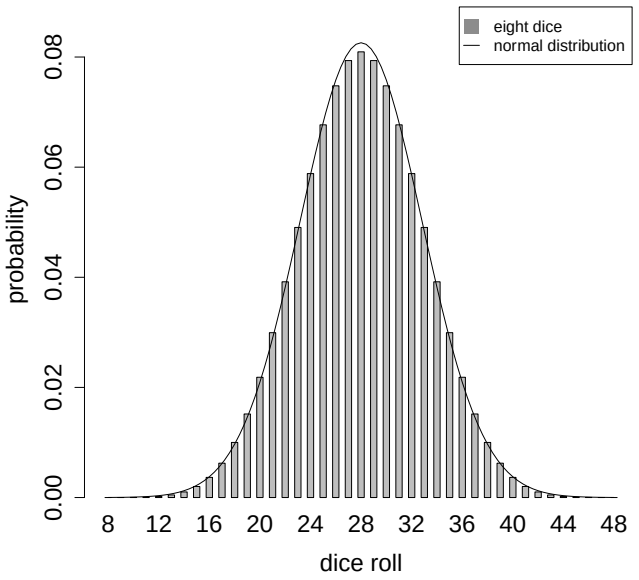
Now, let's add a third die, and see if the probability function looks more normal. Let $Y$ = the sum of *three* dice. It turns out the mean of $Y$ is 10.5 and the variance is 8.75. The probability function for $Y$ is shown in figure (2.5). Also in figure (2.5), the probability function for a normal distribution with $\mu = 10.5$ and $\sigma^2 = 8.75$. Notice the similarity between the two probability functions.

The CLT says that if we add up the result of *enough* dice, the resulting probability function should become normal. Finally, we add up *eight* dice, and show the probability function for both the dice and the normal distribution in figure(2.6), where the mean and variance of the normal probability function has been set equal to that of the sum of the dice.

### 2.5.4 The Chi-square ($\chi^2$) distribution

Suppose that $y$ is normally distributed. If we add or subtract from $y$ we change the mean of $y$, but it still will follow a normal distribution. If we multiply or divide $y$ by a number, we change its variance, but $y$ will still be normal. In fact, this is how we standardize a normal variable (we subtract

Figure 2.6: Probability function for eight dice, and normal distribution




its mean, and divide by its standard deviation).

While a linear transformation (addition, multiplication, etc.) of a normal variable leaves the variable normally distributed, normal variables are not invariant to *non-linear* transformations. If we square a standard normal variable (e.g. $y^2$), it becomes a $\chi^2$ distributed variable. We will use this distribution for the F-test in a later chapter.

## 2.6 Review Questions

1. Define the following terms:

| | | |
|---|---|---|
| outcome | event | random variable |
| discrete variable | continuous variable | parameter |
| CLT | mean | variance |
| probability function | covariance | correlation |

2. Let $X$ be a random variable, where $X = 1$ with probability 0.5, and $X = -1$ with probability 0.5. Let $Y$ be a random variable, where $Y = 0$ if $X = -1$, and if $X = 1$, $Y = 1$ with probability 0.5, and $Y = -1$ with probability 0.5. (a) What is the $\text{Cov}(X, Y)$? (b) Are $X$ and $Y$ independent?

3. Let $X$ be a normal random variable, where $\text{E}[X] = 0$. Remember that a random normal variable has a skewness of zero (the third moment

is zero), so that $\mathrm{E}[X^3] = 0$. Now, let $Y = X^2$. (a) What is the $\mathrm{Cov}(X, Y)$? (b) Are $X$ and $Y$ independent?

4. Use table (2.1). (a) What are the probability functions for $Y$ and $X$ (independent from each other)? (b) What are the mean and variance of $X$? (c) What is $\mathrm{Cov}(X, Y)$? (d) What is $\rho_{XY}$?

## 2.7 Answers

2. The joint probability function for $X$ and $Y$ is:

| | $Y = -1$ | $Y = 0$ | $Y = 1$ |
|---|---|---|---|
| $X = 1$ | 0.25 | 0 | 0.25 |
| $X = -1$ | 0 | 0.5 | 0 |

a) The formula for the covariance of $X$ and $Y$ is:

$$\mathrm{Cov}\,(X, Y) = \mathrm{E}\left[(X - \mu_X)\,(Y - \mu_Y)\right]$$

The mean of $X$ and $Y$ are:

$$\mu_X = 0.5(1) + 0.5(-1) = 0$$

$$\mu_Y = 0.25(-1) + 0.5(0) + 0.25(1) = 0$$

Finally, the covariance is:

$$\mathrm{Cov}\,(X, Y) = \mathrm{E}\,[XY] = 0.25(1)(-1) + 0.5(-1)(0) + 0.25(1)(1) = 0$$

b) Even though the covariance is 0, X and Y are not independent! We can see this by looking at the joint probability function. If we observe the value of $Y$, then we know, with certainty, the value of $X$. That is, if we observe $Y = -1$ or $Y = 1$, then we know that $X = 1$. If we observe $Y = 0$, then we know that $X = -1$. $Y$ can predict the value of $X$, so $X$ and $Y$ are not independent. The point is that covariance measures *linear* association between two variables. In this example, the relationship between $X$ and $Y$ is *non-linear*. If we were to graph the relationship between the two variables, we would see a "U" shape.

3. a) The covariance between $X$ and $Y$ is:

$$\begin{aligned}
\mathrm{Cov}\,[X, Y] &= \mathrm{Cov}\left[X, X^2\right] \\
&= \mathrm{E}\left[\Big(X - \mathrm{E}\,(X)\Big)\Big(X^2 - \mathrm{E}(X^2)\Big)\right] \\
&= \mathrm{E}\left[X^3 - \mathrm{E}\,(X)\,X^2 - X\mathrm{E}(X^2) + \mathrm{E}\,(X)\,\mathrm{E}(X^2)\right] \\
&= \mathrm{E}(X^3) - \mathrm{E}\,(X)\,\mathrm{E}(X^2) - \mathrm{E}\,(X)\,\mathrm{E}(X^2) + \mathrm{E}\,(X)\,\mathrm{E}(X^2) \\
&= 0
\end{aligned}$$

b) $X$ and $Y$ are not independent, since $Y = X^2$. Knowing one variable allows us to know the other variable, perfectly. This is another example of how the covariance between two variables can be zero, even when the variables are clearly related. Covariance is a measure of linear dependence. It is possible to find situations where a non-linear relationship yields zero covariance.

4. a) To get the *uncoditional* probabilities for $Y$ we can sum the columns, and for the probabilities of $X$ we can sum the rows, of table (2.1). The probability function for $Y$ is:

$$\Pr(Y = 1) = 0.77 \quad ; \quad \Pr(Y = 0) = 0.23$$

and for $X$ is:

$$\Pr(X = 1) = 0.25 \quad ; \quad \Pr(Y = 0) = 0.75$$

b)

$$\mathrm{E}[X] = 0.25(1) + 0.75(0) = 0.25$$
$$\mathrm{Var}[X] = 0.25(1 - 0.25)^2 + 0.75(0 - 0.25)^2 = 0.1875$$

c) To get the covariance, we will need the mean of $Y$:

$$\mathrm{E}[Y] = 0.77(1) + 0.23(0) = 0.77$$

Now, the covariance is:

$$\begin{aligned}\mathrm{Cov}(X, Y) &= 0.05(1 - 0.25)(1 - 0.77) \\ &+ 0.20(1 - 0.25)(0 - 0.77) \\ &+ 0.72(0 - 0.25)(1 - 0.77) \\ &+ 0.03(0 - 0.25)(0 - 0.77) \\ &= -0.1425\end{aligned}$$

d) The formula for correlation is given in equation (2.9). We have already calculated $\mathrm{Cov}(X, Y)$ and $\mathrm{Var}[X]$, but we need $\mathrm{Var}[Y]$:

$$\mathrm{Var}[Y] = 0.77(1 - 0.77)^2 + 0.23(0 - 0.77)^2 = 0.1771$$

Now, the correlation is:

$$\rho_{YX} = \frac{\mathrm{Cov}(Y, X)}{\sqrt{\mathrm{Var}(Y)\mathrm{Var}(X)}} = \frac{-0.1425}{\sqrt{0.1875 \times 0.1771}} = -0.7820$$

# 3

# Statistics Review

A statistic is any mathematical function using a *sample* of data. It is just an equation applied to the data. When a statistic is used to estimate a *population* parameter, it is called an *estimator*. One of the main goals of this course is to become familiar with a particular estimator - the *ordinary least squares* estimator, but for this chapter we will review some simpler estimators.

We will discuss the population, and why the sample $y$ should be considered random. Then, we will discuss some estimators. A very important point is that, because $y$ is random, functions of $y$ are also random. Since an estimator is just an equation applied to $y$, the estimator itself is also random. As we know from the previous chapter, random variables have probability functions.

The probability function for an estimator is given a special name - the *sampling distribution*. Obtaining some properties of the estimator from its sampling distribution, such as mean and variance, will tell us whether or not the estimator is "good", and will guide our choice of which estimator to use.

## 3.1 Random Sampling from the Population

A sample of data is a collection of variables. In econometrics, most of these variables are *realizations* of a random process. The numbers that make up (at least some of) the sample values came from a random process. The sample typically appears to us on our computer screen as a "spreadsheet" where each column is a different variable and each row is a different *sample unit*. The sampling "units" could be people, countries, firms, etc.

There are at least two ways to think about where a random sample, $y$, comes from. Both ways make use of the idea of a *population*. The population holds all of the information, the truth. If we knew the entire population, our jobs as statisticians or econometricians would be much easier. Instead we

will obtain only a piece of the puzzle, a *sample* of data from the population.

The first way to think about the population, is that it is a data generating process (dgp). It is a random process that generates the $y$ variables that we observe. It is as if a die is being rolled, generating the numbers in the sample, but we can't quite see what the die looks like. Alternatively, if $y$ is normally distributed, then values in $y$ are generated from equation (2.3), but where $\mu$ and $\sigma^2$ are unknown. This might be a difficult way to think about things.

A second, possibly easier way to think about the population, is to imagine it consisting of all of the data possible. When we obtain economic data, we typically do not observe everyone or everything in the *population* of interest. Instead we observe a *sample* of the population. Hopefully, members of the population will be selected randomly into the sample (otherwise we will have problems).

Suppose we want to know the mean height of a male U of M student. We can not afford to measure the height of every student, so we collect a sample, and hope that it represents the population. Suppose we stand in the University Centre for an hour and measure heights of students. The sample that we will collect is random - we don't know what the heights will be yet. On a different day, at a different time, or in a parallel universe, we will randomly select different students, get different heights, and a different sample.

We will want this sample to be independently and identically distributed (iid). *Indpendent* - none of the random variables in the sample have any connection. Independence would be violated if a basketball team walked through the University Centre and I sampled all of their heights. *Identical* - all of the random variables in the sample come from the same population (or probability function). The *identical* assumption would be violated if I accidentally sampled some Mini U students (grade school students touring campus).

As an example, let's pretend that the entire population of heights is in table (3.1). This is a simplified example of a population - the table should be much larger - usually we assume the population is near-infinite. Let's collect a random sample from this population, say 20 observations (the bold numbers in the table). Our sample is then denoted $y = \{173.9, 171.7, 182.6, 181.5, 162.1, 174.9, 165.7, 182.2, 171.7, 168.1, 189.9, 175.7, 163.4, 186.3, 169.5, 171.9, 173.9, 172.0, 172.7, 172.0\}$. $y$ is random because we *could* have selected different heights from the table.

## 3.2 Estimators and Sampling Distributions

An *estimator* is a way of using the sample data $y$ in order to "guess" something about the population that $y$ comes from. In the example of the heights

Table 3.1: Entire population of heights (in cm). The true (unobservable) population mean and variance are $\mu_y = 176.8$ and $\sigma_y^2 = 39.7$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 177.3 | 170.2 | 187.2 | 178.3 | 170.3 | 179.4 | 181.2 | 180.0 | **173.9** |
| 178.7 | **171.7** | 160.5 | 183.9 | 175.7 | 175.9 | **182.6** | 181.7 | 180.2 |
| **181.5** | 176.5 | **162.1** | 180.3 | 175.6 | **174.9** | **165.7** | 172.7 | 178.9 |
| 175.3 | 178.7 | 175.6 | 166.4 | 173.1 | 173.2 | 175.6 | 183.7 | 181.3 |
| 174.2 | 180.9 | 179.9 | 171.2 | 171.0 | 178.6 | 181.4 | 175.2 | **182.2** |
| **171.7** | 178.4 | **168.1** | 186.0 | **189.9** | 173.4 | 168.7 | 180.0 | 175.1 |
| **175.7** | 180.8 | 176.2 | 170.8 | 177.3 | **163.4** | **186.3** | 177.1 | 191.2 |
| 171.0 | 180.3 | **169.5** | 167.2 | 178.0 | 172.9 | 176.0 | 176.5 | **171.9** |
| 175.1 | 184.2 | 165.3 | 180.2 | 178.3 | 183.4 | **173.9** | 178.6 | 177.9 |
| 184.5 | 184.1 | 180.9 | 187.1 | 179.9 | 167.1 | **172.0** | 167.4 | **172.7** |
| 171.6 | 186.6 | 182.4 | 185.5 | 174.8 | 178.8 | 192.8 | 179.3 | **172.0** |

of male U of M students, we might be interested in knowing the mean height. The mean height would provide the best prediction for the height of the next random student that walks through the door. So, we collect our sample, $y$ = {173.9, 171.7, 182.6, 181.5, 162.1, 174.9, 165.7, 182.2, 171.7, 168.1, 189.9, 175.7, 163.4, 186.3, 169.5, 171.9, 173.9, 172.0, 172.7, 172.0}. How should we use this sample to *estimate* the mean height?

The difference between a population value (such as the population mean or variance), and an estimator (such as the sample mean or variance), is very important. The population mean is the unobservable truth, and is a constant (non-random). The sample mean is an estimator for the population mean, and as we shall see, is a random variable. In this section we want to build up the idea of the *sampling distribution* of an estimator, in order to determine its properties. This will help us to determine if the estimator is "good".

### 3.2.1 Sample mean

A popular choice for estimating the population mean ($\mathrm{E}[y]$ or $\mu_y$) is the *sample mean* (or *sample average*, or just *average*). The sample mean of $y$ is usually denoted by $\bar{y}$. You have seen the equation for the sample mean before:

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \tag{3.1}$$

where $y_i$ denotes the $i^{th}$ observation, and where $n$ denotes the sample size. If we plug in our sample of heights into equation (3.1) we get $\bar{y} = 174.1$.

An important question is: how good is the estimator? That is, how good of a job is the estimator doing at "guessing" the true unobservable thing in

the population? In our specific example: how good is the sample mean at estimating the true population mean of heights? This is an importannt question, because there are many ways that we could use the information in $y$ to try to estimate the mean height. Why is equation (3.1) so popular?

To answer these questions, we need to enter a hypothetical situation, which will likely not be the case in the real world. Let's pretend we can "see" the entire population of heights (all of Table 3.1). If we can see all of Table (3.1), and not just the sample $y$, then we know the true mean height. We just take the average of the entire population, and get 176.8. So, $\bar{y} = 174.1$ is wrong!

Recall that the sample, $y$, is random. Each element of $y$ was selected randomly from the population. We could have selected a different sample of size $n = 20$. For example, in a parallel universe, we could have gotten $y^*$ = {175.9, 175.3, 182.2, 178.6, 175.2, 180.3, 178.3, 183.7, 176.0, 167.4, 178.7, 178.7, 186.0, 175.6, 180.0, 168.7, 178.6, 173.1, 173.2, 187.1}, where the * in $y^*$ denotes that we are in the parallel universe. In this parallel universe, we got $\bar{y}^* = 177.6$. But in every universe, the population (table 3.1), is the same.

So, $\bar{y}$ is a random variable. $\bar{y}$ is random because $y$ is random. We *could* have drawn a different random sample, in which case we would have gotten a different $\bar{y}$. In our example, there are a near infinite number (about $4 \times 10^{20}$) of different samples of size $n = 20$, and $\bar{y}$s, that we could get from the same population. Some of the $\bar{y}$s will be close to the true population mean height of 176.8, others far away. Whether or not $\bar{y}$ is a good idea for estimating the population mean $\mathrm{E}(y)$ can be determined by analyzing all the possible values that $\bar{y}$ can take.

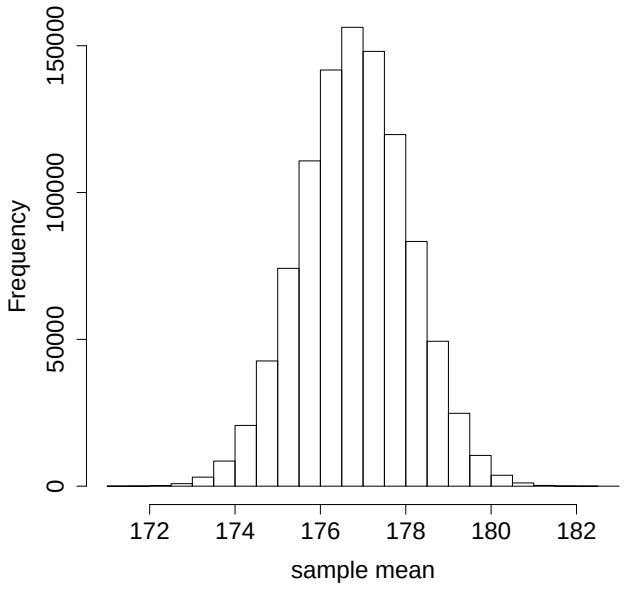
### 3.2.2 Sampling distribution of the sample mean

Recall the discussion on *probability functions* in Chapter 2. A random variable (usually) has a probability function. This probability function describes all the possible values that the random variable can take, assigning a probability to each possibility. The form of the probability function depends on the nature of the random variable.

When the random variable is an *estimator*, then the probability function gets a special name - the *sampling distribution*. That is, a *sampling distribution* is just a fancy name for the probability function of an estimator. The sampling distribution is a hypothetical construct. It describes the probability of outcomes of $\bar{y}$, but in the real world we only get one sample $y$ and one estimate $\bar{y}$.

An alternative way of defining the sampling distribution follows. Imagine that you could draw all possible random samples of size $n = 20$ from the population, calculate $\bar{y}$ each time, and construct a relative frequency diagram (a histogram) for all of the $\bar{y}$s. This relative frequency diagram

Figure 3.1: Histogram for 1 million $\bar{y}$s




would be the sampling distribution of the estimator $\bar{y}$ for $n = 20$.

This alternative definition of the sampling distribution can be approximated using a computer. Using a computer, I have drawn 1 million different random samples of size $n = 20$ from table (3.1), and have calculated $\bar{y}$ each time. (This takes about 10 seconds on a fast computer). I have drawn a histogram using all of the $\bar{y}$s (figure 3.1). Figure (3.1) is a simulated sampling distribution.

Which probability function describes $\bar{y}$? Look again at equation (3.1). Notice the summation operator. $\bar{y}$ involves taking the sum of random variables (the $y_i$s). It turns out that if the sample size is large enough (our $n = 20$ might be a bit too small) then the central limit theorem applies, and $\bar{y}$ is normally distributed (recall the summation of dice). Notice also that figure (3.1) resembles a normal distribution.

We will derive some features of an estimator from its sampling distribution. These features will tell us whether the estimator is "good" or "bad". Some important properties of the estimator are its mean (expected value) and its variance. This may be a strange idea at first. For example, we will take the expected value of the sample mean (which is an estimator for an expected value). That is, we will take the mean of the sample mean (meta!).

Three important properties of an estimator, that will largely guide whether the estimator is "good" or not, are *bias*, *efficiency*, and *consistency*. These properties are partly determined from the sampling distribution of the esti-

mator, and we will now discuss each property in turn.

### 3.2.3 Bias

What happens if we consider the expected value, or the mean, of an estimator? An estimator is random, so it should have a mean. What would we want the expected value of the estimator to be? The thing we are trying to estimate, of course. So, if we are estimating the population mean using the sample mean (equation 3.1), then we want to get the "right" answer on average. That is, we want $\mathrm{E}[\bar{y}] = \mathrm{E}[y]$. If this is true, then I can "expect" to get the right answer when using $\bar{y}$ in many situations.

If $\mathrm{E}[\bar{y}] = \mathrm{E}[y]$, then $\bar{y}$ is said to be unbiased. If $\mathrm{E}[\bar{y}] \neq \mathrm{E}[y]$, then $\bar{y}$ would be a biased estimator; it would not give us the "right" answer on average. Given the popularity of $\bar{y}$ as an estimator for the population mean, you might anticipate that it is an unbiased estimator. The following is a short proof of the unbiasedness of the sample average.

Assume that $y_i \sim (\mu_y, \sigma_y^2)$, and that the $y_i$s are iid. This just says that each random variable, $y_i$, in the sample, has the same population mean ($\mu_y$) and population variance ($\sigma_y^2$). Now, take the expected value of the estimator:

$$\begin{aligned}
\mathrm{E}[\bar{y}] &= \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} y_i\right] \\
&= \frac{1}{n}\mathrm{E}\left[\sum_{i=1}^{n} y_i\right] \\
&= \frac{1}{n}\mathrm{E}[y_1 + y_2 + \cdots + y_n] \\
&= \frac{1}{n}\left(\mathrm{E}[y_1] + \mathrm{E}[y_2] + \cdots + \mathrm{E}[y_n]\right) \\
&= \frac{1}{n}\left(\mu_y + \mu_y + \cdots + \mu_y\right) \\
&= \frac{n\mu_y}{n} = \mu_y
\end{aligned} \tag{3.2}$$

We find that the expected value of $\bar{y}$ is equal to the true unobservable population mean, and so $\bar{y}$ is an unbiased estimator.

### 3.2.4 Efficiency

Suppose that the estimator is unbiased. What happens now if we consider the variance of an estimator? What do want this variance to be? We would

want it to be as small as possible. That is, we would want the estimator to have a high probability of being close to the thing we are trying to estimate. In the case of $\bar{y}$, we should hope that the $\text{Var}[\bar{y}]$ is small so that on average, $\bar{y}$ is close to $\mu_y$.

Efficiency is when an estimator has the smallest variance, compared to all other potential estimators. We will restrict our attention to other estimators that are also linear and unbiased. So, $\bar{y}$ is efficient if $\text{Var}[\bar{y}] \leq \text{Var}[\hat{\mu}_y]$, where $\hat{\mu}_y$ is any other linear unbiased estimator for the population mean of $y$. It turns out that there are many linear and unbiased estimators for the population mean, but that the sample mean has the smallest variance. So, we say that $\bar{y}$ is efficient.

The proof of the efficiency of $\bar{y}$ is omitted, however, an important part of the proof is included. In order to compare the variance of $\bar{y}$ to other potential estimators, we first have to be able to derive it:

$$\begin{aligned}
\text{Var}\left[\bar{y}\right] &= \text{Var}\left[\frac{1}{n}\sum_{i=1}^{n} y_i\right] \\
&= \frac{1}{n^2}\text{Var}\left[\sum_{i=1}^{n} y_i\right] \\
&= \frac{1}{n^2}\text{Var}\left[y_1 + y_2 + \cdots + y_n\right] \\
&= \frac{1}{n^2}\left(\text{Var}\left[y_1\right] + \text{Var}\left[y_2\right] + \cdots + \text{Var}\left[y_n\right]\right) \\
&= \frac{1}{n^2}\left(\sigma_y^2 + \sigma_y^2 + \cdots + \sigma_y^2\right) \\
&= \frac{n\sigma_y^2}{n^2} = \frac{\sigma_y^2}{n}
\end{aligned} \tag{3.3}$$

Note that the $n$ in the denominator means the variance gets smaller as the sample size grows. That is, a larger sample provides an estimate that is on average closer to the true population mean. This is one reason why larger samples are better than smaller ones.

Now that we have derived the mean and variance of $\bar{y}$, and have used the central limit theorem to say that $\bar{y}$ is normally distributed, we can write the full sampling distribution: $\bar{y} \sim N(\mu_y, \sigma_y^2/n)$. Recall that this sampling distribution contains all the knowledge that we can have about the random variable $\bar{y}$. This sampling distribution is not only useful to determine the properties of unbiasedness, efficiency, and consistency, but will also be useful for hypothesis testing.

### 3.2.5 Consistency

*Consistency* is the last statistical property of an estimator that we will consider. An estimator is consistent if, having all possible information in the population, it provides the "right answer" every time. That is, as the sample size grows to infinity, the estimator provides the thing it's trying to estimate with probability 1. Two conditions are required for $\bar{y}$ to be (strongly) consistent: $\lim_{n\to\infty} \mathrm{E}[\bar{y}] = \mu_y$ and $\lim_{n\to\infty} \mathrm{Var}[\bar{y}] = 0$. The first condition says that the bias should disappear as the sample size grows. Since $\bar{y}$ is unbiased this condition is easily met. The second condition says that the variance of the estimator should go to 0 as the sample size grows; this is easily verified by noting the $n$ in the denominator of $\mathrm{Var}[y]$.

Consistency is the most important property for an estimator to have. Without consistency, the estimator is useless. In all, we have shown that $\bar{y}$ is unbiased, efficient, and consistent. Sometimes the acronym BLUE (best linear unbiased estimator) is used to describe such an estimator. That $\bar{y}$ is BLUE is a very good reason to use it as an estimator for $\mu_y$, among the many possibilities.

## 3.3 Hypothesis Tests (known $\sigma_y^2$)

The types of hypotheses we are talking about concern statements about the unobservable population. For example, we might hypothesize that the true population mean height of U of M students is 173 cm. A hypothesis test uses the information in the sample to assess the plausibility of the hypothesis. In general, a hypothesis test begins with a null hypothesis, and an alternative hypothesis. For example:

$$\begin{aligned} H_0 &: \mu_y = \mu_{y,0} \\ H_A &: \mu_y \neq \mu_{y,0} \end{aligned} \tag{3.4}$$

$H_0$ is the null hypothesis. The null hypothesis is "choosing" a value for the population mean, $\mu_y$. The hypothesized value of the population mean is denoted $\mu_{y,0}$. The alternative hypothesis ($H_A$) is two-sided; the null hypothesis is wrong if the population mean ($\mu_y$) is either "too small" or "too big" relative to the hypothesized value. Since most tests in econometrics are two-sided, we will not consider one-sided tests here, although they are very similar.

The hypothesis test concludes with either: (i) "reject" $H_0$ in favour of $H_A$, or (ii) "fail to reject" $H_0$. Which decision is reached ultimately depends on a probability ($p$-value), and on the researcher (you) deciding subjectively whether this probability is small or large. The sample data, and our knowledge of the sampling distribution of the estimator, will determine this probability.

Let's go back to the heights example. From our sample of $n = 20$ we estimated the population mean to be $\bar{y} = 174.1$. Suppose that the null and alternative hypotheses are:

$$\begin{aligned} H_0 &: \mu_y = 173 \\ H_A &: \mu_y \neq 173 \end{aligned} \tag{3.5}$$

Our estimate of 174.1 is clearly different from our hypothesis that the true population mean height is 173. Notice that the difference between what we actually estimated from the sample, and our null hypothesis, is $174.1 - 173 = 1.1$. This difference of 1.1 does not necessarily imply we should reject the null hypothesis. Rather, is this difference big enough to warrant rejection of $H_0$? More accurately, we should only reject $H_0$ if the probability of getting a $\bar{y}$ further away than 1.1 from $H_0$, is small. This probability is called a $p$-value.

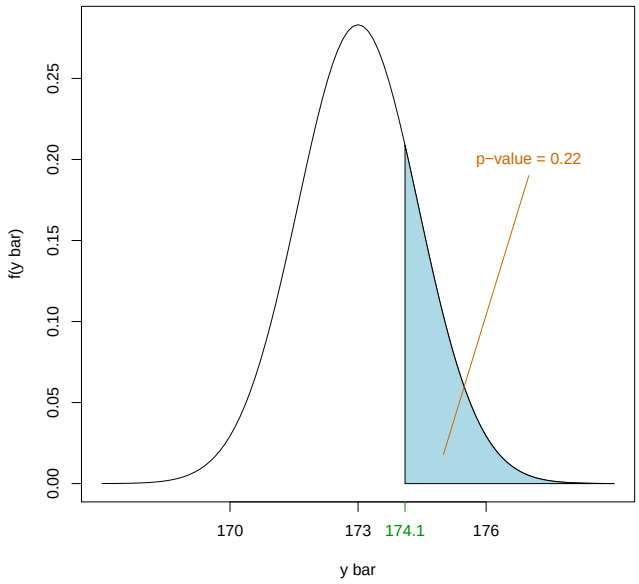
Recall once again that $\bar{y}$ is a random variable. Its value depends on the random sample that we draw from the population. A different sample might give us $\bar{y} = 190$. This would be "worse" for the null hypothesis of 173, than getting the value $\bar{y} = 174.1$. Out of all the samples that we could draw, out of all the parallel universes, what proportion of them would provide a $\bar{y}$ that is further than 1.1 from $H_0$? Imagine that only 4.3% of possible samples from the population were further than 1.1 from $H_0$. We have to decide one of two things. Either we have witnessed a rare event (are living in a strange universe) and the null is true, or the null is false. The actual $p$-value for this example is not 4.3%. We will now discuss how to determine the actual $p$-value for this problem, and for other problems in general.

As we have repeatedly stated, $\bar{y}$ is a random variable. It has a probability function, which we call a sampling distribution (because it's an estimator). We have derived the sampling distribution: $\bar{y} \sim N(\mu_y, \sigma_y^2/n)$. The sampling distribution can be used to calculate various events involving $\bar{y}$. For example, if we want to know the probability that $\bar{y} > 18$, we can draw out the normal curve (provided that we know $\mu_y$ and $\sigma_y^2/n$) and calculate the area under the curve, to the right of 18.

Classical hypothesis testing proceeds by assuming that $H_0$ is true. If $H_0$ is true, then the sampling distribution of $\bar{y}$ is $N(\mu_{y,0}, \sigma_y^2/n)$. That is, if the null hypothesis is correct, the true mean of $\bar{y}$ is $\mu_{y,0}$. To calculate the $p$-value, we still need to know $\sigma_y^2$. For now, we will assume that it is know, but this is an unrealistic assumption. In the real world, we will have to estimate $\sigma_y^2$.

Assuming that we know that $\sigma_y^2 = 39.7$ (again, this is very unrealistic) then we have the variance of the sample average ($\sigma_y^2/n = {}^{39.7}/_{20} = 2.0$), and so the full sampling distribution of the sample mean under the null hypothesis is: $\bar{y} \sim N(173, 2)$. This probability function is drawn in figure (3.2). All

Figure 3.2: Normal distribution with $\mu = 173$ and $\sigma^2 = {}^{39.7}/_{20}$. Shaded area is the probability that the normal variable is greater than 174.1.




that remains is to calculate the probability of obtaining a $\bar{y}$ that is *more adverse* to the null hypothesis than the one we just calculated. Half of this probability is represented by the shaded region in figure (3.2). This is a two sided test, so it doesn't matter if $\bar{y}$ is too large or too small: we need to multiply the *one-sided* $p$-value by 2. So, the $p$-value for our two-sided test is $0.22 \times 2 = 0.44$.

The interpretation of the p-value of 0.44 is as follows. If the null hypothesis of $H_0 = 173$ is true, then there is a 44% chance of observing a $\bar{y}$ that is further away from 173 than the difference of $174.1 - 173 = 1.1$ that we just observed. Would you "reject" or "fail to reject" based on this? Most researchers would fail to reject. There is a high probability of getting a $\bar{y}$ much more adverse to the null, so the null seems plausible.

### 3.3.1 Significance of a test

At what point should we decide that the p-value is too small, and reject the null hypothesis? The choice is somewhat arbitrary, and is up to the researcher (you). Standard choices have been 10%, 5%, and 1%. A pre-decided maximum p-value under which $H_0$ will be rejected is called the *significance level* of the test. It is sometimes denoted by $\alpha$. In the previous example, we fail to reject the null at the 10% significance level. Note that

failing to reject at the 10% level implies that we also fail to reject $H_0$ at the 5% and 1% significance levels.

### 3.3.2 Type I error

Take another look at figure (3.2). Even when the null hypothesis is true and figure (3.2) is the correct sampling distribution for $\bar{y}$, we will sometimes randomly draw a weird sample that makes $H_0$ appear to be "wrong". That is, even when the null is true, in some of the parallel universes we will draw a sample that gives a $\bar{y}$ that is very far from the truth. In these cases, we will erroneously reject the null. If the null hypothesis is falsely rejected, it is called a *type I error*. Type I error is the probability that $H_0$ is rejected when the null is true:

$$\Pr(\text{type I error}) = \Pr(\text{reject } H_0 \mid H_0 \text{ is true}) \tag{3.6}$$

How do we determine what this type I error will be? As soon as we pick the significance of the test, it has been determined. That is, type I error $= \alpha$. When we decide that 5% of $\bar{y}$s that are furthest from $H_0$ are just too rare, we are deciding that we will make a type I error in 5% of the parallel universes (or in 5% of other similar situations). That is, if we conduct thousands of scientific studies where we always use $\alpha = 5\%$, in 5% of those studies where we reject the null, we will be doing so falsely.

In reality, we do not know the population values, so we will never know if we have made a type I error or not. That is, the idea of type I error tells us nothing about the particular sample that we are working with. It only tells us something about what happens through repeated applications of our tested procedure.

### 3.3.3 Type II error

There is another type of error we can make. There are two possibilities for $H_0$: either it is true or false. In type I error, we considered that $H_0$ is actually true. If we consider that $H_0$ is actually false, then we make a *type II error* if we *fail to reject*. The probability of a type II error is:

$$\Pr(\text{type II error}) = \Pr(\text{fail to reject } H_0 \mid H_0 \text{ is false}) \tag{3.7}$$

If $H_0$ is actually false, one of two things can happen: we "reject" or we "fail to reject". The probabilities of both of these events must sum to 1 (something must happen). So:

$$\Pr(1 - \text{type II error}) = \Pr(\text{reject } H_0 \mid H_0 \text{ is false}) \tag{3.8}$$

Equation (3.8) is called the *power* of the test. We want the power to be as high as possible. That is, we do not want to make a type II error, and

we want the probability of rejection to be as high as possible when $H_0$ is actually false.

Determining the type II error (and power) of a test is difficult or impossible. This is because power depends on knowing the unobservable population. The concept is useful, however, when we are trying to find the "best" test available. In may be possible to determine that some ways of testing are more powerful than others, even though we may not know what the actual numbers are.

### 3.3.4 Test statistics

A *test statistic* is a convenient way of assessing the null hypothesis, and provides an easier way to obtain a $p$-value. If we wanted to use the above testing procedure for different problems, we would have to "graph" a different normal curve (similar to the one in figure 3.2), and calculate a different area under the curve, for each testing problem. Decades ago, calculating an area under the normal curve was difficult (now it is easily done by computers). Consequently, a method was devised so that every such testing problem would use the *standard normal curve*. That way, different areas under the curve could be tabulated for various values on the x-axis.

To *standardize* a variable, we subtract its mean and divide by its standard deviation. This creates a new normal random variable from the old one, called a "standard normal" variable. For example, let $y \sim N(\mu_y, \sigma_y^2)$. Create a new variable $z$ where:

$$z = \frac{y - \mu_y}{\sigma_y} \tag{3.9}$$

Now, $z$ is still normally distributed, but has mean 0 and variance 1 since

$$\mathrm{E}[z] = \mathrm{E}[y - \mu_y] = \mathrm{E}[y] - \mu_y = \mu_y - \mu_y = 0$$

and

$$\mathrm{Var}[z] = \mathrm{Var}\left[\frac{y}{\sigma_y}\right] = \frac{\mathrm{Var}[y]}{\sigma_y^2} = \frac{\sigma_y^2}{\sigma_y^2} = 1$$

(refer to the rules of mean and variance).

How is this helpful? Recall the sampling distribution of $\bar{y}$ under the null hypothesis: $\bar{y} \sim N(\mu_{y,0}, \sigma_y^2/n)$. Create a new variable $z$. Subtract $\mu_{y,0}$ (the mean of $\bar{y}$ if the null is true) from $\bar{y}$. Now $z$ has mean 0 (if the null is actually true). Divide by the standard error (standard error = the standard deviation of an estimator) of $\bar{y}$, and $z$ has variance of 1. That is:

$$z = \frac{\bar{y} - \mu_{y,0}}{\sqrt{\sigma_y^2/n}} \sim N(0, 1) \tag{3.10}$$

This is the "$z$ test statistic" for the null hypothesis that $\mu_y = \mu_{y,0}$. If the null is true, then $\bar{y}$ should be close to $\mu_{y,0}$, implying that $z$ should be close

to 0. The probability of observing a $\bar{y}$ further away from $H_0$ than what we just observed from the sample is obtained by plugging $\bar{y}$ and $\mu_{y,0}$ into the $z$ statistic formula, and calculating a probability using the standard normal distribution. From our heights example, the $z$ statistic is:

$$z = \frac{174.1 - 173}{\sqrt{\frac{39.7}{20}}} = 0.78$$

Now, the question: "what is the probability of getting further away than 174.1 from the null hypothesis of 173?" has just been translated to: "What is the probability of a $N(0, 1)$ variable being greater than 0.78 (or less than -0.78)?" So, as you may have guessed:

$$\Pr(z > 0.78) = 0.22 \tag{3.11}$$

Since all such testing problems can be standardized, we only need to calculate the area under the curve for several possible $z$ values. These were tabulated long ago, and are reproduced in Table (3.2).

### 3.3.5 Critical values

Critical values are the most extreme values allowable for the test statistic, before the null hypothesis is rejected. Suppose that we choose a 5% significance level for our test. This means that if we receive a $p$-value that is less than 0.0250 in Table 3.2, we should reject the null hypothesis (since $2.5\% \times 2 = 5\%$). If we use Table 3.2 to find the $z$ statistic that corresponds to a significance level, we are finding the critical value for the test. According to Table 3.2, we see that a $p$-value of 0.0250 corresponds to a $z$ statistic of 1.96. This is the 5% critical value. We know that if the $z$ statistic that we calculate for our test end up being greater than 1.96 or less than -1.96, we will get a $p$-value that is less than 0.05, and we will reject the test.

### 3.3.6 Confidence intervals

A confidence interval corresponds to a significance level. Suppose that the significance level is 5%. Then, the 95% confidence interval contains all of the values for $\mu_{y,0}$ (all values for null hypotheses) that will not be rejected at 5% significance.

What is the probability that our $z$ statistic will be within a certain interval, if the null hypothesis is true? For example, what is the following probability?

$$\Pr\left(-1.96 \leq z \leq 1.96\right)? \tag{3.12}$$

Using Table 3.2, we can figure out that this probability is 0.95. Note that -1.96 and 1.96 are the left and right critical values, respectively, for a

test with 5% significance. Now, to solve for the confidence interval around $\bar{y}$, we will first substitute the formula for the $z$ statistic into equation 3.12:

$$\Pr\left(-1.96 \leq \frac{\bar{y} - \mu_{y,0}}{\sqrt{\sigma_y^2/n}} \leq 1.96\right) = 0.95 \tag{3.13}$$

Finally, we solve equation 3.13 so that the null hypothesis $\mu_{y,0}$ is in the middle of the probability statement:

$$\Pr\left(\bar{y} - 1.96 \times \sqrt{\frac{\sigma_y^2}{n}} \leq \ \mu_{y,0} \ \leq \bar{y} + 1.96 \times \sqrt{\frac{\sigma_y^2}{n}}\right) = 0.95 \tag{3.14}$$

This just says that $1.96 \times \sigma_y^2/n$ is the maximum distance that the null hypothesis can be from the sample average that we calculate, before we would get a $p$-value less than 0.05, and reject the test at the 5% significance level.

An alternative interpretation of the confidence interval (other than containing the set of values for the null that won't be rejected), is the following. Out of many such 95% confidence intervals that we construct in many hypothesis tests, 95% of such intervals will include the true population mean, $\mu_y$. Two common *misinterpretations* of a confidence interval are: (i) there's a 95% probability that $\mu_y$ lies within the interval; and (ii) the confidence interval includes $\mu_y$ 95% of the time. The reason these last two interpretation are wrong has to do with the fact that the confidence interval is *random* and $\mu_y$ is fixed.

## 3.4 Hypothesis Tests (unknown $\sigma_y^2$)

So far we have assumed that $\sigma_y^2$ is known. We needed this $\sigma_y^2$ in order to calculate the variance of $\bar{y}$ (which is $\sigma_y^2/n$), and calculate our $p$-value.

But, if we have to estimate $\mu_y$, it is unlikely that we would know $\sigma_y^2$. That is, if the population mean is unknown, it is likely that the population variance would be unknown as well. Hence, we now need to figure out how to estimate $\sigma_y^2$ from our sample of data, $y$.

### 3.4.1 Estimating $\sigma_y^2$

Recall that the variance for a discrete random variable is defined as:

$$\text{Var}(Y) = \sum_{i=1}^{K} p_i \times (Y_i - \text{E}[Y_i])^2$$

where $Y_i$ are the different values that the random variable can take, and $p_i$ are the probabilities of those values occurring. A sensible way of estimating

$\sigma_y^2$ may be to take the *sample average* of the squared distances, but replacing $\mathrm{E}[Y_i]$ with $\bar{y}$. That is, a natural estimator for $\sigma_y^2$ might be:

$$\hat{\sigma}_y^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 \tag{3.15}$$

When we considered whether or not $\bar{y}$ was a good estimator for $\mu_y$, we first took the expected value of $\bar{y}$, and determined that it was *unbiased*. That is, it turned out that $\mathrm{E}[\bar{y}] = \mu_y$. Well, it turns out that $\hat{\sigma}_y^2$ is a *biased* estimator! We won't derive the expected value here, we will only state it:

$$\mathrm{E}\left[\hat{\sigma}_y^2\right] = \frac{n-1}{n}\sigma_y^2 \tag{3.16}$$

Equation 3.16 says that if we were to use equation 3.15 to estimate the variance of $y$, on average our estimate would be a little bit too small compared to the truth (by a factor of $(n-1)/n$). However, armed with this knowledge, we can construct what is called a *bias corrected* estimator. If we just multiply the right-hand-side of 3.16 by $n/(n-1)$, the bias disappears! That is, if we multiply the estimator $\hat{\sigma}_y^2$ by $n/(n-1)$, the resulting estimator is unbiased. This bias corrected estimator is usually denoted $s_y^2$, where:

$$s_y^2 = \frac{n}{n-1} \times \hat{\sigma}_y^2 = \frac{n}{n-1} \times \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 \tag{3.17}$$

### 3.4.2 The *t*-test

Now that we know how to estimate $\sigma_y^2$, we can estimate the variance of the sample average using:

$$\text{Estimated variance of } \bar{y} = \frac{s_y^2}{n}$$

We can implement hypothesis testing by replacing the unknown $\sigma_y^2$ with its estimator $s_y^2$. The $z$ test statistic now becomes:

$$\frac{\bar{y} - \mu_{y,0}}{\sqrt{s_y^2/n}} = t$$

This is the $t$ statistic. Because we have replaced $\sigma_y^2$ with $s_y^2$ (a random estimator) in the $z$ statistic formula, the form of the randomness of $z$ has changed. The $t$ statistic is no longer a standard normal variable. It follows its own probability distribution, called the $t$ distribution. When performing a $t$ test, the $p$-values are different than in Table 3.2. However, as the sample size grows, the $t$ distribution becomes the standard normal distribution. This means that, for sample sizes of approximately $n > 100$, using the standard normal distribution (Table 3.2) instead of the $t$ distribution, makes

very little difference. For the purposes of this course, we will assume that the sample size is large enough that the $t$ statistic follows a standard normal distribution.

Finally, note that confidence intervals can be constructed, in practice, by replacing the unknown $\sigma_y^2$ in equation 3.14 with the estimator $s_y^2$. As long as the sample size is reasonably large, we do not have to worry about replacing the critical values in the confidence interval formula (for example, 1.96) with critical values from the $t$ distribution. An example of performing a $t$ test and constructing a confidence interval, is left for the Review Questions.

## 3.5 Review Questions

1. Prove that $\bar{y}$ is a random variable. Why might $\bar{y}$ follow a Normal distribution? What is the sampling distribution for $\bar{y}$?

2. Derive the mean and variance of $\bar{y}$. How does this help us determine if $\bar{y}$ is: (i) unbiased; (ii) efficient; and (iii) consistent?

3. Assume that $y_i \sim (\mu_y, \sigma_y^2)$, and that $y_i$ is i.i.d. Let $\tilde{\mu}_y = \frac{y_1+y_n}{2}$. Is $\tilde{\mu}_y$ an unbiased estimator for $\mu_y$? Compare the variance of $\tilde{\mu}_y$ to the variance of $\bar{y}$.

4. Assume that $y_i \sim (\mu_y, \sigma_y^2)$, that $y_i$ is i.i.d., and that the sample size, $n$, is even. Let

$$\hat{\mu}_y = \frac{1}{2n}y_1 + \frac{3}{2n}y_2 + \frac{1}{2n}y_3 + \frac{3}{2n}y_4 + \cdots + \frac{1}{2n}y_{n-1} + \frac{3}{2n}y_n$$

   Is $\hat{\mu}_y$ an unbiased estimator for $\mu_y$? Compare the variance of $\hat{\mu}_y$ to the variance of $\bar{y}$.

5. Refer to the above two questions. Are $\tilde{\mu}_y$ and $\hat{\mu}_y$ consistent estimators for $\mu_y$?

6. Perform a $t$ test of the null hypothesis in equation (3.5), using the heights data from table 3.1. Also, construct 95% and 90% confidence intervals around $\bar{y}$.

## 3.6 Answers

1. The formula for $\bar{y}$ is $1/n \sum_{i=1}^{n} y_i$. It is a linear function of the random $y_i$ values, so it is a random variable itself. $\bar{y}$ might follow a Normal distribution due to the central limit theorem, which (loosely speaking) says that if we add up random variables the resulting sum tends to be Normally distributed. Note the summation operator in the formula

for $\bar{y}$. Finally, the full sampling distribution can be written as: $\bar{y} \sim N(\mu_y, \sigma_y^2/n)$.

2. The mean of $\bar{y}$ is derived in equation (3.2) and the variance in equation (3.3). (i) The mean of $\bar{y}$ tells us that the estimator is unbiased. (ii) The variance of $\bar{y}$ allows us to compare to the variance of all other possible linear and unbiased estimators of $\mu_y$, and determine that $\sigma_y^2/n$ is smallest, and thus $\bar{y}$ is efficient. (iii) The $n$ in the denominator of $\sigma_y^2/n$ shows us that $\bar{y}$ is consistent. We know that the estimator is unbiased, and as the sample size grows, the variance of $\bar{y}$ goes to zero. This means that with a infinitely large sample size, our estimator would give the value $\mu_y$ with probability 1.

3. To derive the bias of the estimator $\tilde{\mu}_y$, we compare its expected value to $\mu_y$:

$$\mathrm{E}\left[\tilde{\mu}_y\right] = \mathrm{E}\left[\frac{y_1 + y_n}{2}\right] = \frac{1}{2}\mathrm{E}\left[y_1 + y_n\right] = \frac{2\mu_y}{2} = \mu_y$$

Since the expected value of the estimator is equal to $\mu_y$, the estimator is unbiased.

The variance of $\tilde{\mu}_y$ is:

$$\mathrm{Var}\left[\tilde{\mu}_y\right] = \mathrm{Var}\left[\frac{y_1 + y_n}{2}\right] = \frac{1}{4}\mathrm{Var}\left[y_1 + y_n\right]$$

The i.i.d. assumption gives us the independence of the $y_i$ values, allowing us to expand within the variance operator:

$$\frac{1}{4}\mathrm{Var}\left[y_1 + y_n\right] = \frac{1}{4}\left(\mathrm{Var}\left[y_1\right] + \mathrm{Var}\left[y_n\right]\right) = \frac{2\sigma_y^2}{4} = \frac{\sigma_y^2}{2}$$

Comparing this variance to the variance of the sample average, we find:

$$\frac{\sigma_y^2}{2} > \frac{\sigma_y^2}{n} \quad ; n > 2$$

which is not surprising result, since we know that $\bar{y}$ is an efficient estimator.

4. Again, we start by taking the expected value of the estimator:

$$\begin{aligned}\mathrm{E}\left[\hat{\mu}_y\right] &= \mathrm{E}\left[\tfrac{1}{2n}y_1 + \tfrac{3}{2n}y_2 + \tfrac{1}{2n}y_3 + \cdots + \tfrac{3}{2n}y_n\right] \\ &= \tfrac{1}{2n}\mu_y + \tfrac{3}{2n}\mu_y + \tfrac{1}{2n}\mu_y + \cdots + \tfrac{3}{2n}\mu_y \\ &= \mu_y\end{aligned}$$

So, $\hat{\mu}_y$ is an unbiased estimator.

Next, we find the variance of $\hat{\mu}_y$, again making use of the independence assumption:

$$\begin{aligned}
\text{Var}\left[\hat{\mu}_y\right] &= \text{Var}\left[\tfrac{1}{2n}y_1 + \tfrac{3}{2n}y_2 + \tfrac{1}{2n}y_3 + \cdots + \tfrac{3}{2n}y_n\right] \\
&= \frac{1}{4n^2}\text{Var}\left[y_1\right] + \frac{9}{4n^2}\text{Var}\left[y_2\right] + \ldots \\
&= \frac{1}{4n^2}\,\sigma_y^2 + \frac{9}{4n^2}\,\sigma_y^2 + \ldots \\
&= \frac{5}{4n}\,\sigma_y^2
\end{aligned}$$

We can see that this variance is larger than the variance of $\bar{y}$, which is another illustration of the efficiency property of $\bar{y}$.

5. $\tilde{\mu}_y$ (for example) is a consistent estimator if $\lim_{n\to\infty}\text{E}[\tilde{\mu}_y] = \mu_y$ and $\lim_{n\to\infty}\text{Var}[\tilde{\mu}_y] = 0$. We have already shown that the estimator is unbiased, so the first condition is satisfied. However, the variance of this estimator does not go to 0 as the sample size increases, so this estimator is not consistent! That is:

$$\lim_{n\to\infty}\frac{\sigma_y^2}{2} = \frac{\sigma_y^2}{2}$$

On the other hand, the estimator $\hat{\mu}_y$ is consistent, since there is an $n$ in the denominator of $\frac{5}{4n}\sigma_y^2$.

6. The null and alternative hypotheses are:

$$\begin{aligned}
H_0 &: \mu_y = 173 \\
H_A &: \mu_y \neq 173
\end{aligned}$$

The sample mean and the sample variance are $\bar{y} = 174.1$ and $s_y^2 = 53.0$. The sample size is $n = 20$. The $t$ statistic is:

$$t = \frac{174.1 - 173}{\sqrt{53.0/20}} = 0.68$$

Assuming that the sample size is large enough (even though $n = 20$ is too small), we can use the standard Normal distribution, and table 3.2 to find that the $p-$value $= 0.2483 \times 2 = 0.5$. We fail to reject the null hypothesis.

The 95% confidence interval is:

$$\bar{y} \pm 1.96 \times \sqrt{s_y^2/n} = 174.1 \pm 1.96 \times 1.63 = [170.9, 177.3]$$

For the 90% confidence interval, we need to change the critical value of 1.96. Using table 3.2, we find the $z$ value which has 5% area under the curve (5% $\times$ 2 = 10% significance, 100% - 10% = 90% confidence). The 10% critical value is 1.64, so the 90% confidence interval is:

$$\bar{y} \pm 1.64 \times \sqrt{s_y^2/n} = 174.1 \pm 1.64 \times 1.63 = [171.4, 176.8]$$

Table 3.2: Area under the standard normal curve, to the right of $z$.

| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .5000 | .4960 | .4920 | .4880 | .4840 | .4801 | .4761 | .4721 | .4681 | .4641 |
| 0.1 | .4602 | .4562 | .4522 | .4483 | .4443 | .4404 | .4364 | .4325 | .4286 | .4247 |
| 0.2 | .4207 | .4168 | .4129 | .4090 | .4052 | .4013 | .3974 | .3936 | .3897 | .3859 |
| 0.3 | .3821 | .3783 | .3745 | .3707 | .3669 | .3632 | .3594 | .3557 | .3520 | .3483 |
| 0.4 | .3446 | .3409 | .3372 | .3336 | .3300 | .3264 | .3228 | .3192 | .3156 | .3121 |
| 0.5 | .3085 | .3050 | .3015 | .2981 | .2946 | .2912 | .2877 | .2843 | .2810 | .2776 |
| 0.6 | .2743 | .2709 | .2676 | .2643 | .2611 | .2578 | .2546 | .2514 | .2483 | .2451 |
| 0.7 | .2420 | .2389 | .2358 | .2327 | .2296 | .2266 | .2236 | .2206 | .2177 | .2148 |
| 0.8 | .2119 | .2090 | .2061 | .2033 | .2005 | .1977 | .1949 | .1922 | .1894 | .1867 |
| 0.9 | .1841 | .1814 | .1788 | .1762 | .1736 | .1711 | .1685 | .1660 | .1635 | .1611 |
| 1.0 | .1587 | .1562 | .1539 | .1515 | .1492 | .1469 | .1446 | .1423 | .1401 | .1379 |
| 1.1 | .1357 | .1335 | .1314 | .1292 | .1271 | .1251 | .1230 | .1210 | .1190 | .1170 |
| 1.2 | .1151 | .1131 | .1112 | .1093 | .1075 | .1056 | .1038 | .1020 | .1003 | .0985 |
| 1.3 | .0968 | .0951 | .0934 | .0918 | .0901 | .0885 | .0869 | .0853 | .0838 | .0823 |
| 1.4 | .0808 | .0793 | .0778 | .0764 | .0749 | .0735 | .0721 | .0708 | .0694 | .0681 |
| 1.5 | .0668 | .0655 | .0643 | .0630 | .0618 | .0606 | .0594 | .0582 | .0571 | .0559 |
| 1.6 | .0548 | .0537 | .0526 | .0516 | .0505 | .0495 | .0485 | .0475 | .0465 | .0455 |
| 1.7 | .0446 | .0436 | .0427 | .0418 | .0409 | .0401 | .0392 | .0384 | .0375 | .0367 |
| 1.8 | .0359 | .0351 | .0344 | .0336 | .0329 | .0322 | .0314 | .0307 | .0301 | .0294 |
| 1.9 | .0287 | .0281 | .0274 | .0268 | .0262 | .0256 | .0250 | .0244 | .0239 | .0233 |
| 2.0 | .0228 | .0222 | .0217 | .0212 | .0207 | .0202 | .0197 | .0192 | .0188 | .0183 |
| 2.1 | .0179 | .0174 | .0170 | .0166 | .0162 | .0158 | .0154 | .0150 | .0146 | .0143 |
| 2.2 | .0139 | .0136 | .0132 | .0129 | .0125 | .0122 | .0119 | .0116 | .0113 | .0110 |
| 2.3 | .0107 | .0104 | .0102 | .0099 | .0096 | .0094 | .0091 | .0089 | .0087 | .0084 |
| 2.4 | .0082 | .0080 | .0078 | .0075 | .0073 | .0071 | .0069 | .0068 | .0066 | .0064 |
| 2.5 | .0062 | .0060 | .0059 | .0057 | .0055 | .0054 | .0052 | .0051 | .0049 | .0048 |
| 2.6 | .0047 | .0045 | .0044 | .0043 | .0041 | .0040 | .0039 | .0038 | .0037 | .0036 |
| 2.7 | .0035 | .0034 | .0033 | .0032 | .0031 | .0030 | .0029 | .0028 | .0027 | .0026 |
| 2.8 | .0026 | .0025 | .0024 | .0023 | .0023 | .0022 | .0021 | .0021 | .0020 | .0019 |
| 2.9 | .0019 | .0018 | .0018 | .0017 | .0016 | .0016 | .0015 | .0015 | .0014 | .0014 |
| 3.0 | .0013 | .0013 | .0013 | .0012 | .0012 | .0011 | .0011 | .0011 | .0010 | .0010 |
| 3.1 | .0010 | .0009 | .0009 | .0009 | .0008 | .0008 | .0008 | .0008 | .0007 | .0007 |
| 3.2 | .0007 | .0007 | .0006 | .0006 | .0006 | .0006 | .0006 | .0005 | .0005 | .0005 |
| 3.3 | .0005 | .0005 | .0005 | .0004 | .0004 | .0004 | .0004 | .0004 | .0004 | .0003 |
| 3.4 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0002 |

# 4

# Ordinary Least Squares (OLS)

In this chapter, we discuss a method to estimate the marginal effect of one variable on another. Economic models typically posit that one variable *causes* or *determines* another variable. Seldom (or never) does the economic model *quantify* the marginal effect. We need data and econometrics in order to estimate a *number* for the marginal effect.

We begin the chapter with two motivating examples. They are meant to show that many simple economic models can be represented through the equation for a line. We then proceed to estimate this line uses data. The method that we use to fit a straight line through data points is *ordinary least squares* (OLS) or just *least squares*. We will make some simplifying assumptions, and discuss the properties of the OLS estimator.

## 4.1 Motivating Example 1: Demand for Liquor

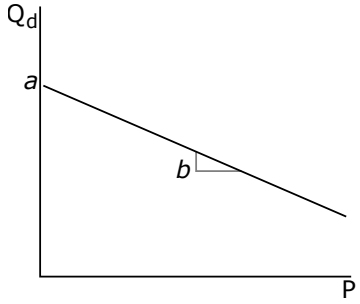
How much less alcohol will people consume if we raise the price? In first-year microeconomics you learned about the law of demand. The quantity demanded of a product should depend on its price (and other things):

$$Q_d = a + bP \tag{4.1}$$

where $a$ is the intercept of the demand "curve", and $b$ is the slope. See figure 4.1. You learned that the slope of the demand curve, $b$, depends on the type of good. For example, necessities such as medicine should have relatively flatter demand curves than luxuries such as a diamonds.

Estimating the slope of the demand curve is important for policy makers who might want to affect the quantity demanded of a good. For example, we might want to reduce consumption of alcohol or cigarettes by increasing price (taxing them). But before we fiddle with the price of these products,

Figure 4.1: A typical demand "curve". Note this is an "inverse" demand curve (quantity demanded is on the vertical axis, and price on the horizontal axis).




we should estimate how much quantity demanded will change given a change in price (if it changes at all).

Using data from Prest (1949), we plot the yearly (from 1870 to 1938) per-capita consumption of spirits (in proof gallons), and the relative price of spirits (deflated by a cost-of-living index). See figure 4.2. How should we fit a line through the data in figure 4.2? If we can pick a "good" line, then we will have a good estimate for the slope, $b$. This estimated $b$ could then be used to determine how much alcohol consumption will decrease if we increase the tax on alcohol by \$1, for example. Note that $b$ is the marginal effect of a change in price of spirits, on the quantity demanded of spirits, holding all else constant.

## 4.2 Motivating Example 2: Marginal Propensity to Consume

This example uses data on total disposable income and consumption (in millions of Pounds) from 1971-1985 (quarterly) in the U.K. (Verbeek and Marno, 2008). The data is shown in figure 4.3.

An increase in consumption is induced by an increase in income, but not all of the increase in income is consumed. Marginal propensity to consume is the proportion of an increase in disposable income that individuals spend on consumption:

$$MPC = \frac{\Delta C}{\Delta Y} \tag{4.2}$$

where $\Delta C$ is the change in consumption "caused" by the change in income, $\Delta Y$. John Maynard Keynes supposed that the $MPC$ should be less than one, but without data and econometrics there is no way to put an actual number to the $MPC$.

Figure 4.2: Per capita consumption, and price, of spirits. Choosing a line through the data necessarily chooses the slope of the line, $b$, which determines how much $Q_d$ decreases for an increase in $P$.

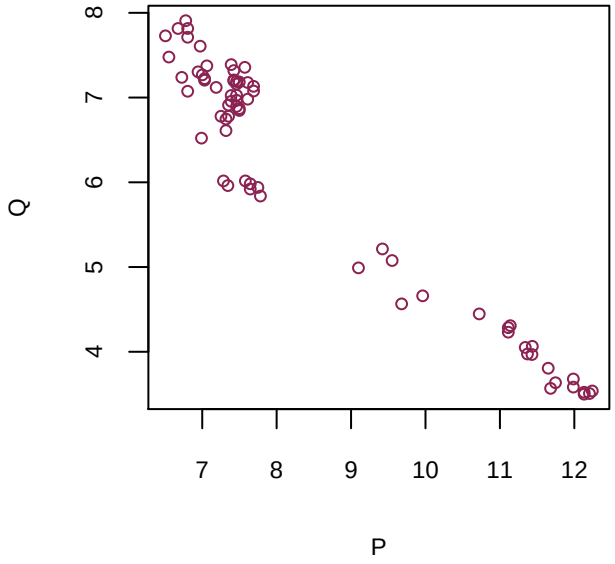
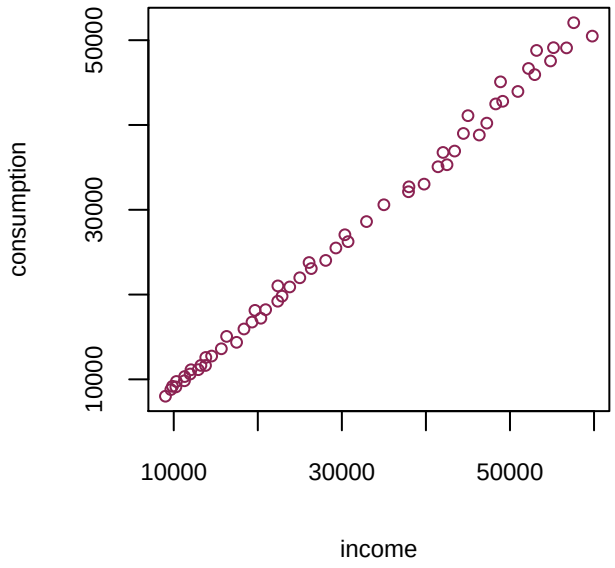



Figure 4.3: Income and consumption in the U.K. (Verbeek and Marno, 2008).



We can also write the relationship between consumption and disposable income through the equation of a line:

$$C = a + MPC \times Y \tag{4.3}$$

where $a$ is again the intercept of the line (representing the amount of consumption with disposable income of zero), and where this time $MPC$ is the *slope* of the line. Remember that $MPC$ is the thing we are trying to estimate.

One of the points we are trying to make here is that many economics models can be represented by the equation of a straight line. If we can figure out how to estimate the line, then we have an estimate for the slope (the marginal effect), which is of great practical usefulness.

The next question is: how should we fit a line through data points (like the ones in figures 4.2 and 4.3)? Before we determine how to pick the line, however, we need to introduce some definitions and general notation.

## 4.3 The Linear Population Regression Model

The general regression model is:

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i \tag{4.4}$$

- $X$ is called the *independent* variable or *regressor*. It is the variable that is assumed to *cause* the $Y$ variable. In the "Demand for Liquor" example, this variable was *price* ($P$). See equation 4.1. In the $MPC$ example the regressor was *income*. See equation 4.3.
- $Y$ is the *dependent* variable. This variable is assumed to be caused by $X$ (it *depends* on $X$). In the demand example the dependent variable was *quantity demanded* ($Q_d$) and in the $MPC$ example it was *consumption* ($C$).
- $\beta_0$ is the population intercept. It was labelled $a$ in both examples. It is unobservable, but we can try to estimate it.
- $\beta_1$ is the population slope. When $X$ increases by 1, $Y$ increases by $\beta_1$. This is the primary object of interest, and is unobservable. We want to estimate $\beta_1$. $\beta_1$ is interpreted as the marginal effect in many economics models.
- $\epsilon$ is the regression *error* term. It consists of all the other factors or variables that determine $Y$, other than the $X$ variable. All of these other variables causing $Y$ are combined into $\epsilon$. $\epsilon$ is considered to be a random variable since we can not observe it.

- $i = 1, \ldots, n$. The subscript $i$ denotes the observation. $n$ is the sample size. For example, $Y_4$ refers to the fourth $Y$ observation in the data set.

### 4.3.1 The importance of $\beta_1$

Note that in equation 4.4, the object of interest is $\beta_1$. It is the thing we are trying to estimate. It is the causal, or marginal effect, of $X$ on $Y$. That is, a change in $X$ of $\Delta X$ causes a $\beta_1$ change in $Y$:

$$\frac{\Delta Y}{\Delta X} = \beta_1$$

### 4.3.2 The importance of $\epsilon$

$\epsilon$ (epsilon) is the random component of the model. Without $\epsilon$, statistics/econometrics is not required. $\epsilon$ represents all of the other things that determine $Y$, other than $X$. They are all added up and lumped into this one random variable. Because we can not observe all of these other factors, we consider them to be random. The fact that $\epsilon$ is random makes $Y$ random as well.

Later, we will make some assumptions about the randomness of $\epsilon$, that will ultimately determine the properties of the way that we choose to estimate $\beta_1$.

### 4.3.3 Why it's called a population model

Equation 4.4 is called a "population" model because it represents the true, but unknown way in which the $Y$ variable is "created" or "determined". $\beta_0$ and $\beta_1$ are unknown (and so is $\epsilon$). We will observe a sample of $Y$ and $X$, and use the sample to try to figure out the $\beta$s.

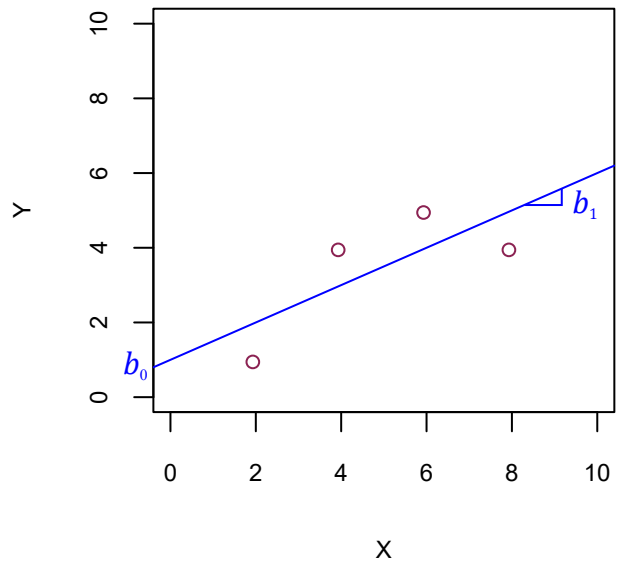
## 4.4 The estimated model

Our primary goal is to estimate $\beta_1$ (the marginal effect of $X$ on $Y$), but to do so we'll also have to estimate $\beta_0$. This estimated intercept and slope will define a straight line. These estimates will be denoted $b_0$ and $b_1$, the OLS intercept and slope.

Let's start with a very simple example using data that I made up: $Y = \{1, 4, 5, 4\}$, $X = \{2, 4, 6, 8\}$. The data, and estimated OLS line, are shown in figure 4.4. The OLS estimated intercept is $b_0 = 1$, and the estimated slope is $b_1 = 0.5$.

We still don't know how to get $b_0$ and $b_1$! Before we decide how to fit a straight line through some data points, we need to define two terms first.

Figure 4.4: A simple data set with the estimated OLS line in blue. $b_0$ is the OLS intercept, and $b_1$ is the OLS slope.




### 4.4.1 OLS predicted values ($\hat{Y}_i$)

The OLS predicted (or fitted) values, are the values for $Y$ that we get when we "plug" the $X$ values back into the estimated OLS line. These predicted $Y$ values are denoted by $\hat{Y}$. We can find each predicted value, $\hat{Y}_i$, by plugging each $X_i$ into the estimated equation.

In general, the estimated equation (or line) is written as:

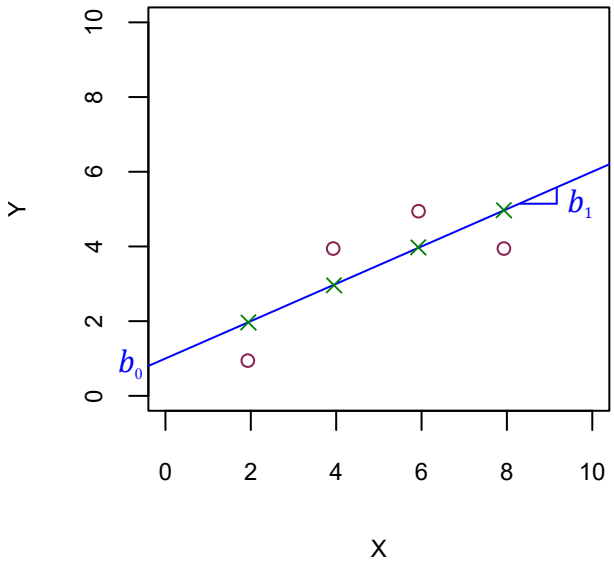
$$\hat{Y}_i = b_0 + b_1 X_i. \tag{4.5}$$

For our simple example, equation 4.5 becomes $\hat{Y}_i = 1 + 0.5X_i$, and each OLS predicted values is:

$$\begin{aligned}
\hat{Y}_1 &= 1 + 0.5(2) = 2 \\
\hat{Y}_2 &= 1 + 0.5(4) = 3 \\
\hat{Y}_3 &= 1 + 0.5(6) = 4 \\
\hat{Y}_4 &= 1 + 0.5(8) = 5
\end{aligned}$$

These OLS predicted values are added to the plot in figure 4.5. Notice how each predicted value lies on the blue line, directly above or below the data point.

Figure 4.5: The OLS predicted values shown by ×.




### 4.4.2 OLS residuals ($e_i$)

An OLS predicted value tells us what the estimated model predicts for $Y$ when given a particular value of $X$. When we plug in the sample values for $X$ (as we did in the previous section), we see that the predicted values ($\hat{Y}_i$) don't quite line up with the actual $Y_i$ values. The differences between the two are the OLS *residuals*. The OLS residuals are like prediction errors, and are determined by:

$$e_i = Y_i - \hat{Y}_i \tag{4.6}$$

Using equation 4.6 for our simple example, each OLS residual is:

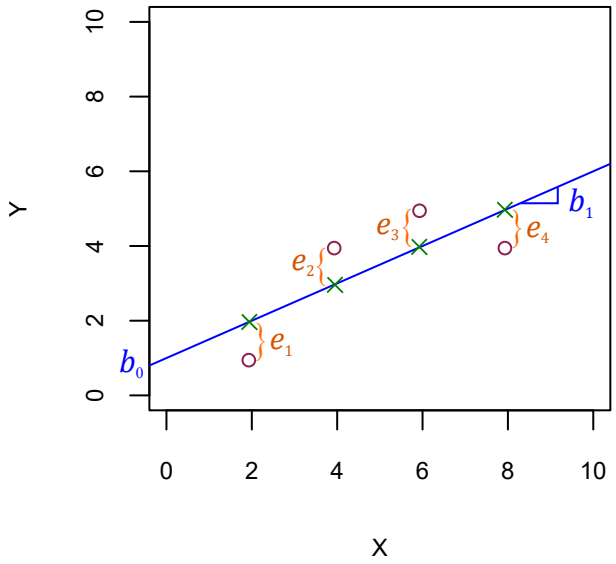
$$\begin{aligned} e_1 &= 1 - 2 = -1 \\ e_2 &= 4 - 3 = 1 \\ e_3 &= 5 - 4 = 1 \\ e_4 &= 4 - 5 = -1 \end{aligned}$$

These OLS residuals are indicated in figure 4.6. They are the vertical distances between the actual data points (the circles) and the OLS predicted values (the ×).

Each data point ($Y_i$) is equal to its predicted value, plus its residual. That is, we can rearrange equation 4.6 and write:

$$Y_i = \hat{Y}_i + e_i$$

Figure 4.6: The OLS residuals ($e_i$) are the vertical distances between the actual data points (circles) and the OLS predicted values ($\times$).




or, using equation 4.5 for the definition of $\hat{Y}_i$:

$$Y_i = b_0 + b_1 X_i + e_i, \tag{4.7}$$

which will be useful in the next chapter. Note that equation 4.7 is the observable counterpart to the unobservable population model in equation 4.4.

## 4.5 How to choose $b_0$ and $b_1$, the OLS estimators

Now that we have defined the OLS residuals ($e_i$), we can define the OLS estimators $b_0$ and $b_1$ by coming up with an equation that will tell us how to use the $X$ and $Y$ data.

The OLS estimators are defined in the following way. They are the values for $b_0$ and $b_1$ that minimize the sum of squared vertical distances between the OLS line and the actual data points ($Y_i$). These vertical distances have already been defined as the OLS residuals ($e_i$). So the "objective" is to choose $b_0$ and $b_1$ so that $\sum_{i=1}^{n} e_i^2$ is minimized. This is an optimization problem from calculus. Formally stated, the OLS estimator is the solution to the minimization problem:

$$\min_{b_0, b_1} \sum_{i=1}^{n} e_i^2 \tag{4.8}$$

Substituting the value for $e_i$ (equation 4.6) into equation 4.8:

$$\min_{b_0,b_1} \sum_{i=1}^{n} \left(Y_i - \hat{Y}_i\right)^2$$

and substituting in the value for $\hat{Y}_i$ (from equation 4.5) we get:

$$\min_{b_0,b_1} \sum_{i=1}^{n} (Y_i - b_0 - b_1 X_i)^2 \tag{4.9}$$

To solve this minimization problem, we take the partial derivatives of $\sum_{i=1}^{n} e_i^2$ with respect to $b_0$ and $b_1$, set those derivatives equal to zero, and solve for $b_0$ and $b_1$. That is, we need to solve the two equations:

$$\frac{\partial \left(\sum_{i=1}^{n} e_i^2\right)}{\partial b_0} = 0$$

$$\frac{\partial \left(\sum_{i=1}^{n} e_i^2\right)}{\partial b_1} = 0$$

We leave the derivation for an exercise, and only write the solution here:

$$b_1 = \frac{\sum_{i=1}^{n} \left[\left(Y_i - \bar{Y}\right)\left(X_i - \bar{X}\right)\right]}{\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2} \tag{4.10}$$

$$b_0 = \bar{Y} - b_1 \bar{X}$$

These equations tell us how to pick a line (by picking an intercept and slope) in order to minimize the sum of squared vertical distances between the chosen line and each data point. The next question is, why should we choose a line in such a way?

## 4.6 The Assumptions and Properties of OLS

So, what's so great about OLS? There are many other ways that we could fit a line through some data points:

- instead of *vertical* distances, we could minimize the sum of *horizontal* or *orthogonal* distances
- instead of taking the sum of *squared* distances, we could take the sum of *absolute* distances
- we could divide the sample into two parts, get the average $Y$ and $X$ coordinates, and connect the dots

- we could pick (randomly or not) any two different data points and connect them

The main point here is that there are many ways that we could fit a line, so we should wonder why OLS is so special. Some of these alternatives above are obviously silly, but some lead to alternative estimators that have merit in various situations.

Recall that estimators are random variables (see Chapter 3). The OLS slope and intercept estimators have sampling distributions, with a mean and a variance. The reason why we use OLS is because these random estimators have good statistical properties (under certain assumptions). Here, we list the assumptions, and return to them at various stages throughout the book.

### 4.6.1 The OLS assumptions

A1 The population model is linear in the $\beta$s.

A2 There is no perfect multicollinearity between the $X$ variables.

A3 The random error term, $\epsilon$, has mean zero.

A4 $\epsilon$ is identically and independently distributed.

A5 $\epsilon$ and $X$ are independent.

A6 $\epsilon$ is Normally distributed.

### 4.6.2 The properties of OLS

Provided that the above six assumptions hold:

- The OLS estimator is unbiased.
- The OLS estimator is efficient.
- The OLS estimator is consistent.
- The OLS estimator is Normally distributed.

Note that not all assumptions are needed for each of the above four properties. Additionally, some of the assumptions A1 - A6 are often unrealistic. Testing for the validity of these assumptions, re-evaluating the properties of the OLS estimator in the absence of each assumption, and figuring out how to recover unbiasedness, efficiencyand consistency, would lead to some different estimators, and would form the basis for future econometrics courses.

## 4.7 Review Questions

1. Let the sample data be $Y = \{5, 2, 2, 3\}$ and $X = \{5, 3, 5, 3\}$.

   a) Write down the population model.

   b) Calculate the OLS estimated slope and intercept, using equation 4.10.

   c) Interpret these estimates.

   d) Calculate the OLS predicted values and residuals.

   e) Using R, verify your answer in part (b).

2. How are the formulas for $b_1$ and $b_0$ derived?

3. Explain why, even if assumption A.6 does not hold, the OLS estimator may still be normally distributed.

4. Why is the $\epsilon$ term needed in equation 4.4?

5. Download the MPC data from: http://home.cc.umanitoba.ca/~godwinrt/3040/data/mpc.csv. Use R to aid in the following exercises.

   a) Write down the population model you are trying to estimate. Describe the components of this model.

   b) Plot the data.

   c) Calculate the OLS estimated slope and intercept.

   d) Interpret these estimates.

   e) Add the estimated regression line to the plot of the data.

## 4.8 Answers

1. a) The assumed population model is $Y_i = \beta_0 + \beta_1 + \epsilon$. It is assumed that the $X$ variable "causes" the $Y$ variable. The $Y$ and $X$ data has been given to us. $\beta_0$ and $\beta_1$ are unknown parameters to be estimated. $\epsilon$ represents all the other factors (or variables) that cause $Y$ but that are unobserved.

   b)

$$\bar{Y} = 3, \quad \bar{X} = 4$$

$$b_1 = \frac{(5-3)(5-4) + (2-3)(3-4) + (2-3)(5-4) + (3-3)(3-4)}{(5-4)^2 + (3-4)^2 + (5-4)^2 + (3-4)^2}$$

$$= 0.5$$

$$b_0 = 3 - 0.5 \times 4 = 1$$

c) $b_1$ is the estimated slope, or marginal effect. Numerically, the values $b_1 = 0.5$ means that it is estimated that when $X$ increases by 1, $Y$ will increase by 0.5. $b_0$ is the estimated intercept. Numerically, when $X$ is 0, it is estimated that $Y$ is 1.

d)

$$\bar{Y_1} = 1 + 0.5(5) = 3.5$$
$$\bar{Y_1} = 1 + 0.5(3) = 2.5$$
$$\bar{Y_1} = 1 + 0.5(5) = 3.5$$
$$\bar{Y_1} = 1 + 0.5(3) = 2.5$$
$$e_1 = 5 - 3.5 = 1.5$$
$$e_2 = 2 - 2.5 = -0.5$$
$$e_3 = 2 - 3.5 = -1.5$$
$$e_4 = 3 - 2.5 = -0.5$$

e) In R, enter the following three commands:

```
y <- c(5,2,2,3)
x <- c(5,3,5,3)
lm(y ~ x)
```

and you should see the following output:

```
Call:
lm(formula = y ~ x)

Coefficients:
(Intercept)            x
        1.0          0.5
```

2. The formulas for the OLS estimator are derived by minimizing the sum of squared OLS residuals. This involves solving an optimization problem in calculus. The derivatives of the sum of squared residuals, with respect to $b_0$ and $b_1$, are set equal to 0 and solved, providing the formulas in equation 4.10.

3. If assumption A.6 holds, then the OLS estimators will be Normally distributed. This is because, by the population model (equation 4.4), $Y$ is a linear function of $\epsilon$, hence $Y$ is also Normally distributed. Furthermore, because $b_1$ and $b_0$ are linear functions of $Y$, they are also Normally distributed.

   However, even without A.6, the OLS estimator may still be Normally distributed. This is again due to the central limit theorem. Look again at the formula for the OLS estimator (equation 4.10) and note the summation sign. Since the OLS estimator involves summing the

random variable $Y$, as long as the sample size is large enough, the resulting sum should be Normally distributed.

4. The error term is needed in order to represent all of the other factors that influence $Y$, besides the $X$ variable. Since these other factors (or variables) are unobserved, we consider them to be random, and add them all up into one term. $\epsilon$ represents the randomness in the population model, without which there would be no need for statistics or econometrics.

5. a) The population model that we are trying to estimate is the consumption model from equation 4.3: $C = \beta_0 + \beta_1 \times Y + \epsilon$, where $C$ is the independent variable (the "$Y$" variable), $\beta_1$ is the MPC, $Y$ is the independent variable (the "$X$" variable), $\epsilon$ represents all the other variables that determine $C$, and where $\beta_0$ doesn't have much economic interest.

   b) First, you must load the data into R using the following two commands (in R, each command should be on a single line):

```
mpcdata <- read.csv("http://home.cc.umanitoba.ca/
     ~godwinrt/3040/data/mpc.csv")
attach(mpcdata)
```

   Once the data has been loaded, enter the following command (on a single line), in order to plot the data:

```
plot(income,consumption, main="Consumption and
     Income in the U.K.")
```

   c) In order to calculate the OLS estimates for the intercept and slope, run the following command in R:

```
lm(consumption ~ income)
```

   d) The estimated slope on income is the estimated marginal propensity to consume. That is, when *income* increases by 1, it is estimated that *consumption* will increase by 0.869. The estimated intercept of 176.848 is the amount of consumption when income (or GDP) is zero, and since GDP is never zero, the intercept doesn't hold much economic interest.

   e) In order to add the estimated regression line to your plot of data, use the following command (choose your own colour!):

```
abline(lm(consumption ~ income), col = "red")
```

# 5

# OLS Continued

In this chapter, we discuss three extensions of OLS. First, we introduce the regression R-square, which is a way to evaluate how well the estimated OLS regression line fits the data. Second, we discuss how to test a null hypothesis involving the $\beta$s (usually $\beta_1$). Third, we discuss the use of dummy variables in econometric models.

## 5.1 R-squared

R-squared is a "measure of fit" of the regression line. It is a number between 0 and 1 (as long as the model contains an intercept) that indicates how close the data points are to the estimated line. More accurately, the regression R-squared ($R^2$) is the portion of variance in the $Y$ variable that can be explained by variation in the $X$ variable.

Look again at the assumed population model:

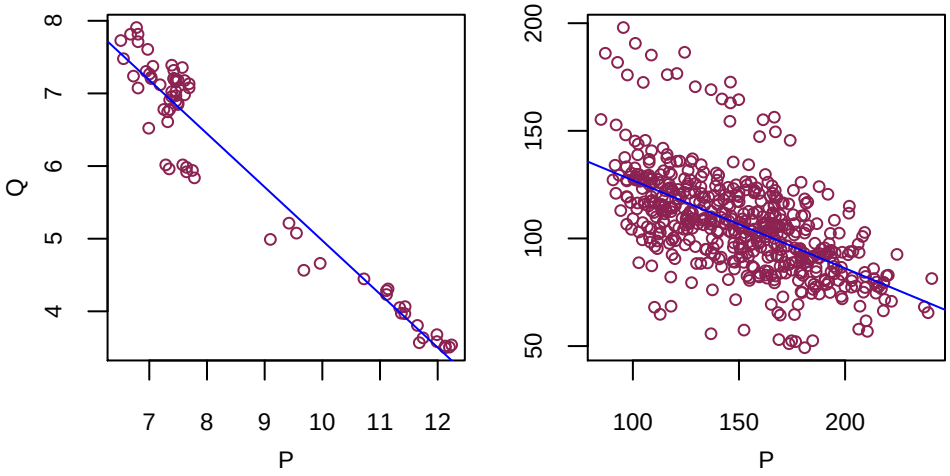
$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

The assumption is that changes in $X$ lead to changes in $Y$. We are using the observed changes in both variables to choose the regression line (via OLS). But, changes in $X$ aren't the only reason that $Y$ changes. There are unobservable variables in the error term ($\epsilon$) that lead to changes in $Y$. How much of the changes in $Y$ are coming from $X$ (not $\epsilon$)? $R^2$ helps answers this question.

The $R^2$ can also be thought of as an overall measure of how well the model explains the $Y$ variable. That is, we are using information in $X$ to explain or *predict* $Y$ by estimating a model. How well does the estimated regression line "fit" the data? How well does the model explain the $Y$ variable? $R^2$ provides a measure to address these questions. Let's reiterate the interpretations of $R^2$ before we derive it. $R^2$ measures:

- how well the estimated model explains the $Y$ variable.

Figure 5.1: Which estimated regression line fits better? Demand for spirits (left) and demand for cigarettes (right). We might expect the regression on the left to have a higher $R^2$.




- how well changes in $X$ explain changes in $Y$.
- how well the estimated regression line "fits" the data.
- the portion of the variance in $Y$ that can be explained using the estimated model.

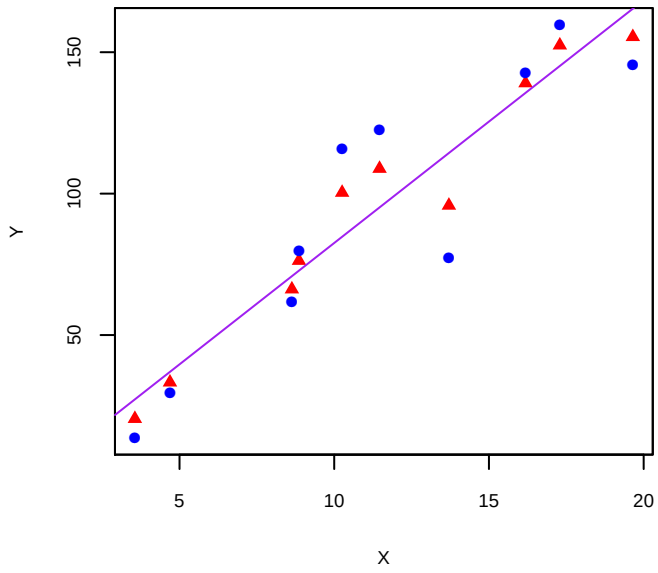
Figure 5.1 shows the estimated OLS regression line fitted to both the demand for spirits and demand for cigarettes data. The estimated regression line seems to fit the data better, or explain more of the variation in $Q$, for spirits rather than for cigarettes. We will find that the $R^2$ is indeed higher for the spirits data. In some sense, the $R^2$ can be used to compare OLS regressions.

Figure 5.2 shows a hypothetical situation where, if all data moves vertically further away from the estimated regression line, the regression line stays the same, but the $R^2$ decreases. That is, both the red (triangles) and blue (circles) provide the same estimated $b_1$, but the line fits the red data better. Changes in $X$ account for more of the changes in $Y$ for the red data. For the blue data, the *unobserved factors* (in $\epsilon$) are accounting for more of the changes (or variation) in $Y$.

### 5.1.1 The $R^2$ formula

Now, we will derive the $R^2$ statistic, beginning with the definition: "R-squared is the portion of variance in $Y$ that can be explained using the

Figure 5.2: Two different data sets. The estimated regression line for both data sets is the same. The blue data points (circles) are twice as far (vertically) from the regression line as are the red data points (triangles). For red data, $R^2 = 0.95$. For blue data, $R^2 = 0.82$.




estimated model." The population model is (equation 4.4):

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

The estimated model is (equation 4.7):

$$Y_i = b_0 + b_1 X_i + e_i$$

Recall that the OLS predicted value is (equation 4.5):

$$\hat{Y}_i = b_0 + b_1 X_i$$

So:

$$Y_i = \hat{Y}_i + e_i \tag{5.1}$$

Equation 5.1 shows that each $Y_i$ value has two parts: a part that can be explained by OLS ($\hat{Y}_i$), and a part that cannot ($e_i$). To get $R^2$, we'll start by taking the sample variance of both sides of equation 5.1. This will break the variance in $Y$ up into two parts: variance the we *can* explain (variance in $\hat{Y}_i$), and variance that we *can't* explain (variance in $e_i$).

Recall that in Chapter 3, when we wanted to estimate the variance of $y$, we used equation 3.17, which is the sample variance:

$$s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

Taking the sample variance of both sides of equation 5.1 we get (there is no sample covariance because $\hat{Y}_i$ and $e_i$ are independent):

$$s_Y^2 = s_{\hat{Y}}^2 + s_e^2$$

Or:

$$\frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(\hat{Y}_i - \bar{\hat{Y}}\right)^2 + \frac{1}{n-1}\sum_{i=1}^{n}(e_i - \bar{e})^2 \tag{5.2}$$

To simplify equation 5.2, we'll make use of three algebraic properties:

- the $(n-1)$ cancel out
- $\bar{\hat{Y}} = \bar{Y}$
- $\bar{e} = 0$

Using these three properties, equation 5.2 becomes:

$$\sum\left(Y_i - \bar{Y}\right)^2 = \sum\left(\hat{Y}_i - \bar{Y}\right)^2 + \sum(e_i)^2 \tag{5.3}$$

Notice that the terms in equation 5.3 are "sums of squares", and equation 5.3 is often written as:

$$TSS = ESS + RSS \tag{5.4}$$

where:

- TSS - total sum of squares
- ESS - explained sum of squares
- RSS - residual sum of squares

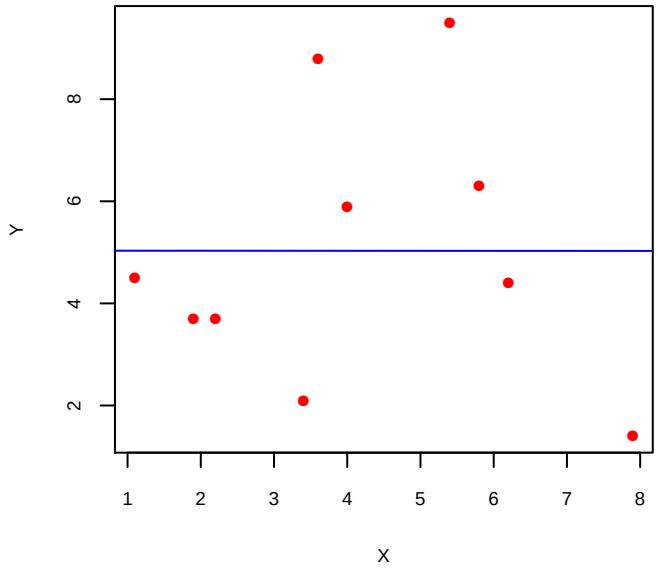
Now, we return to our definition of $R^2$: "the portion of variance in $Y$ that can be explained using the estimated model." This portion is written as:

$$R^2 = \frac{ESS}{TSS} \tag{5.5}$$

We can also re-write the formula for $R^2$ using equation 5.4:

$$R^2 = 1 - \frac{RSS}{TSS} \tag{5.6}$$

Figure 5.3: The estimated regression line is essentially flat: $b_1 = 0$. Observed changes in $X$ are not at all helpful in predicting changes in $Y$. There is "no fit", and $R^2 = 0.00$.




### 5.1.2 "No fit" and "perfect fit"

What is the worst possible situation, in terms of the "fit" of the estimated regression line? If the $X$ variable cannot explain any of the changes/variation in the $Y$ variable, then the estimated model (the estimated regression line) will be useless.

If the $X$ observations are not useful in explaining changes in the $Y$ observations (that is, if the sample $X$ and $Y$ data are *independent*), then $b_1 = 0$. In this case, we have a situation of "no fit", where $R^2 = 0$. See figure 5.3.

To see algebraically why $R^2 = 0$ when $b_1 = 0$, we start by looking at equation 4.5 again:

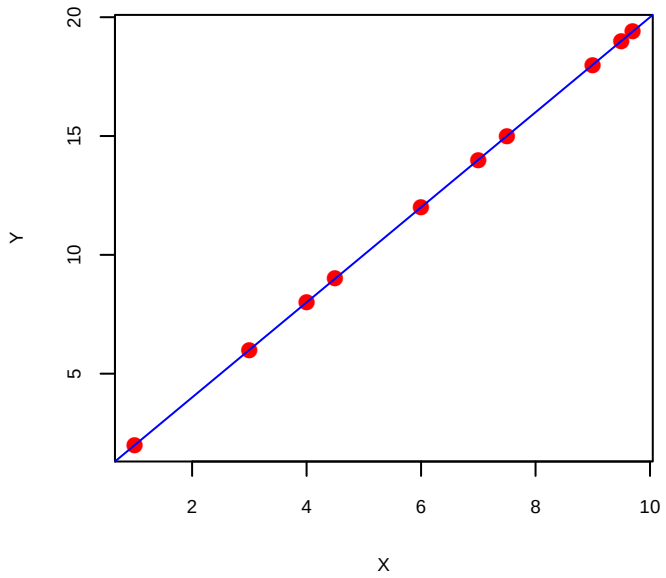
$$\hat{Y}_i = b_0 + b_1 X_i$$

So, if $b_1 = 0$ then each predicted $\hat{Y}_i$ value is equal to just $b_0$ (all the predicted values are the same). Additionally, when $b_1 = 0$, by looking at the equation for the OLS intercept estimator, we see that:

$$b_0 = \bar{Y} - b_1 X_i = \bar{Y}$$

This mean that, if $b_1 = 0$, each predicted value is equal to the sample average

Figure 5.4: The estimated regression line exactly passes through each data point. Observed changes in $X$ perfectly predict changes in $Y$. There is "perfect fit", and $R^2 = 1$.




of $Y$: $\hat{Y}_i = \bar{Y}$. Hence, $ESS = 0$:

$$ESS = \sum \left(\hat{Y}_i - \bar{Y}\right)^2 = \sum \left(\bar{Y} - \bar{Y}\right)^2 = 0,$$

and $R^2 = 0$.

Now, let's consider the opposite extreme: a situation where we have a "perfect fit". Imagine that observed changes in $X$ could perfectly predict a change in $Y$. That is, if we knew the value of $X$, we would exactly know the value of $Y$ with certainty. What would our sample of data have to look like in order for this to be the case? See figure 5.4.

In order for the estimated regression line to fit the data perfectly, all of the observed data points must line up in a straight line. If this were so, the estimated line would pass through each data point, the OLS predicted values ($\hat{Y}_i$) would be exactly equal to the actual values ($Y_i$), and there would be no prediction error ($e_i = 0 \ \forall \ i$). Algebraically, $\hat{Y}_i = Y_i$, so that $ESS = TSS$, and $R^2 = 1$.

The two cases that we have just considered, "no fit" and "perfect fit", are extremes. They should not actually occur in practice. In reality, the fit of the line will be somewhere between these two extremes. If the worst that can happen is "no fit" and the best is " perfect fit", then $0 \leq R^2 \leq 1$.

## 5.2 Hypothesis testing

We'll begin this section by looking at the variance of the OLS slope estimator ($\text{Var}[b_1]$). There are three reasons to get this formula:

1. Looking at it will provide insight into what determines the accuracy (a smaller variance) of the estimator.
2. It is required to prove that OLS is an efficient estimator, and therefore is BLUE.
3. It is needed for hypothesis testing.

### 5.2.1 The variance of $b_1$

In chapter 3, we derived the variance of the estimator, $\bar{y}$. Similarly, $b_1$ is a random variable, since it is obtained from a formula involving the random sample $\{Y_i, X_i\}$, and it is common to consider the variance of a random variable. However, deriving the variance of the OLS estimator is too difficult for this course, and we simply write the result:

$$\text{Var}[b_1] = \frac{\sigma_\epsilon^2}{\sum X_i^2 - \frac{\left(\sum X_i\right)^2}{n}}, \tag{5.7}$$

where $\sigma_\epsilon^2$ is the variance of the error term $\epsilon$, $n$ is the sample size, and in the denominator we see something that looks like the sample variance of $X_i$. From equation 5.7, it can be seen that:

- $\text{Var}[b_1]$ decreases as $n$ increases.
- $\text{Var}[b_1]$ decreases as the sample variation in $X$ increases.
- $\text{Var}[b_1]$ decreases as variation in $\epsilon$ decreases.

We want our estimator to have as low a variance as possible! A lower variance means that, on average, we have a higher probability of being close to the "rights answer" (provided the estimator is unbiased). These factors that lead to a lower $\text{Var}[b_1]$ make sense:

- If we have more information (larger $n$), it should be "easier" to pick the right regression line.
- Since we are using changes in $X$ to try to explain changes in $Y$, the bigger changes in $X$ that we observe, the easier it is to pick the regression line.
- The less unobservable changes there are (in $\epsilon$ that are causing changes in $Y$, the easier it is to pick the regression line.

We could discuss a similar formula for $\text{Var}[b_0]$ as well, however, there is rarely any economic interest in the model's intercept that we omit the discussion.

A final note. $\text{Var}[b_1]$ is required in order to prove that OLS is *efficient* (the Gauss-Markov theorem). Proving that an estimator is efficient requires that its variance is shown to be the smallest among all other possible candidate estimators (in the Gauss-Markov theorem other candidate estimators are linear and unbiased ones). The Gauss-Markov theorem is very important because it provides the reason for why OLS should be used: provided (some of) assumptions A1-A6 hold, OLS is the best linear unbiased estimator (BLUE) possible for estimating $\beta_1$.

### 5.2.2 Test statistics and confidence intervals

Hypothesis testing in the context of OLS usually involves $\beta_1$. That is, usually we want to test if a marginal effect is equal to some value. For example, do similarly qualified women earn less than men? Are the returns to education the same for men and women? If we raise the taxes on cigarettes, will consumption decrease? These are all questions that can be answered by forming a null and alternative hypothesis, collecting data, estimating, and rejecting or failing to reject the null. In the context of OLS, a two-sided null and alternative hypothesis looks like:

$$H_0 : \beta_1 = \beta_{1,0}$$
$$H_A : \beta_1 \neq \beta_{1,0}$$

A common hypothesis in economics is where the marginal effect is zero ($X$ does not cause $Y$), so that the above null and alternative become:

$$H_0 : \beta_1 = 0$$
$$H_A : \beta_1 \neq 0$$

As in chapter 3, we will begin with the $z$-test. In general, the $z$-statistic is determined by:

$$z\text{−statistic} = \frac{\text{estimate} - \text{value of } H_0}{\sqrt{\text{Var}[\text{estimator}]}} \tag{5.8}$$

This $z$-statistic is Normally distributed with mean 0 and variance 1 ($z \sim N(0,1)$), if $H_0$ is true and $\bar{Y}$ is Normal. In chapter 3, when our test involved the population mean, equation 5.8 became:

$$z = \frac{\bar{y} - \mu_{Y,0}}{\sqrt{\sigma_Y^2/n}}$$

In OLS, when we are testing the slope (marginal effect) of the model, equation 5.8 becomes:

$$z = \frac{b_1 - \beta_{1,0}}{\sqrt{\text{Var}\left[b_1\right]}},$$

where $b_1$ is the estimate that we actually get from the sample, $\beta_{1,0}$ is the hypothesized value of the slope, and $\text{Var}\left[b_1\right]$ is given by equation 5.7.

As was the case in chapter 3, however, it is not realistic that we would know the variance of $b_1$. By looking again at equation 5.7, we see that the unknown part is the variance of the error term, $\sigma_\epsilon^2$. If we could estimate $\sigma_\epsilon^2$, we would have an estimate for the variance of $b_1$, and we could use a $t$-test instead of a $z$-test.

Recall that the population model is:

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i,$$

and that the estimated model is:

$$Y_i = b_0 + b_1 X_i + e_i$$

Each unobservable part in the population model ($\beta_0$, $\beta_1$, $\epsilon_i$) has an observable counter-part in the estimated model. So, if we want to know something about $\epsilon$ we can use $e$. In fact, an estimator for the variance of $\epsilon$ is the *sample variance* of the OLS residuals:

$$s_\epsilon^2 = \frac{1}{n-2}\sum_{i=1}^{n}(e_i - \bar{e})^2 = \frac{1}{n-2}\sum_{i=1}^{n} e_i^2 \tag{5.9}$$

Why is the $-2$ in the denominator of equation 5.9? Recall that, in chapter 3, when we wanted to estimate $\sigma_y^2$ we used the sample variance of $y$:

$$s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

and that the $-1$ in the denominator was a degrees-of-freedom correction, so that the estimator is unbiased. We only had $(n-1)$ pieces of information available to estimate $\sigma_y^2$, after we had used up a piece of information to get $\bar{y}$. The story is similar in equation 5.9. In order to get the OLS residuals, we first have to estimate *two* things ($b_0$ and $b_1$):

$$e_i = Y_i - \hat{Y}_i = Y_i - (b_0 + b_1 X_i)$$

This uses up two pieces of information, leaving $(n-2)$ remaining when we are using the $e_i$. Now that we have an estimator for $\sigma_\epsilon^2$, we have an estimator

for $\text{Var}[b_1]$ (we just replace the unknown $\sigma_\epsilon^2$ with $s_\epsilon^2$):

$$\hat{\text{Var}}[b_1] = \frac{s_\epsilon^2}{\sum X_i^2 - \frac{\left(\sum X_i\right)^2}{n}}$$

And now, the $t$-statistic for testing $\beta_1$ is obtained by substituting $\hat{\text{Var}}[b_1]$ for $\text{Var}[b_1]$ in the $z$-statistic formula:

$$t = \frac{b_1 - \beta_{1,0}}{\sqrt{\hat{\text{Var}}[b_1]}} \tag{5.10}$$

The denominator of 5.10 is often called the *standard error* of $b_1$ (like a standard deviation), and equation 5.10 is often written instead as:

$$t = \frac{b_1 - \beta_{1,0}}{\text{s.e.}[b_1]} \tag{5.11}$$

where $\text{s.e.}[b_1]$ stands for the estimated standard error of $b_1$.

If the null hypothesis is true, the $t$-statistic in equation 5.11 follows a $t$-distribution with degrees of freedom $(n-k)$, where $k$ is the number of $\beta$s we have estimated (two). To obtain a $p$-value we should use the $t$-distribution, however, if $n$ is large, then the $t$-statistic follows the standard Normal distribution. For the purposes of this course, we shall always assume that $n$ is large enough such that $t \sim N(0,1)$. To obtain a $p$-value, we can use the same table that we used at the end of chapter 3 (see Table 3.2).

### 5.2.3 Confidence intervals

Confidence intervals are obtained very similarly to how they were in chapter 3. The 95% confidence interval for $b_1$ is:

$$b_1 \pm 1.96 \times \text{s.e.}[b_1] \tag{5.12}$$

The 95% confidence interval can be interpreted as follows: (i) if we were to construct many such intervals (hypothetically), 95% of them would contain the true value of $\beta_1$; (ii) all of the values that we could choose for $\beta_{1,0}$ that we would fail to reject at the 5% significance level.

We can get the 90% confidence interval by changing the 1.96 in equation 5.12 to 1.65, and the 99% C.I. by changing it to 2.58, for example.

## 5.3 Dummy Variables

A *dummy variable* is a variable that takes on one of two values (usually 0 or 1). A dummy variable is also sometimes called a *binary variable* or a *dichotomous variable*. We will consider that the independent variable

(the regressor or "$X$" variable) in our population model (equation 4.4) is a dummy variable, where:

$$D_i = \begin{cases} 0, & \text{if individual } i \text{ belongs to group } A \\ 1, & \text{if individual } i \text{ belongs to group } B \end{cases}$$

Dummy variables are useful for estimating differences between groups, where groups "A" and "B" can take on many definitions. For example, in labour economics and many other areas of economics, it is common to use a dummy variable to identify the *gender* of the individual.

### 5.3.1 A population model with a dummy variable

Now, let's consider a population model with a dummy:

$$Y_i = \beta_0 + \beta_1 D_i + \epsilon_i, \tag{5.13}$$

where $D_i = 0$ if the individual is female, $D_i = 1$ if the individual is male, and $Y_i$ is the wage of the individual. How do we interpret $\beta_1$ from equation 5.13? Since $D_i$ is not a continuous variable, $\beta_1$ is not a marginal effect, and we cannot take the derivative of $Y$ with respect to $D$ when $D$ is non-continuous. Instead, let's use *conditional expectations* to find the interpretation of $\beta_1$.

Let's consider the expected wage of a male worker:

$$\mathrm{E}\left[Y_i | D_i = 1\right] = \beta_0 + \beta_1(1) + \mathrm{E}\left[\epsilon_i\right] = \beta_0 + \beta_1 \tag{5.14}$$

We have simply substituted in the population model (equation 5.13) for $Y_i$, substituted in $D_i = 1$, and made use of assumption A.3 ($\mathrm{E}\left[\epsilon_i\right] = 0$). Now, let's consider the expected wage of a female worker:

$$\mathrm{E}\left[Y_i | D_i = 0\right] = \beta_0 + \beta_1(0) + \mathrm{E}\left[\epsilon_i\right] = \beta_0 \tag{5.15}$$

What is the difference between these two conditional expectations (equations 5.14 and 5.15)? $\beta_1$! That is:

$$\mathrm{E}\left[Y_i | D_i = 1\right] - \mathrm{E}\left[Y_i | D_i = 0\right] = \beta_1 \tag{5.16}$$

So, when the "$X$" variable is a dummy variable, the attached $\beta$ is interpreted as the difference in population means between the two groups.

### 5.3.2 An estimated model with a dummy variable

OLS works just fine when the right-hand-side variable is a dummy variable. The estimated model will be the same as it was before:

$$Y_i = b_0 + b_1 D_i + e_i, \tag{5.17}$$

where everything has the same interpretation as before, except that $b_1$ is the *estimated* difference in population mean of $Y$ between the two groups as defined by the dummy variable. In fact, it turns out that:

- $b_0$ is the *sample* mean ($\bar{Y}$) for $D_i = 0$
- $b_0 + b_1$ is the *sample* mean for $D_i = 1$
- $b_1$ is the difference in sample means (be careful of the sign)

This means that, instead of using OLS, we could just divide the sample into two parts (using $D_i$), and calculate two sample averages! So why should we use OLS? At this stage, it looks like we are making things more complicated than they need to be. However, in the next chapter, we will add more $X$ variables, so that we will not be able to get the same results by dividing the sample into two.

### 5.3.3 Example: Gender and wages using the CPS

The current population survey (CPS) is a monthly detailed survey conducted in the United States. It contains information on many labour market and demographic characteristics. In this section, we will use a subset of data from the 1985 CPS, to estimate the differences in wages between men and women.

The data is available from the R package `AER` (Kleiber and Zeileis, 2008). To load this package, and the CPS data into R, use the following commands:

```
install.packages("AER")
library(AER)
data("CPS1985")
attach(CPS1985)
```

You will see many variables in the dataset. For now, we look at only a few:

- wage - hourly wage
- education - number of years of education
- gender - dummy variable for gender

To run an OLS regression of `wage` on `gender`, use the following command:

```
summary(lm(wage ~ gender))
```

You should see the following output:

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)    9.9949     0.2961   33.75  < 2e-16 ***
genderfemale  -2.1161     0.4372   -4.84  1.7e-06 ***
---
```

```
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 5.034 on 532 degrees of freedom
Multiple R-squared:  0.04218,   Adjusted R-squared:  0.04038
F-statistic: 23.43 on 1 and 532 DF,  p-value: 1.703e-06
```

From this output, you should be able to answer the following questions:

- What is the sample mean wage for males and for females?
- What is the interpretation of $b_1$?

We stated earlier that the results we obtain from regressing on a dummy variable are equivalent to what we would obtain by dividing the sample into two parts (by gender). Let's verify this using the CPS data. In R, take the sample mean wage of males only:

```
mean(wage[gender=="male"])
```

and the sample mean wage of female workers only:

```
mean(wage[gender=="female"])
```

The difference is equal to $b_1$, which is -2.1161.

## 5.4 Reporting regression results

We end this chapter with a concise and conventional way of reporting regression results. If you were to see the results of an OLS regression in an economics paper or report, you would not see the ugly R output above. If there are many variables in the regression (see the next chapter), the results may be displayed in a table. However, if there are only a few variables in the regression, it is convenient to report results in an equation with two lines.

For example, when we regress `wage` on `gender`:

```
summary(lm(wage ~ gender))
```

we could report the regression results as follows:

$$\begin{aligned} \widehat{wage} =& 10.00 - 2.12 \times gender, \; R^2 = 0.042 \\ & (0.30) \quad (0.44) \end{aligned} \tag{5.18}$$

Equation 5.18 conveys the estimated $\beta$s, as well as the estimated standard errors, and the $R^2$. Verify that you know where all of these numbers are coming from in the R output.

## 5.5 Review Questions

1. Derive the following expression for $R^2$:

$$R^2 = \frac{ESS}{TSS},$$

and show that $R^2$ can be rewritten as:

$$R^2 = 1 - \frac{RSS}{TSS}$$

2. Using diagrams, explain why $0 \leq R^2 \leq 1$.

3. Using equation 5.7, explain why having a larger sample is better.

4. Explain what s.e.$[b_1]$ is.

5. Using equation 5.13, explain how to interpret $\beta_0$ and $\beta_1$.

6. The following question refers to the regression of `wage` on `gender` using the CPS data. The estimated results, equation 5.18, are repeated here:

$$\begin{aligned} \hat{wage} =& 10.00 - 2.12 \times gender, \ R^2 = 0.042 \\ & (0.30) \quad (0.44) \end{aligned}$$

   a) What is the estimated wage-gender gap?
   b) What is the sample mean wage for males and for females?
   c) Test the hypothesis that there is no wage-gender gap.
   d) Construct a 90% confidence interval for the wage-gender gap.
   e) Interpret the value for $R^2$.
   f) Another researcher uses the same data, but defines the dummy variable in the *opposite* way. What will be the estimated values for $b_0$ and $b_1$?

7. This question uses the CPS data set, which can be loaded into R using the following commands:

```
install.packages("AER")
library(AER)
data("CPS1985")
attach(CPS1985)
```

   a) Estimate the returns (in hourly wages) of an additional year of education. Summarize your results concisely in an equation.

b) Test the hypothesis that the returns to education are zero.

c) Construct a 95% confidence interval for the returns to education.

d) Interpret the value of $R^2$.

e) What does the estimated model predict the hourly wages will be for high school graduates and for university graduates?

f) What is the estimated value, in terms of hourly wage, of obtaining an undergraduate degree?

## 5.6 Answers

1. A definition for $R^2$, in words, is: the portion of variance in $Y$ that can be explained by the estimated model. Each $Y$ observation can be written as a sum of two parts (a part that can be explained using the $X$ variable, and the left over unexplainable part):

$$Y_i = \hat{Y}_i + e_i$$

Taking the sample variance of both sides we get:

$$\hat{\text{var}}[Y_i] = \hat{\text{var}}[\hat{Y}_i] + \hat{\text{var}}[e_i]$$

Note that there is no sample covariance between $\hat{Y}$ and $e$ because they are *independent*. Using the formula for sample variance (from chapter 3, equation 3.17) into the above equation, we get:

$$\frac{\sum(Y_i - \bar{Y})^2}{n-1} = \frac{\sum(\hat{Y}_i - \bar{\hat{Y}})^2}{n-1} + \frac{\sum(e_i - \bar{e})^2}{n-1} \tag{5.19}$$

Now, we make three simplifications to the above:

- the $(n-1)$ cancel
- $\bar{\hat{Y}} = \bar{Y}$ (the sample mean of the OLS predicted values equals the sample mean of the actual values)
- $\bar{e} = 0$ (the OLS residuals sum to 0)

Equation 5.19 becomes:

$$\sum(Y_i - \bar{Y})^2 = \sum(\hat{Y}_i - \bar{Y}_i)^2 + \sum e_i^2$$

The terms in the above equation are "sums-of-squares", so that:

$$TSS = ESS + RSS \tag{5.20}$$

Where $TSS$ is the total sum-of-squares (from the total sample variance of $Y$), $ESS$ is the explained sum-of-squares (from the sample variance of the OLS predicted values), and $RSS$ is the residual sum-of-squares (from the sample variance of the OLS residuals).

Returning to our original definition of $R^2$: "the portion of variance in $Y$ that can be explained by the estimated model", we get:

$$R^2 = \frac{ESS}{TSS}. \tag{5.21}$$

To get an alternate equation, we solve 5.20 for $ESS$:

$$ESS = TSS - RSS$$

and substitute into $R^2$:

$$R^2 = \frac{ESS}{TSS} = \frac{TSS - RSS}{TSS} = 1 - \frac{RSS}{TSS} \tag{5.22}$$

2. This question is answered by considering two extreme cases: (i) the $X$ variable has no explanatory power, and (ii) the $X$ variable can perfectly explain $Y$. (i) is a situation of "no fit", drawn in figure 5.3, and would occur if $b_1 = 0$. In this situation, each OLS predicted value will be equal to $\bar{Y}$, so $ESS$ will equal 0, and so $R^2$ will also equal 0. (ii) is a situation of "perfect fit", drawn in figure 5.4. All data points are on the estimated regression line. $ESS = TSS$, $RSS = 0$, and so $R^2 = 1$.

3. Using equation 5.7, we just need to see that as $n$ increases, the variance of the OLS estimator decreases.

4. In order to perform hypothesis testing, an estimate for the variance of the OLS estimator is required. If equation 5.7 is to be used in practice, we must replace the unknown $\sigma_\epsilon^2$ with the estimator $s_\epsilon^2 psilon = \sum e_i^2/n - 2$. When we take the square-root of this quantity, it is called the *standard error* of $b_1$ (or $s.e.[b_1]$ for short). That is,

$$s.e.[b_1] = \sqrt{\frac{s_\epsilon^2}{\sum X_i^2 - \frac{\left(\sum X_i\right)^2}{n}}}$$

5. The interpretation of $\beta_1$, when the independent variable is a dummy variable, is obtained by taking the conditional expectation of $Y$ for each of the two possible values that the dummy variable can take. We repeat equation 5.16:

$$\mathrm{E}\left[Y_i | D_i = 1\right] - \mathrm{E}\left[Y_i | D_i = 0\right] = \beta_1$$

6. a) The estimated wage-gender gap is the coefficient in front of the `gender` dummy variable (where it is understood that `gender` = 1 if the worker is female). So, the estimated wage-gender gap is -2.12, meaning that on average, women earn $2.12 less than men, according to this sample data.

   b) The sample mean wage for mean is $b_0 = 10.00$, and for women is $b_0 + b_1 = 10.00 - 2.12 = 7.78$.

   c) The null hypothesis is that the differences in wages between men and women is zero. In terms of the population model, this would mean that $\beta_1 = 0$.

$$H_0 : \beta_1 = 0$$
$$H_A : \beta_1 \neq 0$$

   The $t$-test statistic for this null hypothesis is:

$$t = \frac{b_1 - \beta_{1,0}}{\text{s.e.}\,[b_1]} = \frac{-2.12 - 0}{0.44} = -4.82$$

   The associated $p$-value is 0.00. We reject the null hypothesis. The estimated wage-gender gap is statistically significant.

   d) The 90% confidence interval for the wage-gender gap is:

$$-2.12 \ \pm \ 1.65 \times 0.44 = (-2.85, -1.39)$$

   e) Gender explains 4.2% of the variation in wages.

   f) $b_0 = 7.78$ and $b_1 = 2.12$.

7. a) Use the following command:

```
summary(lm(wage ~ education))
```

   and you should see the following output:

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.74598    1.04545  -0.714    0.476
education    0.75046    0.07873   9.532   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.754 on 532 degrees of freedom
Multiple R-squared:  0.1459,    Adjusted R-squared:  0.1443
F-statistic: 90.85 on 1 and 532 DF,  p-value: < 2.2e-16
```

   Some of this information is summarized as follows:

$$\underset{(1.05)}{\hat{wage} = -0.75} + \underset{(0.08)}{0.75} \times education, \ R^2 = 0.146$$

The estimated returns to education are \$0.75 in hourly wages per year of education.

b) From the R output we can see that the `education` variable is highly statistically significant. The $p$-value for the test is 0 (to sixteen decimal places).

c) The 95% confidence interval is:

$$0.75 \ \pm \ 1.96 \times 0.079 = (0.60, 0.91)$$

d) Years of education can explain 14.6% of the differences in wages.

e) Assuming that a high school graduate has 12 years of education, the predicted wage is:

$$\hat{wage} = -0.75 + 0.75(12) = 8.25$$

and assuming that university graduates have 16 years of education the predicted wage is:

$$\hat{wage} = -0.75 + 0.75(16) = 11.25$$

f) The predicted difference in wages between university and high school graduates is \$11.25 - \$8.25 = \$3.

# 6

# Multiple Regression

Multiple regression refers to having more than one "$X$" variable (more than one regressor). From now on, we will typically be dealing with population models of the form:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_k X_{ki} + \epsilon_i \tag{6.1}$$

where $k$ is the number of regressors in the model, and the total number of $\beta$s to be estimated is $(k+1)$. This new model allows for $Y$ to be explained used *multiple* variables. That is, there can now be many $X$s that are causal determinants of $Y$.

## 6.1 House prices

Should I build a fireplace in my home before I sell it? To motivate the need for a multiple regression model, we begin with an example. Let's try to determine the value of a fireplace using data on house prices. The data are from the New York area, 2002-2003, and are from Richard De Veaux of Williams College.

To load the data into R, use the following two commands:

```
houses <- read.csv("http://home.cc.umanitoba.ca/
          ~godwinrt/3040/data/houseprice.csv")
attach(houses)
```

The variables in the dataset are shown in table 6.1.

We are interested in the effect of the variable `Fireplaces` on `Price`. Let's get some summary statistics for `Fireplaces`. Enter the command:

```
summary(Fireplaces)
```

and you should see the output:

```
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
 0.0000  0.0000  1.0000  0.6019  1.0000  4.0000
```

Table 6.1: Description of the variables in the house price data set.

| | |
|---|---|
| Price | the price of the house in dollars |
| Lot.Size | the size of the property in acres |
| Waterfront | dummy variable equal to 1 if house is on the water |
| Age | number of years since the house was built |
| Central.Air | dummy variable equal to 1 if house has air conditioning |
| Living.Area | the size of the house in square feet |
| Bedrooms | number of bedrooms |
| Fireplaces | number of fireplaces |
| Bathrooms | number of bathrooms (half-bathrooms are 0.5) |
| Rooms | total number of rooms in the house |

The houses in the sample have anywhere from 0 to 4 fireplaces, with the average being 0.6. For convenience, let's instead measure `Price` in thousands of dollars:

```
Price <- Price/1000
```

Next, let's see the sample mean price, conditional on the number of fireplaces:

```
mean(Price[Fireplaces == 0])
[1] 174.6533
mean(Price[Fireplaces == 1])
[1] 235.1629
mean(Price[Fireplaces == 2])
[1] 318.8214
mean(Price[Fireplaces == 3])
[1] 360.5
mean(Price[Fireplaces == 4])
[1] 700
```

We see that the average house price increases quite dramatically as the number of fireplaces increase. It's looking like I should build that fireplace! It should be no surprise that the two variables are correlated:

```
cor(Price, Fireplaces)
[1] 0.3767862
```

Now, let's try estimating the population model:

$$Price = \beta_0 + \beta_1 Fireplaces + \epsilon$$

where $\beta_0$ would be the price of a house with 0 fireplaces, and $\beta_1$ is the increase in house price for an additional fireplace. The R command to estimate this model via OLS in R, and the resulting output, are as follows:

```
summary(lm(Price ~ Fireplaces))
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  171.824      3.234   53.13   <2e-16 ***
Fireplaces    66.699      3.947   16.90   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 91.21 on 1726 degrees of freedom
Multiple R-squared:  0.142,     Adjusted R-squared:  0.1415
F-statistic: 285.6 on 1 and 1726 DF,  p-value: < 2.2e-16
```

What is the estimated marginal effect of `Fireplaces` on `Price`? Take a minute to google the cost of fireplace installation. As an economist, this should trouble you deeply. If the estimated value of an additional fireplace is \$66,700, and if it only costs \$10,000 to install a fireplace, we should see lots of houses with many fireplaces. Something is wrong here. To conclude this section, think about what the main determinant of house price should be.

## 6.2 Omitted variable bias

The above OLS estimator ($b_1$ in the house prices example) is suffering from omitted variable bias. Omitted variable bias (OVB) occurs when one or more of the variables in the random error term ($\epsilon$) are related to one or more of the $X$ variables. Recall that $\epsilon$ contains all of the variables that determine $Y$, but that are unobserved (or omitted). Also, recall that one of the assumptions required for OLS to be a "good" estimator is A.5: $\epsilon$ and $X$ are independent. If A.5 is not true, the OLS estimator can be biased (giving the wrong answer on average).

Suppose that there are two variables that determine $Y$: $X$ and $Z$. Also suppose that $X$ and $Z$ are correlated (not independent). When $X$ changes, $Y$ changes. But when $X$ changes, $Z$ changes too (because $Z$ and $X$ are related), and this change in $Z$ also causes a change in $Y$. If $Z$ is omitted so that we only observe $X$ and $Y$, then we cannot attribute changes in $X$ directly to changes in $Y$. The changes in $Z$ will "channel" through $X$. The OLS estimator for the effect of $X$ on $Y$ will be biased, unless the $Z$ variable is included.

### 6.2.1 House prices revisited

What is the important omitted variable from the above house prices example? It seems like the estimated effect of `Fireplaces` on `Price` is too large. In fact, it may be that the number of fireplaces is just indicating the *size* of the house, which is really important for price!

Let's add the `Living.Area` variable to our population model:

$$Price = \beta_0 + \beta_1 Fireplaces + \beta_2 Living.Area + \epsilon$$

The R command and associated output is:

```
summary(lm(Price ~ Fireplaces + Living.Area))
```

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 14.730146   5.007563   2.942  0.00331 **
Fireplaces   8.962440   3.389656   2.644  0.00827 **
Living.Area  0.109313   0.003041  35.951  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 68.98 on 1725 degrees of freedom
Multiple R-squared:  0.5095,    Adjusted R-squared:  0.5089
F-statistic: 895.9 on 2 and 1725 DF,  p-value: < 2.2e-16
```

Several results have changed with the addition of the `Living.Area` variable:

- The estimated value of an additional fireplace has dropped from \$66,699 to \$8,962.
- The $R^2$ has increased from 0.142 to 0.5095.
- The estimated intercept has changed by a lot (but this is unimportant).
- There is a new estimated $\beta$: $b_2 = 0.11$. This means that, it is estimated that an additional square-foot of house size increases price by \$110.

So, what is going on here? From the first regression, the results are:

$$\begin{aligned} \hat{Price} =& 171.82 + 66.70 \times Fireplaces, \; R^2 = 0.142 \\ & (3.23) \quad (3.95) \end{aligned}$$

and from the second regression:

$$\begin{aligned} \hat{Price} =& 14.73 + 8.96 \times Fireplaces + 0.11 \times Living.Area, \; R^2 = 0.511 \\ & (5.01) \quad (3.39) \qquad\qquad (0.003) \end{aligned}$$

Why has the estimated effect of `Fireplace` on `Price` changed so much? `Living.Area` is an important variable. Arguably, the most important factor in determining house price is the size of the house. Houses that have more fireplaces tend to be larger. (There usually aren't two fireplaces in one room, for example). So, `Fireplaces` and `Living.Area` are correlated:

```
cor(Fireplaces, Living.Area)
[1] 0.4737878
```

When `Living.Area` is *omitted* from the regression, its effect on `Price` becomes mixed up in the effect of `Fireplaces` on `Price`. That is, when the house has more fireplaces, that means it's a larger house, so there are two reasons for a higher price. Lots of fireplaces is just indicating the house is large!

This is an example of omitted variable bias (OVB). When `Living.Area` is omitted, the OLS estimator is biased (in this case the effect of more fireplaces on house price is estimated to be way too large). OVB provides an important motivation for the multiple regression model: even though we may only be interested in estimating one marginal effect, we still should include other variables that are correlated to $X$, otherwise our estimator is biased. OVB is solved by adding the extra variables to the equation, thus *controlling* for their effect.

## 6.3 OLS in multiple regression

### 6.3.1 Derivation

The OLS estimators, $b_0, b_1, \ldots, b_k$, are derived similarly to how they were in chapter 4 (when we only had one $X$ variable). The formulas are obtained by choosing $b_0, b_1, \ldots, b_k$ so that the sum of squared residuals is minimized:

$$\min_{b_0, b_1, \ldots, b_k} \sum_{i=1}^{n} e_i^2$$

This involves taking $(k+1)$ derivatives, setting them all equal to zero, and solving the system of equations. The formulas become too complicated to write, unless we use matrices (which we won't do here).

Now that we have multiple $X$ variables, many concepts that we have already discussed become much more difficult to *visualize*. For example, the estimated model:

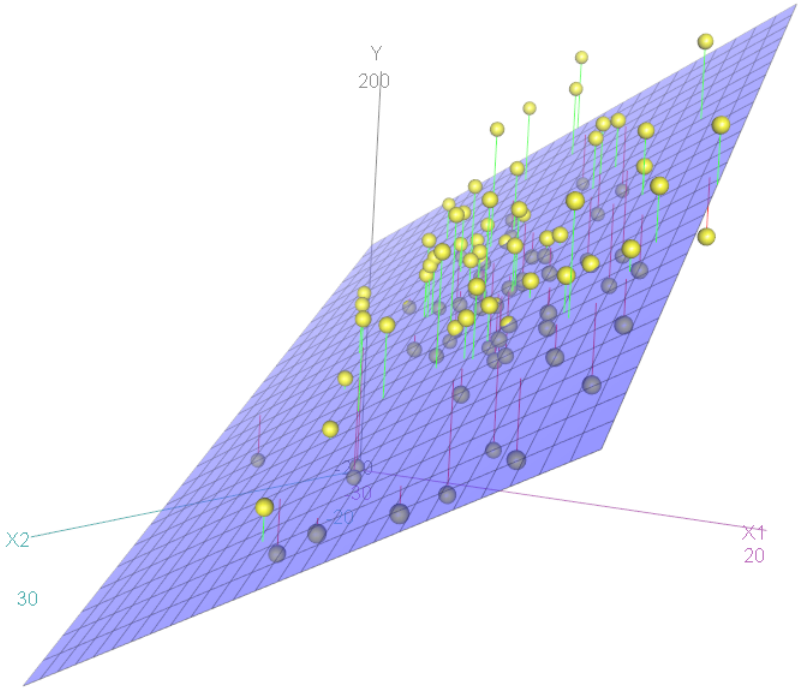
$$\hat{Y}_i = b_0 + b_1 X_{1i} + b_2 X_{2i} + \cdots + b_k X_{ki} \tag{6.2}$$

can not be interpreted as a line! A line (with an intercept and slope) can be drawn in two dimensional space. The estimated model in equation 6.2 has $k$ dimensions (and is a $k$-dimensional hyperplane). However, if we have only two $X$ variables:

$$\hat{Y}_i = b_0 + b_1 X_{1i} + b_2 X_{2i}$$

then we can still represent the estimated model in 3-dimensional space (see figure 6.1).

Figure 6.1: An OLS estimated regression plane (two $X$ variables). The plane is chosen so as to minimize the sum of squared vertical distances indicated in the figure. The figure was drawn using the `scatter3d` function from the `rgl` package.



### 6.3.2 Interpretation

Let's look at a population model with two $X$ variables:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \epsilon_i \tag{6.3}$$

- $Y$ is still the dependent variable
- $X_1$ and $X_2$ are the independent variables (the regressors)
- $i$ still denotes an observation number
- $\beta_0$ is the population intercept
- $\beta_1$ is the effect of $X_1$ on $Y$, holding all else constant ($X_2$)
- $\beta_2$ is the effect of $X_2$ on $Y$, holding all else constant ($X_1$)
- $\epsilon$ is the regression error term (containing all the omitted factors that effect $Y$)

Nothing substantial has changed. $\beta_1$, for example, is the marginal effect of $X_1$ on $Y$, while holding $X_2$ constant. In the fireplaces example, by inlcuding `Living.Area` in the regression we are able to find the marginal effect of fireplaces while holding house size constant. When we add more variables to the model, the interpretation of the $\beta$s remains the same.

## 6.4 OLS assumption A2: no perfect multicollinearity

In this section, we pay special attention to assumption A2, which has only now become relevant in the context of the multiple regression model.

**A2 There is no perfect multicollinearity between the $X$ variables.**

This assumptions means that no two $X$ variables (or combinations of the variables) can have an exact linear relationship. For example, exact linear relationship between $X$s are:

- $X_1 = X_2$
- $X_1 = 100X_2$
- $X_1 = 1 + X_2 - 3X_3$

In these examples, you can figure out what one of the $X$s will be, if you know the other $X$s. This situation is usually called perfect multicollinearity. The data contains redundant information. This shouldn't be much of a problem, except that the OLS formula doesn't allow all of the estimators to be calculated (the problem is similar to trying to divide by zero).

Using R, let's see what happens when we try to include an $X$ variable that is a perfectly linear relationship with another $X$ variable. We'll use the house price data again. The `Living.Area` variable measures the size of the house in square feet. Suppose that there was another variable in the data set that measured house size in square metres (1 square foot = 0.0929 square metre). We can create this variable in R using:

```
House.Size <- 0.0929 * Living.Area
```

and now let's include it in our OLS estimation:

```
summary(lm(Price ~ Fireplaces + Living.Area + House.Size))
```

```
Coefficients: (1 not defined because of singularities)
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 14.730146   5.007563   2.942  0.00331 **
Fireplaces   8.962440   3.389656   2.644  0.00827 **
Living.Area  0.109313   0.003041  35.951  < 2e-16 ***
House.Size         NA         NA      NA       NA
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 68980 on 1725 degrees of freedom
Multiple R-squared:  0.5095,    Adjusted R-squared:  0.5089
F-statistic: 895.9 on 2 and 1725 DF,  p-value: < 2.2e-16
```

Notice the error message "1 not defined because of singularities", and the row of "NA"s (not available). So, R recognized that there was a problem, and dropped the redundant variable, but not all econometric software has been this clever.

Some common examples of where the assumption of "no perfect multicollinearity" is violated in practice are when the same variable is measure in different units (such as square feet and square metres, or dollars and cents), and in the *dummy variable trap*.

### 6.4.1 The dummy variable trap

The dummy variable trap occurs when one too many dummy variables are included in the equation. For example, suppose that we have a dummy variable `female` that equals 1 if the worker is female. Suppose that we also have a variable `male` that equals 1 if the worker is male. There is an exact linear combination between the two variables:

$$female = 1 - male$$

If you know the value for the variable `male`, then you automatically know the value for `female`. Including both `male` and `female` in the equation would be a violation of assumption A2, and would be referred to as the dummy variable trap for this example. That is, OLS would not be able to estimate all of the $\beta$s in the equation:

$$wage = \beta_0 + \beta_1 \times male + \beta_2 \times female + \epsilon$$

The `male` and `female` dummy variables is a simple example, in other situations it is much easier to fall into the "trap". For example, suppose that you are provided data on a worker's location by province or territory. That is, each worker has a `Location` variable that takes on one of the values: $\{AB, BC, MB, NB, NL, NS, NT, NU, ON, PE, QC, SK, YT\}$. How should this variable be used? Typically, a series of dummy variables would be created from the `Location` variable:

$$\begin{aligned} Alberta &= 1 \text{ if } Location = AB; 0 \text{ otherwise} \\ British.Columbia &= 1 \text{ if } Location = BC; 0 \text{ otherwise} \\ Manitoba &= 1 \text{ if } Location = MB; 0 \text{ otherwise} \\ &\vdots \\ Yukon &= 1 \text{ if } Location = YT; 0 \text{ otherwise} \end{aligned}$$

So, we could create 13 dummy variables from the `Location` variable, but if we included all of them in the regression, we would fall into the dummy variable trap! Instead, one of the provinces/territories must be left out of the equation. Whichever group is left out, it becomes the *base group*, to which comparisons are made.

The solution to perfect multicollinearity, then, is to identify the redundant variable(s), and simply drop it from the equation.

As a final note, it is *not* a violation of "no perfect multicollinearity" if we take a non-linear transformation of a variable in the data set. For example, if we create a new variable $X_2$ where $X_2 = X_1^2$, this is ok! In fact, we will make use of non-linear transformations in chapter 8.

### 6.4.2 Imperfect multicollinearity

Imperfect multicollinearity is when two (or more) variables are *almost* perfectly related. That is, they are very highly correlated. Suppose that the true population model is (remember, we don't actually know this in practice):

$$Y = 2X_1 + 2X_2 + \epsilon$$

and that the correlation between $X_1$ and $X_2$ is 0.99. Regress $Y$ on $X_1$:

```
summary(lm(Y ~ X1))
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -4.4165     3.8954  -1.134    0.263
X1            4.0762     0.4698   8.676 2.13e-11 ***
```

The estimated standard error is small, so that the $t$-statistic is large, and we are sure that $X_1$ is statistically significant. However, the estimated $\beta_1$ is twice as big as it should be. This is because of omitted variable bias. So, we add $X_2$ to the equation:

```
summary(lm(Y ~ X1 + X2))
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   -4.676      3.956  -1.182    0.243
X1             1.958      4.075   0.481    0.633
X2             2.128      4.066   0.523    0.603
```

Now, the estimated $\beta$s are closer to their true value of 2, but both appear to be statistically insignificant! (Note the large standard errors and small $t$-statistics.)

The problem here is that, because $X_1$ and $X_2$ are highly correlated, it is difficult to attribute changes in one of the $X$ variables to changes in $Y$, because both $X_1$ and $X_2$ are almost always changing together in a similar fashion. That is, the *ceteris paribus* assumption (all else equal), is not feasible when the variables are highly correlated. $\beta_1$ is the effect of $X_1$ on $Y$, *holding* $X_2$ *constant*. But, because of the correlation, the data can not provide us such a *ceteris paribus* environment.

The problem of imperfect multicollinearity shows up in the large standard errors for the estimated $\beta$s of the affected variables. Adding and dropping the affected variables may result in large swings in the estimated coefficients. Imperfect multicollinearity makes us unsure of our estimated results. The problem is difficult to address. We cannot drop one of the correlated variables, due to the problem of omitted variable bias. In fact, there is very little to be done here. We need more *information*, but presumably the sample size $n$ cannot be increased. As long as the variables we are interested in studying are not part of the multicollinearity problem (and the ones that are part of the problem are there to avoid OVB), then multicollinearity is not an issue.

## 6.5 Adjusted R-squared

We should no longer use $R^2$ in the multiple regression model. This is because when we add a new variable to the model, $R^2$ must always increase (or at best stay the same). This means that we could keep adding "junk" variables to the model to arbitrarily inflate the $R^2$. This is not a good property for a

"measure of fit" to have. Instead, we will use "adjusted R-squared", denoted by $\bar{R}^2$.

### 6.5.1 Why $R^2$ must increase when a variable is added

To see why $R^2$ must always increase when a variable is added, we begin by looking again at the formula:

$$R^2 = \frac{ESS}{TSS} = 1 - \frac{RSS}{TSS} = 1 - \frac{\sum e_i^2}{TSS}$$

and again at the minimization problem that defines the OLS estimators:

$$\min_{b_0, b_1, \ldots, b_k} \sum_{i=1}^{n} e_i^2$$

When we add another $X$ variable, the minimized value of $\sum_{i=1}^{n} e_i^2$ must get smaller! OLS picks the values for the $b$s so that the sum of squared vertical distances are minimized. If we give OLS another option for minimizing those distances, the distances have to get smaller (or at the worst stay the same). So, adding a variable means $RSS$ decreases, so $R^2$ increases. The only way that $R^2$ stays the same is if OLS chooses a value of 0 for the associated slope coefficient, which never happens in practice.

As an example, let's try adding a nonsense variable to the house price model: random dice rolls. Using R, 1728 die rolls are simulated (to match the house price sample size of $n = 1728$), are recorded as a variable `Dice`, and added to the regression. Notice the difference in "Multiple R-squared" ($R^2$) and "Adjusted R-squared" ($\bar{R}^2$) between the two regressions:

```
summary(lm(Price ~ Fireplaces + Living.Area))
```

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 14.730146   5.007563   2.942  0.00331 **
Fireplaces   8.962440   3.389656   2.644  0.00827 **
Living.Area  0.109313   0.003041  35.951  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 68.98 on 1725 degrees of freedom
Multiple R-squared:  0.5095,    Adjusted R-squared:  0.5089
F-statistic: 895.9 on 2 and 1725 DF,  p-value: < 2.2e-16
```

```
summary(lm(Price ~ Fireplaces + Living.Area + Dice))
```

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 12.105383   6.072084   1.994  0.04635 *
Fireplaces   8.829436   3.394526   2.601  0.00937 **
Living.Area  0.109378   0.003042  35.954  < 2e-16 ***
Dice         0.743506   0.972575   0.764  0.44469
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 68.99 on 1724 degrees of freedom
Multiple R-squared:  0.5097,    Adjusted R-squared:  0.5088
F-statistic: 597.3 on 3 and 1724 DF,  p-value: < 2.2e-16
```

The variable `Dice` has no business being in the regression of house prices, and we fail to reject the null hypothesis that its effect is zero, yet the $R^2$ increases. The adjusted R-squared ($\bar{R}^2$) decreases, however.

### 6.5.2 The $\bar{R}^2$ formula

Adjusted R-squared ($\bar{R}^2$) is a measure-of-fit that can either increase or decrease when a new variable is added. $\bar{R}^2$ is a slight alteration of the $R^2$ formula. It introduces a penalty into $R^2$ that depends on the number of $X$ variables in the model. (Remember that the number of $X$s in the model is denoted by $k$.)

$$\bar{R}^2 = 1 - \frac{RSS \ / \ (n-k-1)}{TSS \ / \ (n-1)} \tag{6.4}$$

The $\bar{R}^2$ formula is such that when a variable is added to the model, $k$ goes up, which tends to make $\bar{R}^2$ smaller. We know from the previous discussion, however, that whenever a variable is added, $RSS$ must decrease. So, whether or not $\bar{R}^2$ increases or decreases depends on whether the new variable improves the fit of the model enough to beat the penalty incurred by $k$.

The justification for the $(n-k-1)$ and $(n-1)$ terms is from a degrees-of-freedom correction. How many things do we have to estimate before we can calculate $RSS$? $k+1$ $\beta$s must first be estimated before we can get the OLS residuals, and $RSS$. If you want to use $RSS$ for something else (such as a measure of fit), we recognize that we don't have $n$ pieces of information left in the sample, we have $(n-k-1)$. A similar argument can be made for the $(n-1)$ term in equation 6.4.

## 6.6 Review Questions

1. Explain why the estimated value for $\beta_1$ changes so much between the equations:

$$Price = \beta_0 + \beta_1 Fireplaces + \epsilon$$

and

$$Price = \beta_0 + \beta_1 Fireplaces + \beta_2 Living.Area + \epsilon$$

2. What are the two conditions that will make an omitted variable cause OLS to be biased?

3. Explain how the OLS estimators, $b_0, b_1, \ldots, b_k$, are derived in the multiple regression model. (Explain how the equations for $b_0, b_1, \ldots, b_k$ are obtained.)

4. For the model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon,$$

   explain the interpretation of $\beta_1$ and $\beta_2$.

5. Why is perfect multicollinearity a problem for OLS estimation?

6. Explain how the "dummy variable trap" is a situation of perfect multicollinearity.

7. Explain what imperfect multicollinearity is, and how it poses a problem for OLS estimation.

8. Why does $R^2$ always increase when a variable is added to the model?

9. Explain where the $(n-k-1)$ and $(n-1)$ terms in $\bar{R}^2$ come from.

10. An estimated model with two $X$ variables, and from a sample size of $n = 27$, yields $R^2 = 0.5882$. Calculate $\bar{R}^2$.

11. This question again uses the CPS data set, which can be loaded into R using the following commands:

```
install.packages("AER")
library(AER)
data("CPS1985")
attach(CPS1985)
```

    a) Regress `wage` on `education`, `age`, and `gender`, and report your results.
    b) Why has the estimated returns to education changed from the exercise in chapter 5?
    c) Are the variables statistically significant?
    d) Test the hypothesis that there is no wage-gender gap.
    e) What is the predicted wage for a 40 year-old female worker with 12 years of education?
    f) What is the predicted wage for a 40 year-old male worker with 12 years of education? What is the difference from the previous question?

g) Why are the $R^2$ and $\bar{R}^2$ so similar for this regression?

h) Interpret the value of $\bar{R}^2$.

i) Try adding the variable `experience` to the regression. Are all the variables still statistically significant? What is going on here?

## 6.7 Answers

1. The estimated value changes so much due to *omitted variable bias*. `Living.Area` is an important determination of house price, and is correlated with `Fireplaces` (larger houses have more fireplaces). The effect of house size is "channeling" through the number of fireplaces. The omission of `Living.Area` is causing the OLS estimator in the first equation to be biased (and inconsistent).

2. If the omitted variable is (i) a determinant of the dependent ($Y$) variable; and (ii) is correlated with one or more of the included ($X$) variables.

3. The OLS estimators in the multiple regression model are derived similarly to how they were in chapter 4. $b_0, b_1, \ldots, b_k$ are chosen so as to minimize the sum of squared residuals. Solving for $b_0, b_1, \ldots, b_k$ involves solving a calculus minimization problem.

4. $\beta_1$ is the marginal effect of $X_1$ on $Y$, holding $X_2$ constant. Similar for $\beta_2$. To prove this, we can take the partial derivative of $Y$ with respect to (say) $X_1$:

$$\frac{\partial Y}{\partial X_1} = 0 + \beta_1 + 0 + 0 = \beta_1$$

   This tells us that the change in $Y$ resulting from a change in $X_1$, is $\beta_1$, and that these changes are independent from changes in $X_2$.

5. Perfect multicollinearity is a problem because the OLS estimator is not *defined*. That is, our computer software will be unable to calculate all of the OLS estimators.

6. The "dummy variable trap" is when a redundant dummy variable is included in the regression. This is a case of perfect multicollinearity: there is an exact linear relationship between the dummy variables. For example, suppose that I had a two dummy variables:

$$attended = \begin{cases} 1, & \text{if the student attended class} \\ 0, & \text{if the student did not attend class} \end{cases}$$

and

$$skipped = \begin{cases} 1, & \text{if the student skipped class} \\ 0, & \text{if the student did not skip class} \end{cases}$$

Including both of these variables in the equation would result in perfect multicollinearity because there is an exact linear relationship between the two variables:

$$attended = 1 - skipped$$

7. Imperfect multicollinearity is when two (or more) variables are highly correlated. In this situation, OLS can be imprecise (have high variance) because it is difficult to tell which of the two correlated variables is causing the change in $Y$. The problem of imperfect multicollinearity shows up in large standard errors and confidence intervals, and large swings in the estimated $\beta$s as the affected variables are added to and dropped from the model.

8. The $b$s in OLS are chosen so as to minimize the sum of squared residuals. When a variable is added to the model, a $b$ is added to the minimization problem, giving one more way to minimize $RSS$. So, $RSS$ must increase (or possibly stay the same) when another $b$ is added. By the formula for $R^2$, it can easily be seen that $R^2$ must increase.

9. The justification for the $(n-k-1)$ and $(n-1)$ terms are due to degrees-of-freedom. The amount of information in the $RSS$ statistic is $(n-k-1)$ since $k+1$ $\beta$s must first be estimated by OLS. In the $TSS$ statistic, one thing must be estimated first ($\bar{Y}$), so the amount of information left over is $(n-1)$.

10.

$$\begin{aligned} R^2 &= 1 - \frac{RSS}{TSS} = 0.5882 \\ \frac{RSS}{TSS} &= 1 - R^2 = 1 - 0.5882 = 0.4118 \\ \bar{R}^2 &= 1 - \frac{RSS \ / \ (n-k-1)}{TSS \ / \ (n-1)} \\ &= 1 - 0.4118 \frac{(n-1)}{(n-k-1)} \\ &= 1 - 0.4118 \left(\frac{26}{24}\right) = 0.5539 \end{aligned}$$

11. a) `summary(lm(wage ~ education + age + gender))`

Table 6.2: Regression results using the CPS data.

| Dependent variable: wage | |
|---|---|
| Regressor | Estimate (standard error) |
| education | 0.827*** |
| | (0.075) |
| age | 0.113*** |
| | (0.017) |
| female | -2.335*** |
| | (0.388) |
| intercept | -4.843*** |
| | (1.244) |
| $n = 534$ | |
| $\bar{R}^2 = 0.249$ | |
| *** denotes significance at the 0.1% level | |

b) The estimated returns to education have changed from 0.751 to 0.827. The formula for each OLS estimator ($b$) depends on all of the variables in the regression. So, when the $X$ variables change the estimated results will change (unless the sample correlation between the variables is exactly 0, which is never the case in practice). The fact that the results change may indicate that the regression from chapter 5 was suffering from omitted variable bias.

c) Yes (see the $p$-values in R).

d) This hypothesis has already been tested for us. We reject at the 0.1% significance level.

e)

$$\hat{wage} = -4.843 + 0.827(12) + 0.113(40) - 2.335(1) = 7.266$$

f)

$$\hat{wage} = -4.843 + 0.827(12) + 0.113(40) - 2.335(0) = 9.601$$

The difference between the two predicted values ($9.601 - 7.266 = 2.335$) is equal to the estimated gender-wage gap.

g) $R^2$ and $\bar{R}^2$ differ by the term:

$$\frac{(n-1)}{(n-k-1)}$$

As $n$ grows, the difference between $R^2$ and $\bar{R}^2$ disappears. In the CPS data, the sample size is reasonably large at $n = 534$, and $k$ is only equal to 3, making the two measures-of-fit quite similar.

h) 24.9% of the variation in wages can be explained using the three variables in the model.

i) When we add `experience` to the model:

```
summary(lm(wage ~ education + age + gender
                                + experience))
```

all variables except the female dummy variable lose statistical significance. This is due to imperfect multicollinearity. Age, education, and experience, are all closely related.

# 7

# Joint Hypothesis Tests

Now that we have multiple $X$ variables and $\beta$s in our population model, we might want to test hypotheses that involves two or more of the $\beta$s at once. In these cases, we (typically) do not use $t$-tests. Instead, we will use the $F$-test.

## 7.1 Joint hypotheses

The types of hypotheses we are now considering involve multiple coefficients ($\beta$s). For example:

$$\begin{aligned} H_0 &: \beta_1 = 0,\ \beta_2 = 0 \\ H_A &: \beta_1 \neq 0 \text{ and/or } \beta_2 \neq 0 \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} H_0 &: \beta_1 = 1, \beta_2 = 2, \beta_4 = 5 \\ H_A &: \beta_1 \neq 1 \text{ and/or } \beta_2 \neq 2 \text{ and/or } \beta_4 \neq 5 \end{aligned} \tag{7.2}$$

Note that the null hypothesis is wrong if *any* of the individual hypotheses about the $\beta$s are wrong. In the latter example, if $\beta_2 \neq 2$, then the whole thing is wrong. Hence the use of the "and/or" operator in $H_A$. It is common to omit all the "and/or" and simply write "not $H_0$" for the alternative hypothesis.

A joint hypothesis specifies a value (imposes a restriction) for two or more coefficients. Use $q$ to denote the number of restrictions ($q = 2$ for hypothesis 7.1, and $q = 3$ for hypothesis 7.2).

### 7.1.1 Model selection

If we fail to reject hypothesis 7.1, this implies that we should drop $X_1$ and $X_2$ from the model. That is, if variables are insignificant, we might want to exclude them from the model. If we wish to drop multiple variables from the

model at once, that means we are hypothesizing that all of the associated $\beta$s are jointly equal to zero.

Why would we want to drop (or omit) variables from the model? There are two main reasons:

- A simpler model is always better. The same reasons that we wish to have simple models in economics also apply to econometrics. Simple models are easier to understand, easier to work with. They focus on the things we are trying to explain.
- The fewer $\beta$s that we try to estimate, the more information is available for each. That is, the variance of the remaining OLS estimators will be smaller after we drop $X$ variables.

We have to be careful when we drop variables, however! The cost of wrongly dropping a variable is high. We can end up with omitted variable bias. So, we should be careful and err on the side of caution, since it is generally held that the cost of *wrongly omitting a variable* (omitted variable bias) is higher than the cost of *wrongly including a variable* (a loss of efficiency).

## 7.2 Example: CPS data

Load the CPS data (you don't need the first line of code if you have already installed the `AER` package):

```
install.packages("AER")
library(AER)
data("CPS1985")
attach(CPS1985)
```

Regress `wage` on `education`, `gender`, `age`, and `experience`:

```
summary(lm(wage ~ education + gender + age + experience))
```

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   -1.9574     6.8350  -0.286    0.775
education      1.3073     1.1201   1.167    0.244
genderfemale  -2.3442     0.3889  -6.028 3.12e-09 ***
age           -0.3675     1.1195  -0.328    0.743
experience     0.4811     1.1205   0.429    0.668
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.458 on 529 degrees of freedom
Multiple R-squared:  0.2533,    Adjusted R-squared:  0.2477
F-statistic: 44.86 on 4 and 529 DF,  p-value: < 2.2e-16
```

In the above regression, both `age` and `experience` appear to be statistically *insignificant* (the $p$-values in the table are 0.743 and 0.668, respectively).

That is, the null hypothesis $H_0 : \beta_3 = 0$ cannot be rejected, and neither can the null hypothesis $H_0 : \beta_4 = 0$. This suggests that `age` and `experience` could be dropped from the model. However, to drop both of these variables we actually need to test the joint hypothesis:

$$H_0 : \beta_3 = 0,\ \beta_4 = 0$$
$$H_A : \beta_3 \neq 0 \text{ and/or } \beta_4 \neq 0$$

$t$-tests won't work for this hypothesis. Instead we will use the $F$-test.

## 7.3 The failure of the *t*-test in joint hypotheses

A natural idea for testing $H_0 : \beta_3 = 0,\ \beta_4 = 0$ (for example), is to reject $H_0$ if either $|t_3| > 1.96$ and/or $|t_4| > 1.96$. There are two problems with this. First, the type I error will not be 5%, unless we increase the critical value (showing this is left as an exercise). A much bigger problem is that $t_3$ and $t_4$ are likely *not* independent (they are correlated).

For example, in the population model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \epsilon, \tag{7.3}$$

if $X_3$ and $X_4$ are correlated, then the OLS estimators $b_3$ and $b_4$ will also be correlated with each other (recall OVB and how adding a variable to the model changes all the estimates - the formula for each $b$ depends on *all* the $X$ variables). If $b_3$ and $b_4$ are correlated then $t_3$ and $t_4$ are correlated!

In population model 7.3, suppose that $X_3$ and $X_4$ are positively correlated. Consider the null $H_0 : \beta_3 = 0,\ \beta_4 = 0$. Given the sign of the correlation between $X_3$ and $X_4$ (positive), it is more likely that $b_3$ and $b_4$ have the same sign (both positive or both negative). It is less likely that one of the coefficients would be estimated to be negative, and the other positive. Seeing opposite signs in the estimated coefficients would be additional evidence against the null hypothesis that is not taken into account by looking at the individual $t$-statistics.

We need a test that will take into account the correlations between all the variables that are involved in the test. Such a test is the $F$-test.

## 7.4 The *F*-test

The $F$-test takes into account the correlations between the OLS estimators. Suppose the null hypothesis is still $H_0 : \beta_3 = 0,\ \beta_4 = 0$. Since we are testing exactly two $\beta$s, the $F$-statistic formula can be written as:

$$F = \frac{1}{2}\frac{t_3^2 + t_4^2 - 2r_{t_3,t_4}t_3t_4}{1 - r_{t_3,t_4}^2}$$

where $r_{t_3,t_4}$ is the estimated correlation between $t_3$ and $t_4$. The larger the $F$-statistic, the more likely we are to reject the null. The purpose of showing this formula here is to highlight that the $F$-test takes into account the correlation between $t_3$ and $t_4$. The formula becomes much too complicated when we are testing more than two $\beta$s.

To obtain a more convenient formula for the $F$-test statistic, we need the idea of a *restricted* and *unrestricted* model. The *restricted* model is obtained by incorporating the values chosen for the $\beta$s in the null hypothesis into the population model. That is, the null hypothesis chooses certain values for some of the $\beta$s, and when those values are substituted into the full population model, we get a restricted model. In the alternative hypothesis, the population model is fully unrestricted. That is, none of the $\beta$s are chosen beforehand, and all values can be chosen by OLS. To summarize:

- restricted model - the model under the null hypothesis. Some $\beta$s are chosen in the null, and substituted into the population model.
- unrestricted model - the model under the alternative hypothesis. All $\beta$s are free to be chosen by the estimation procedure (OLS).

The $F$-test can be implemented by estimating these two models, and using some summary statistics from the regression. The intuition is that, if the restrictions are true (if $H_0$ is true), then the "fit" of the two models should be similar. Alternatively, if the restrictions are false (the null is false), then the unrestricted model should "fit" much better than the restricted model. We can measure the fit of the two models using the residual sum-of-squares, or the $R^2$.

One version of the $F$-statistic formula is:

$$F = \frac{(RSS_r - RSS_u)/q}{RSS_u/(n - k_u - 1)} \tag{7.4}$$

where:

- $RSS_r$ is the residual sum-of-squares from the restricted model
- $RSS_u$ is the residual sum-of-squares from the unrestricted model
- $q$ is the number of restrictions being tested
- $k_u$ is the number of $X$ variables in the unrestricted model, or the number of $\beta$s (not counting the intercept)

Recall that the unrestricted model *must* fit better than the restricted model (OLS has more options for minimizing $RSS$). Also, note that the $F$-statistic must be a positive number, since $RSS$ is a sum-of-squares.

If the restrictions are true, then OLS should (approximately) choose values for the $\beta$s that are already in the null hypothesis. The restricted and unrestricted models will be similar, $(RSS_r - RSS_u)$ will be small (close to zero), the $F$-statistic will be close to zero, and we will tend to fail to reject the null. Alternatively, if the null is false, $(RSS_r - RSS_u)$ will be large, leading to a large $F$-statistic, and a tendency to reject.

Another (possibly more convenient and intuitive) formulation of the $F$-statistic involves the $R^2$ (*not* the adjusted $R^2$). We can solve $R^2$ for $RSS$ using the formula:

$$R^2 = 1 - \frac{RSS}{TSS}$$

and re-write the $F$-statistics formula in equation 7.4 as:

$$F = \frac{(R_u^2 - R_r^2)/q}{(1 - R_u^2)/(n - k_u - 1)} \tag{7.5}$$

where:

- $R_r^2$ is the (unadjusted) $R^2$ from the restricted model
- $R_u^2$ is the (unadjusted) $R^2$ from the unrestricted model
- $q$ and $k_u$ are as before

Table 7.1: $\chi^2$ critical values for the $F$-test statistic.

| $q$ | 5% critical value |
|---|---|
| 1 | 3.84 |
| 2 | 3.00 |
| 3 | 2.60 |
| 4 | 2.37 |
| 5 | 2.21 |

Remember that whenever we add a $\beta$ to the model that $R^2$ has to increase. This was the whole reason that we needed to use adjusted R-square ($\bar{R}^2$) instead. However, if the fit of the model doesn't change much when the restrictions are imposed, the $R^2$ will be similar between the two models, leading to a small $F$-statistic, and a tendency to fail to reject $H_0$. Alternatively, if imposing the restrictions makes a big difference in terms of the fit of the model, the $F$-statistic will be large and we will tend to reject $H_0$.

The $F$-test statistic that we have been discussing follows an $F$ distribution with $q$ and $(n - k_u - 1)$ degrees of freedom. If the sample size $n$ is large, however, the $F$-statistic follows a $\chi^2$ (chi-square) distribution with $q$ degrees of freedom (similar to how the $t$-statistic follows a Normal distribution for large $n$). In this book we assume that $n$ is large enough for this to

be true. The $F$-statistic critical values for 5% significance, and for large $n$, are given in table 7.1. If the $F$-statistic exceeds the 5% critical value, the null hypothesis should be rejected at 5% significance.

## 7.5 Confidence sets

Confidence intervals may be used to test hypotheses that involve only one $\beta$. If the value chosen for $\beta$ by the null hypothesis is within the confidence interval, we will fail to reject. In fact, one of the definitions for a confidence interval is that it is the interval that contains all values that can be chosen for a null hypothesis, that won't be rejected.

If our null hypothesis involves two $\beta$s, as in $H_0 : \beta_1 = 0,\ \beta_2 = 0$ for example, then the idea of a confidence interval would be extended to a *confidence set*. The confidence set would contain all the pairs of values for $\beta_1$ and $\beta_2$ that could be jointly chosen under the null hypothesis, where the null hypothesis would not be rejected.

### 7.5.1 Example: confidence intervals and a confidence set

Consider the model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \epsilon$$

which has been estimated by OLS:

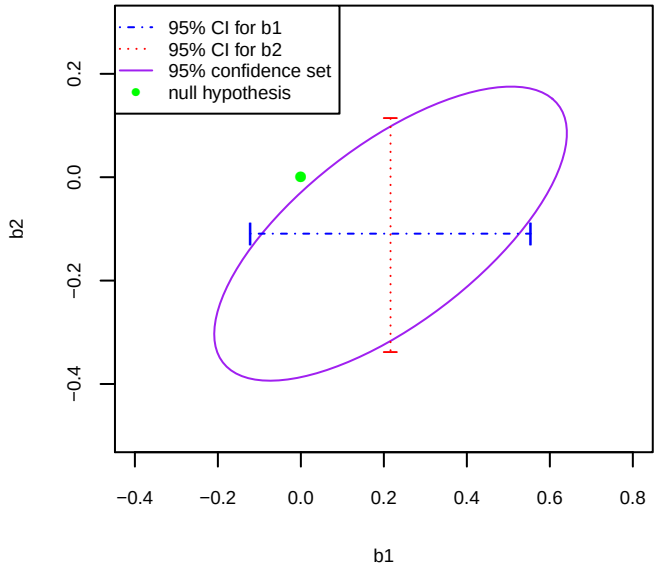
```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -0.6246     0.4660  -1.340    0.182
X1            0.2161     0.1723   1.255    0.211
X2           -0.1092     0.1153  -0.946    0.345
X3            2.9384     0.1092  26.914   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

The 95% confidence interval around $b_1$ is $0.2161 \pm 1.96 \times 0.1723 = [-0.12,\ 0.55]$. The null hypothesis of $H_0 : \beta_1 = 0$ cannot be rejected at the 5% significance level since the value 0 is contained in the interval. By looking at the R output, we can tell that the 95% confidence interval contains 0 given that the $p$-value of 0.211 is greater than 0.05. Similarly, the confidence interval around $b_2$ is $-0.1092 \pm 1.96 \times 0.1153 = [-0.34,\ 0.12]$, and contains 0. Both $X_1$ and $X_2$ appear to be statistically insignificant, according to their individual confidence intervals.

Similar to why individual $t$-tests should not be used to test a joint hypothesis, neither should individual confidence intervals be used. In order to test the hypothesis:

$$H_0 : \beta_1 = 0,\ \beta_2 = 0$$
$$H_A : \text{not } H_0$$

Figure 7.1: Individual confidence intervals, and the confidence set.




using a predetermined set of values, we should use a confidence set containing all the *pairs* of $\beta_1$ and $\beta_2$ that won't be rejected. For this example, it turns out that the null hypothesis is not within the 95% confidence set, so that we reject the null hypothesis that both variables are statistically insignificant. We should not drop them from the model. This is a bit surprising considering the individual confidence intervals. The individual confidence intervals, and the confidence set for $b_1$ and $b_2$, are shown in figure 7.1.

The confidence set in figure 7.1 is a rotated ellipse. The angle of rotation is determined by the correlation between $X_1$ and $X_2$. Calculating the confidence intervals is easy, calculating the confidence set is not. Confidence sets are not typically used in practice in econometrics. The purpose of discussing them in this section was to reinforce the idea that the correlation between the variables must be considered when performing a joint hypothesis test.

## 7.6 Calculating the *F*-test statistic

To implement an $F$-test, we can estimate the *restricted* and *unrestricted* model, and compare the two. Using the previous data, we will test the hypothesis:

$$H_0 : \beta_1 = 0,\ \beta_2 = 0$$
$$H_A : \text{not } H_0$$

The full unrestricted model (under the alternative hypothesis) is:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \epsilon$$

The restricted model (under the null hypothesis) is:

$$Y = \beta_0 + \beta_3 X_3 + \epsilon$$

In R, we start by estimating these two models, and saving them:

```
unrestricted <- lm(Y ~ X1 + X2 + X3)
restricted <- lm(Y ~ X3)
```

Then, we can use the `anova` command to perform the $F$-test directly:

```
anova(restricted, unrestricted)
```

```
Analysis of Variance Table

Model 1: Y ~ X3
Model 2: Y ~ X1 + X2 + X3
  Res.Df    RSS Df Sum of Sq      F  Pr(>F)
1    198 8805.1
2    196 8472.7  2    332.37 3.8444 0.02303 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The $F$-statistic is 3.84, which is larger than the 5% critical value of 3.00 (see table 7.1). The $p$-value is 0.02303. We reject the null hypothesis at the 5% significance level.

To calculate the $F$-statistic using equation 7.5:

$$F = \frac{(R_u^2 - R_r^2)/q}{(1 - R_u^2)/(n - k_u - 1)}$$

we need the $R^2$ from the two models. From the unrestricted model, the $R^2$ is 0.7921:

```
summary(unrestricted)
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -0.6246     0.4660  -1.340    0.182
X1            0.2161     0.1723   1.255    0.211
X2           -0.1092     0.1153  -0.946    0.345
X3            2.9384     0.1092  26.914   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.575 on 196 degrees of freedom
Multiple R-squared:  0.7921,    Adjusted R-squared:  0.7889
F-statistic: 248.9 on 3 and 196 DF,  p-value: < 2.2e-16
```

and from the restricted model the $R^2$ is 0.784:

```
summary(restricted)
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -0.5924     0.4719  -1.255    0.211
X3            2.9604     0.1104  26.804   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.669 on 198 degrees of freedom
Multiple R-squared:  0.784,     Adjusted R-squared:  0.7829
F-statistic: 718.5 on 1 and 198 DF,  p-value: < 2.2e-16
```

We are testing two restrictions ($q = 2$), and $n = 200$, so that the $F$-statistic is:

$$F = \frac{(R_u^2 - R_r^2)/q}{(1 - R_u^2)/(n - k_u - 1)} = \frac{(0.7921 - 0.784)/2}{(1 - 0.7921)/(200 - 3 - 1)} = 3.82$$

The number that we get by calculating the $F$-statistic using $R^2$ is a little different than from the `anova` command due to rounding.

## 7.7 The overall *F*-test

Regression software almost always reports the results of an "overall" $F$-test, whenever a model is estimated. The null and alternative hypotheses for this overall $F$-test is:

$$\begin{gathered} H_0: \ \beta_1 = \beta_2 = \cdots = \beta_k = 0 \\ H_A: \text{at least one } \beta \neq 0 \end{gathered} \tag{7.6}$$

Again, $k$ denotes the number of $X$ variables in the model. This null hypothesis says that none of the $X$ variable can explain the $Y$ variable. It is a test to see if the estimated model is garbage. The intercept ($\beta_0$) is not included in the null hypothesis, otherwise there would be nothing to estimate, and if $\beta_0 = 0$ then the mean of $Y$ is also zero (a somewhat silly hypothesis in most cases). The overall $F$-test statistic is reported in the bottom line of R ouptut. In the previous two examples the overall $F$-test statistic is 248.9 and 718.5, with associated $p$-values of 0 (to 16 decimal places). There is evidence that at least one $X$ variable explains $Y$.

We also take this opportunity to point out that, when $q = 1$, the $t$-test and $F$-test provide identical results. In fact, when $q = 1$, $F = t^2$. This can be verified from the previous R output. The $t$-statistic on `X3` is 26.804, and $26.804^2 = 718.5$ (the overall $F$-statistic).

## 7.8 R output for OLS regression

We can now understand all of the R output from OLS estimation, except for "residual standard error". This is just the sample standard deviation of

the OLS residuals. It is also used as a measure of fit, and is also sometimes called the root mean-squared-error. The residual standard error is:

$$\sqrt{\frac{\sum e_i^2}{n-k-1}}$$

We have not discussed this elsewhere in the book, but mention it here as a matter of finality. We now know what everything is in the standard R output for OLS estimation.

## 7.9 Review Questions

1. Explain what is meant by a joint hypothesis, and provide an example.
2. Explain what the restricted and unrestricted models are in a joint hypothesis test.
3. Explain why $t$-tests can't be used to test a joint hypothesis.
4. Calculate the type I error (which is also the significance) when testing:

$$H_0 : \beta_3 = 0,\ \beta_4 = 0$$
$$H_A : \text{not } H_0$$

   using two individual $t$-tests with critical value 1.96, and assuming that the $t$-statistics are independent.

5. Use the CPS data. Let the full unrestricted population model be:

$$wage = \beta_0 + \beta_1 education + \beta_2 gender + \beta_3 age + \beta_4 experience + \epsilon$$

   a) Use $t$-tests to test the null hypothesis: $H_0 : \beta_3 = 0,\ \beta_4 = 0$.
   b) Use the `anova` command to test the null hypothesis from part (a).
   c) Use the $R^2$ from the unrestricted and restricted models to calculate the $F$-statistic for the null hypothesis in part (a). Use table 7.1 to decide whether to reject or fail to reject.
   d) Roughly sketch the confidence set for $b_3$ and $b_4$.
   e) Test the null hypothesis: $H_0 : \beta_1 = 0,\ \beta_2 = 0,\ \beta_3 = 0,\ \beta_4 = 0$.
   f) Using this data, and a null hypothesis of your choosing, verify that $t^2 = F$.

## 7.10 Answers

1. A joint hypothesis is a null hypothesis that involves two or more parameters ($\beta$s). That is, the null hypothesis is *jointly* specifying the values of two or more $\beta$s. See equations 7.1 and 7.2 for examples.

2. One way of conducting a joint hypothesis test is to estimate two separate models. The population model can be considered as the *unrestricted* model under the alternative hypothesis. It is unrestricted since none of the values are chosen (by $H_0$), and all $\beta$s are free to be estimated. The null hypothesis, $H_0$, however, is choosing (restricting) some of the values of the $\beta$s. When the restrictions under $H_0$ are incorporated into the population model, we get a *restricted* model.

3. $t$-tests are typically not used to test joint hypotheses for two reasons. (i) The usual critical values (such as 1.96 for 5% significance) would have to be adjusted. (ii) The estimators that are used in the hypothesis test (the OLS estimators $b$) are likely not-independent (e.g. correlated). This means that the individual $t$-statistics are also likely to be correlated. Unless this correlation is taken into account,

4. We will calculate the type I error assuming that the $t$-statistics are independent. Using two individual $t$-tests, the null hypothesis would be rejected if either, or both, of the $t$-statistics exceed 1.96 in absolute value. There are four possible outcomes: (i) both $t$-statistics are less than 1.96 (in absolute value), (ii) both are greater than 1.96, (iii) $|t_3| > 1.96$ and $|t_4| \leq 1.96$, (iv) $|t_3| \leq 1.96$ and $|t_4| > 1.96$. Only in (i) do we fail to reject the null. The probability of (i) occurring is $0.95 \times 0.95 = 0.9025$. So the probability of rejecting $H_0$ when it is true (the type I error) is the probability of (ii), (iii) and (iv), which is 1 minus the probability of (i), or 0.0975 (not 0.05). We could get the "right" type I error by increasing the critical value from 1.96. This, however, does not solve the larger problem of the dependence between the $t$-statistics.

5. Load the CPS data (you don't need the first line of code if you have already installed the `AER` package):

```
install.packages("AER")
library(AER)
data("CPS1985")
attach(CPS1985)
```

   a) First we need to estimate the model. Regress `wage` on `education`, `gender`, `age`, and `experience` (put the R code all on one line):

```
summary(lm(wage ~ education + gender
                                  + age + experience))
```

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   -1.9574     6.8350  -0.286    0.775
education      1.3073     1.1201   1.167    0.244
genderfemale  -2.3442     0.3889  -6.028 3.12e-09 ***
age           -0.3675     1.1195  -0.328    0.743
experience     0.4811     1.1205   0.429    0.668
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.458 on 529 degrees of freedom
Multiple R-squared:  0.2533,    Adjusted R-squared:  0.2477
F-statistic: 44.86 on 4 and 529 DF,  p-value: < 2.2e-16
```

From the R output, we see that the individual $t$-statistics on `age` and `experience` are small (-0.328 and 0.429, with $p$-values 0.743 and 0.668). This indicates that we should fail to reject the null hypothesis.

b) We need to estimate a restricted model (under the null hypothesis):

```
restricted <- lm(wage ~ education + gender)
```

and an unrestricted model (under the alternative hypothesis):

```
unrestricted <- lm(wage ~ education + gender
                                                + age + experience)
```

and use the `anova` command to get the relevant $F$-statistic:

```
anova(restricted, unrestricted)
```

```
Analysis of Variance Table

Model 1: wage ~ education + gender
Model 2: wage ~ education + gender + age + experience
  Res.Df   RSS Df Sum of Sq      F    Pr(>F)
1    531 11425
2    529 10511  2    914.27 23.007 2.625e-10 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

The $F$-statistic is 23.007 with a $p$-value of 0.000. We reject the null hypothesis. This is the opposite result of what the $t$-statistics would indicate.

c) We can find the $R^2$ from the restricted model using the command:

```
summary(restricted)
```

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.21783    1.03632   0.210    0.834
education     0.75128    0.07682   9.779  < 2e-16 ***
```

```
genderfemale -2.12406    0.40283  -5.273 1.96e-07 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.639 on 531 degrees of freedom
Multiple R-squared:  0.1884,    Adjusted R-squared:  0.1853
F-statistic: 61.62 on 2 and 531 DF,  p-value: < 2.2e-16
```

So, $R_r^2 = 0.1884$. The $R^2$ from the unrestricted model is $R_u^2 = 0.2533$ (see the R output in part (a)). We are testing two restrictions, so that $q = 2$. The sample size is $n = 534$. The number of $X$ variables in the unrestricted model is 4, so that $k_u = 4$. We can now calculate the $F$-statistic using equation 7.5:

$$F = \frac{(R_u^2 - R_r^2)/q}{(1 - R_u^2)/(n - k_u - 1)} = \frac{(0.2533 - 0.1884)/q}{(1 - 0.2533)/(534 - 4 - 1)} = 22.989$$

This is very close to the $F$-statistic that was obtained using the `anova` command in part (b). Using table 7.1, we see that the relevant 5% critical value is 3.00. Since $22.989 > 3.00$, we reject the null hypothesis at the 5% significance level.

d) The main feature of the confidence ellipse is that it should be rotated about the origin. See figure 7.1 for an example. The rotation of the ellipse reflects the non-independence of the estimators, $b_3$ and $b_4$.

e) The null hypothesis in this question is referring to the "overall $F$-test". This $F$-test statistic is calculated for us when we use the `summary` command. From the output in part (a), this $F$-statistic is 44.86 with $p$-value 0.000. We reject the null hypothesis.

f) The $F$-test and $t$-test are equivalent when $q = 1$. Specifically, $t^2 = F$. Note that the 5% critical value for $q = 1$ in the $F$-test (3.84) is the square of the 5% critical value in the $t$-test (1.96).

To verify the equivalence of the $F$-test and $t$-test, we'll calculate the $F$-statistic for a null hypothesis where $q = 1$, and make sure that it is the square of the corresponding $t$-statistic.

Note that, in the R output in part (a), the $t$-statistic on `education` is 1.167. So, for the test:

$$H_0 : \beta_1 = 0$$
$$H_A : \beta_1 \neq 0$$

The $F$-statistic should be $F = 1.167^2 = 1.362$. Estimate the restricted model under this null hypothesis, and use the `anova` command:

```
restricted2 <- lm(wage ~ gender + age + experience)
anova(restricted2, unrestricted)
```

```
Analysis of Variance Table

Model 1: wage ~ gender + age + experience
Model 2: wage ~ education + gender + age + experience
  Res.Df   RSS Df Sum of Sq     F Pr(>F)
1    530 10538
2    529 10511  1    27.063 1.362 0.2437
```

# 8

# Non-Linear Effects

Many models in economics involve *non-linear effects*. A non-linear effect just means that the effect of one variable on another is *not constant*. For example, diminishing marginal utility says that as more is consumed, eventually there is less of an increase to utility than previous. The effect of quantity consumed on utility is *not constant* (there is a non-linear relationship between quantity and utility). Increasing and decreasing returns to scale is another example of a non-linear effect that you may have encountered in your first-year economics courses. Increasing returns to scale implies that when the inputs of production are doubled, output would more than double. The prevalence of the terms "marginal" and "increasing" or "decreasing" in many of our economic models would suggest a need to handle non-linearity.

## 8.1 The linear model

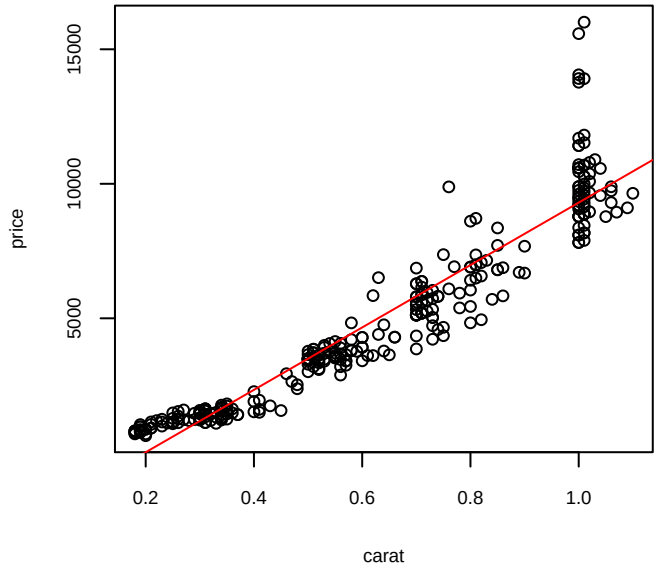
The models we have seen so far have been linear. In the population model:

$$Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_k + \epsilon$$

the change in $Y$ due to a change in $X_1$ (for example) is: $\Delta Y / \Delta X_1 = \beta_1$. This effect of $X_1$ on $Y$ is *constant*. For many relationships between variables, this is unreasonable.

As an example of how the linear model does *not* work, we use the `Diamond` data from the `Ecdat` R package (data originally from Chu, 2001). A plot of the *price* and *carats* of diamonds are shown in figure 8.1, with the OLS estimated line included in the plot. The relationship between *price* and *carats* appears to be non-linear. The effect of *carat* on *price* appears to be small when then diamond is small, and gets large as the size of the diamond grows. The reason for this might be that large diamonds are more *rare*. A larger diamond can always be cut into smaller diamonds, but two diamonds cannot be combined to make a larger one. The linear model says

Figure 8.1: Price of diamonds, and carats, with OLS estimated line.




that the effect of *carat* on *price* is constant, no matter how large or small the diamond is to begin with.

Ideally, we would like an estimated model that is capable of capturing the half "U" shape that we see in the diamonds plot, and other such non-linear shapes. If the true relationship between the two variables is non-linear, then the linear model is *misspecified*. OLS is biased and inconsistent. For situations like this, we need to specify a population model that allows for the marginal effect of $X$ on $Y$ to change depending on the value of $X$.

## 8.2 Polynomial regression model

A non-linear relationship between two variables can be approximated using a *polynomial* function. The validity of the approximation is based on a Taylor series expansion. A population model with a polynomial is:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + \beta_3 X_1^3 + \cdots + \beta_r X_1^r + \epsilon \tag{8.1}$$

Equation 8.1 has a polynomial of degree $r$ in $X_1$. If $r = 2$ we get a quadratic equation, and if $r = 3$ we get a cubic equation. Note that this is just the linear model that we have been using all along, but some of the regressors are powers of $X_1$. Other variables ($X_2, X_3$, etc.) can be added as usual. With the polynomial, estimation by OLS, and hypothesis testing, is the same as usual. Including powers of $X_1$ in the model as additional regressors is *not* a violation of no perfect multicollinearity (assumption A.2), because the relationship between the regressors is not linear.

### 8.2.1 Interpreting the $\beta$s in a polynomial model

The $\beta$s in the polynomial model become much more difficult to interpret. This is the point in including them. We are trying to model a (more complicated) non-linear relationship. Let's take a population model with a quadratic term (usually squaring is sufficient to model the non-linear effect):

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_2^2 + \epsilon \tag{8.2}$$

In equation 8.2, $\beta_1$ is the marginal effect of $X_1$ on $Y$, but the marginal effect of $X_2$ on $Y$ depends on both $\beta_2$ *and* $\beta_3$. That is, $\beta_2$ and $\beta_3$ don't make much sense by themselves. If we take the partial derivative of $Y$ with respect to $X_2$, we get:

$$\frac{\partial Y}{\partial X_2} = \beta_2 + 2\beta_3 X_2$$

This derivative tells us that the squared term ($X_2^2$) allows the effect of $X_2$ on $Y$ to *depend on the value of* $X_2$. A change in $Y$ due to a change in $X_2$ is not constant, but depends on the value of $X_2$.

Including the squared term is just a mathematical "trick" for approximating the non-linear relationship. For example, if $\beta_2$ is positive, then a negative $\beta_3$ means there is a diminishing effect, and a positive $\beta_3$ means there is an increasing effect. OLS is free to choose values for $\beta_2$ and $\beta_3$ to best capture any non-linear relationship.

In order to obtain an interpretation for our estimated polynomial model, we can consider specific OLS predicted values. If we consider a lot of predicted values, we can plot them out in the data and see our estimated equation. If we calculate at least two pairs of predicted values, and take the differences between them, we can get an idea about how the estimated effect depends on the value of the $X$ variable. This is illustrated in a following example.

### 8.2.2 Determining $r$

To determine the degree ($r$) of the polynomial, we can use a series of $t$-tests. We can start with a polynomial of degree $r$, and test the null hypothesis $H_0 : \beta_r = 0$. If we fail to reject (implying that $X^r$ is not needed) then we re-estimate the model with a polynomial of degree $r - 1$. The process repeats until the null hypothesis is rejected. However, in most econometrics models only squared terms are used if needed; very rarely are there cubed (or higher) terms. Testing for the degree of $r$ is illustrated in the following example.

### 8.2.3 Modelling the non-linear relationship in the Diamond data

We start by loading up the Diamond data:

```
install.packages("Ecdat")
library(Ecdat)
data(Diamond)
attach(Diamond)
```

and estimating the *linear* model, $price = \beta_0 + \beta_1 carat + \epsilon$:

```
summary(lm(price ~ carat))
```

```
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  -2298.4       158.5  -14.50   <2e-16 ***
carat        11598.9       230.1   50.41   <2e-16 ***
```

It is estimated that an increase in *carat* of 1 is associated with an increase in the *price* of a diamond by \$11598.9. It might be more sensible to consider the smaller increase of 0.1 carats: an increase of 0.1 carats is associated with an increase in price of \$1160. This effect is the same whether the diamond is small or large to begin with.

In order to allow for the effect of *carat* on *price* to depend on the size of the diamond, we can include a quadratic term, and estimate the population model $price = \beta_0 + \beta_1 carat + \beta_2 carat^2 + \epsilon$. The first thing we need to do is to create the new variable $carat^2$. We can do this in R using:

```
carat2 <- carat^2
```

where ^ is the power operator (shift-6 on most keyboards). The above line of R code creates a new variable by squaring the old variable. I called the new variable `carat2`, but you can call it whatever you want. We can now estimate a quadratic population model simply by including this new variable in our estimation command:

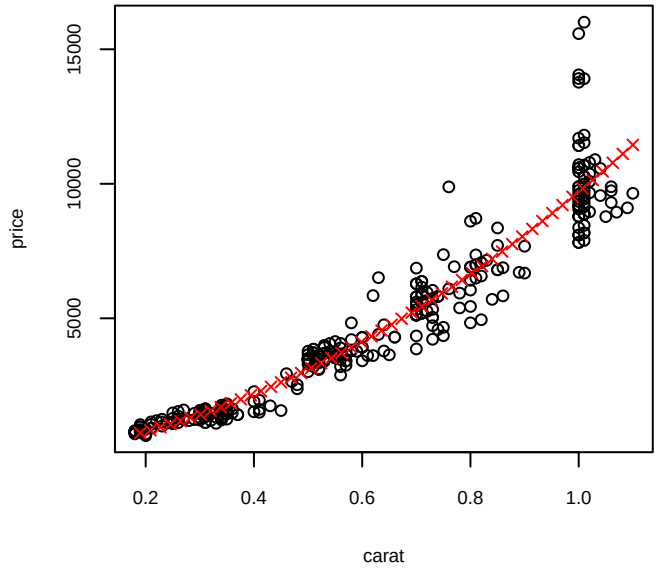
```
summary(lm(price ~ carat + carat2))
```

```
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   -42.51     316.37  -0.134   0.8932
carat        2786.10    1119.61   2.488   0.0134 *
carat2       6961.71     868.83   8.013  2.4e-14 ***
```

Notice that `carat2` is highly statistically significant. There is evidence that the effect is non-linear.

The positive sign on `carat2` means that we have estimated an *increasing* marginal effect. How do we interpret our estimated $\beta$s further? That is, what is the estimated effect of *carats* on *price*? The key is to calculate some OLS *predicted values*, to consider some specific scenarios. In figure 8.2, I calculate 50 OLS predicted values by choosing values for *carat* at regular intervals, and plot them over the Diamond data. Notice that our estimated equation captures the half "U" shape, and seems to fit the data well.

Figure 8.2: Diamond data, with estimated quadratic model.




The predicted values used in figure 8.2 were obtained by substituting different values for *carat* into the estimated equation:

$$\hat{price} = -42.51 + 2786.10carat + 6967.71carat^2 \tag{8.3}$$

Now, let's focus on two specific scenarios: the effect of an increase in *carats* when (i) the diamond is small, and (ii) the diamond is large. Let's consider an increase of 0.1 in *carats* when the diamond is (i) 0.2 *carats* in size, and (ii) 1 *carat* in size. We need two predicted values for each scenario. For (i), we get the predicted values for $carat = 0.2$ and for $carat = 0.3$:

$$\hat{price}|_{carat=0.2} = -42.51 + 2786.10(0.2) + 6967.71(0.2)^2 = 793$$
$$\hat{price}|_{carat=0.3} = -42.51 + 2786.10(0.3) + 6967.71(0.3)^2 = 1420$$

and take the difference between these two predicted values:

$$\hat{price}|_{carat=0.3} - \hat{price}|_{carat=0.2} = 1419.88 - 793.18 = 627$$

So, the predicted effect of an increase in *carats* of 0.1, when the diamond is 0.2 *carats*, is \$627.

Now we consider the effect of a 0.1 increase in *carats* for (ii) a large diamond:

$$\hat{price}|_{carat=1} = -42.51 + 2786.10(1) + 6967.71(1)^2 = 9705$$
$$\hat{price}|_{carat=1.1} = -42.51 + 2786.10(1.1) + 6967.71(1.1)^2 = 11446$$

and again take the difference between the two predicted values:

$$\hat{price}|_{carat=1.1} - \hat{price}|_{carat=1} = 11446 - 9705 = 1741$$

The predicted effect of an increase in *carats* is larger, when the diamond is larger. That is, the estimated effect of a 0.1 increase in *carats* is \$1741.

The important point of this exercise is the following. The estimated effect of *carats* on *price* is much different depending on whether the diamond is large or small (\$627 when $carats = 0.2$ vs. \$1741 when $carats = 1$. The linear model estimates a constant effect of \$1160, which misses out on important non-linearities.

Finally, we determine the appropriate degree of the polynomial in `carat` (we probably should have begun with this). Let's estimate a cubic model: $price = \beta_0 + \beta_1 carat + \beta_2 carat^2 + \beta_3 carat^3 + \epsilon$. We'll need a new variable:

```
carat3 <- carat^3
```

and to add it to the regression:

```
summary(lm(price ~ carat + carat2 + carat3))
```

```
(Intercept)    786.3      765.4   1.027   0.3051
carat        -2564.2     4636.9  -0.553   0.5807
carat2       16638.9     8185.3   2.033   0.0429 *
carat3       -5162.5     4341.9  -1.189   0.2354
```

The cubed variable, `carat3`, is insignificant (with $p$-value 0.2354). The quadratic specification is sufficient for capturing the non-linear relationship between *carat* and *price*. It is often the case that a quadratic specification is good enough.

## 8.3 Logarithms

Another way to approximate the non-linear relationship between $Y$ and $X$ is by using logarithms. Logarithms can be used to approximate a percentage change. If one or more percentage changes are involved in the relationship between two variables, it is a type of non-linear effect. To see this, consider a 1% increase in 100 (which is 1), and a 1% increase in 200 (which is 2). The same 1% increase has a different effect depending on the starting value.

### 8.3.1 Percentage change

Let's be explicit about what is meant by a percentage change. A percentage change in $X$ is:

$$\frac{\Delta X}{X} \times 100 = \frac{X_2 - X_1}{X_1} \times 100$$

where $X_1$ is the starting value of $X$, and $X_2$ is the final value.

### 8.3.2 Logarithm approximation to percentage change

The approximation to percentage changes using logarithms is:

$$\log(X + \Delta X) - \log(X) \times 100 \approx \frac{\Delta X}{X} \times 100$$

or

$$\log(X_2 - X_1) \times 100 \approx \frac{X_2 - X_1}{X_1} \times 100$$

So, when $X$ changes, the change in $\log(X)$ is approximately equal to a percentage change in $X$. The approximation is more accurate the smaller the change in $X$. Table 8.1 shows variation percentage changes in $X$, and the approximate change using the log function. The approximation does not work well for changes above 10%.

Table 8.1: Percentage change, and approximate percentage change using the log function.

| Change in $X$ | Percentage change: $\frac{X_2 - X_1}{X_1} \times 100$ | Approximated percentage change: $(\log X_2 - \log X_1) \times 100$ |
|---|---|---|
| $X_1 = 1, X_2 = 2$ | 100% | 69.32% |
| $X_1 = 1, X_2 = 1.1$ | 10% | 9.53% |
| $X_1 = 1, X_2 = 1.01$ | 1% | 0.995% |
| $X_1 = 5, X_2 = 6$ | 20% | 18.23% |
| $X_1 = 11, X_2 = 12$ | 9.09% | 8.70% |
| $X_1 = 11, X_2 = 11.1$ | 0.91% | 0.91% |

### 8.3.3 Logs in the population model

The log function can be used in our population model so that the $\beta$s have various *percentage changes* interpretations. There are three ways we can introduce the log function into our models. The three different possibilities arise from taking logs of the left-hand-side variable, one or more of the right-hand-side variables, or both. Table 8.2 shows these three cases.

Table 8.2: Three population models using the log function.

| Population model | Population regression function |
|---|---|
| I. linear-log | $Y = \beta_0 + \beta_1 \log X + \epsilon$ |
| II. log-linear | $\log Y = \beta_0 + \beta_1 X + \epsilon$ |
| III. log-log | $\log Y = \beta_0 + \beta_1 \log X + \epsilon$ |

For each of the three different population models in table 8.2, $\beta_1$ has a different percentage change interpretation. We don't derive the interpre-

tations of $\beta_1$, but instead list them for the three different cases in table 8.2:

- linear-log: a 1% change in $X$ is associated with a $0.01\beta_1$ change in $Y$.
- log-linear: a change in $X$ of 1 is associated with a $100 \times \beta_1\%$ change in $Y$.
- log-log: a 1% change in $X$ is associated with a $\beta_1\%$ change in $Y$. $\beta_1$ can be interpreted as an *elasticity*.

### 8.3.4 A note on $R^2$

$R^2$ and $\bar{R}^2$ measure the proportion of variation in the dependent variable ($Y$) that can be explained using the $X$ variables. When we take the log of $Y$ in the log-linear or log-log model, the variance of $Y$ changes. That is, $\text{Var}[\log Y] \neq \text{Var}[Y]$. We cannot use $R^2$ or $\bar{R}^2$ to compare models with different dependent variables. That is, we should not use $R^2$ to decide between two models, where the dependent variable is $Y$ in one, and $\log Y$ in the other.

### 8.3.5 Log-linear model for the CPS data

It is common to use the log of *wage* as the dependent variable, instead of just *wage*. This allows for the factors that determine differences in wages be associated with approximate percentage changes in *wage*. In the following, we'll see an example of a log-linear model estimated using the CPS data. Start by loading the data:

```
library(AER)
data("CPS1985")
attach(CPS1985)
```

and estimate a log-linear model $\log(wage) = \beta_0 + \beta_1 education + \beta_2 gender + \beta_3 age + \beta_4 experience + \epsilon$:

```
summary(lm(log(wage) ~ education + gender + age
        + experience))
```

```
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   1.15357    0.69387   1.663    0.097 .
education     0.17746    0.11371   1.561    0.119
genderfemale -0.25736    0.03948  -6.519 1.66e-10 ***
age          -0.07961    0.11365  -0.700    0.484
experience    0.09234    0.11375   0.812    0.417
```

The interpretation of the estimated coefficient on `education`, for example, is that a 1 year increase in *education* is associated with a 17.8% increase in *wage*. The interpretation of the coefficient on the dummy variable `genderfemale` is a bit more tricky. It is estimated that women make

$(100 \times (\exp(-0.257) - 1) = -22.7\%)$ 22.7% less than men. For simplicity, however, we can say that women make approximately 25.7% less than men, but you should know that this interpretation is actually wrong.

The advantage of using log *wage* as the dependent variable is that it allows the estimated model to capture non-linear effects. The 25.7% decrease in wages for women means that the dollar difference in wages between females and males in high-paying jobs (such as medicine) is larger than the dollar difference in wages between females and males in lower-paying jobs.

## 8.4 Interaction terms

Interaction terms can model a type of non-linear effect between variables. They are useful when the effect of $X$ on $Y$ may depend on a different $X$ variable. Typically, one of the variables in the interaction term is a *dummy variable* (denote the dummy variable $D$). When the other variable is continuous (call it $X$), the interaction term $(D \times X)$ allows for a different linear effect between the two groups (the groups defined by $D$). When both of the variables in the interaction term are dummy variables $(D_1 \times D_2)$, we get something called a "difference-in-difference". Finally, both of the variables in the interaction can be continuous $(X_1 \times X_2)$, but this situation is somewhat rare and we do not discuss it here.

### 8.4.1 Motivating example

To motivate the usefulness of interaction terms, we use a *hypothetical* data set. This data was *created*, and should not be taken seriously, or to inform policy.

Suppose that 500 marijuana users are surveyed in different locations, and the variables in the data are:

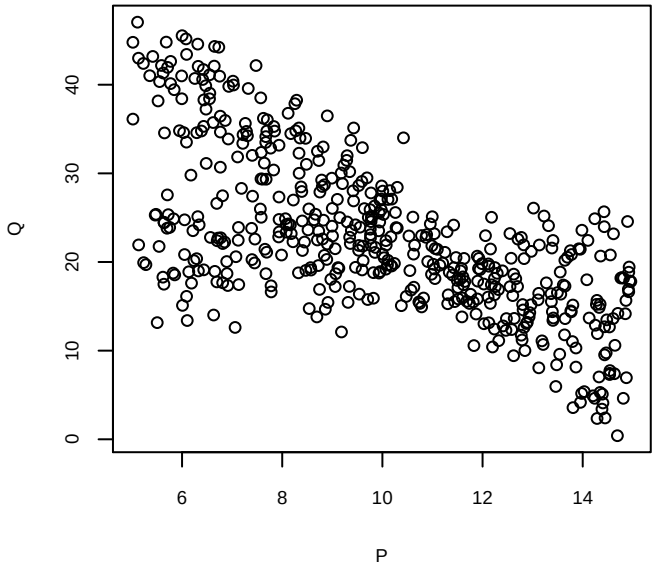
- $Q$ - the quantity of marijuana consumed, in grams, per month
- $P$ - the average price per gram in the individual's location
- $adult = 1$ if the individual is an adult, $= 0$ if the individual is a teenager

A plot of price versus quantity is shown in figure 8.3. Do you notice anything? Perhaps this data may be better explained using two separate regression lines. For now, however, let's ignore the *adult* variable. Estimate a regression of $Q$ on $P$:

```
summary(lm(Q ~ P))
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  44.2152     1.0776   41.03   <2e-16 ***
P            -2.1634     0.1041  -20.78   <2e-16 ***
```

Figure 8.3: Plot of the hypothetical demand for marijuana data.




It is estimated that an increase in price of \$1/gram reduces consumption by 2.16 grams/month. This estimated regression line is added to the plot of the data in figure 8.4. We see that we are get an "average" regression line for the two groups.

Ideally, we would like a separate regression line for the two groups (adults and teenagers), since the effect of price on consumption may differ for the two. To highlight this idea, the data is plotted, making note of which group each data point belongs to. In figure 8.5, we clearly see that the two groups should be treated separately.

Let's try to separate the groups by adding the dummy variable to our regression:

```
summary(lm(Q ~ P + adult))
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 46.21319    1.02971  44.880   <2e-16 ***
P           -2.12242    0.09712 -21.854   <2e-16 ***
adult       -4.81124    0.54975  -8.752   <2e-16 ***
```

The estimated coefficent on $P$ means that an increase in price of \$1/gram decreases consumption by 2.12 grams/month (similar to before). The coefficient on *adult* is interpreted to mean that, on average, adults consume 4.81 grams/month less than teenagers. Graphically, we have two different regression lines that have the same slope, but different intercepts (46.21 for teenagers, and 46.21 - 4.81 for adults). The two different regression lines are plotted in figure 8.6.

Figure 8.4: Marijuana data, with estimated regression line from $Q = \beta_0 + \beta_1 P + \epsilon$ added to the plot.

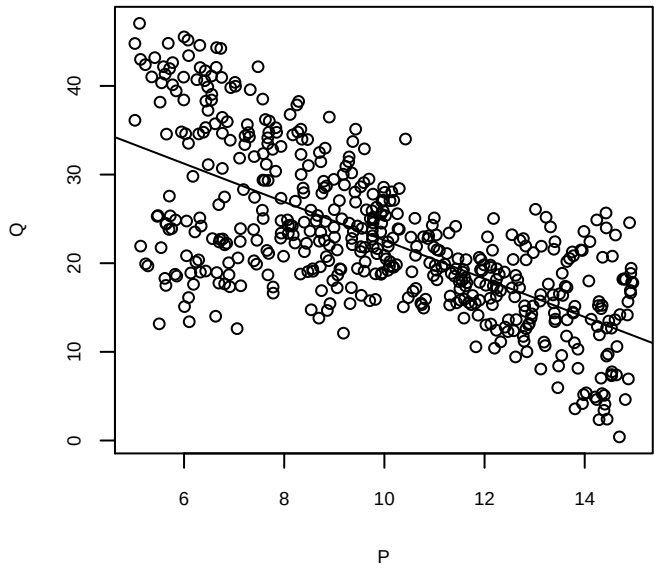



Figure 8.5: Marijuana data plotted by age group.

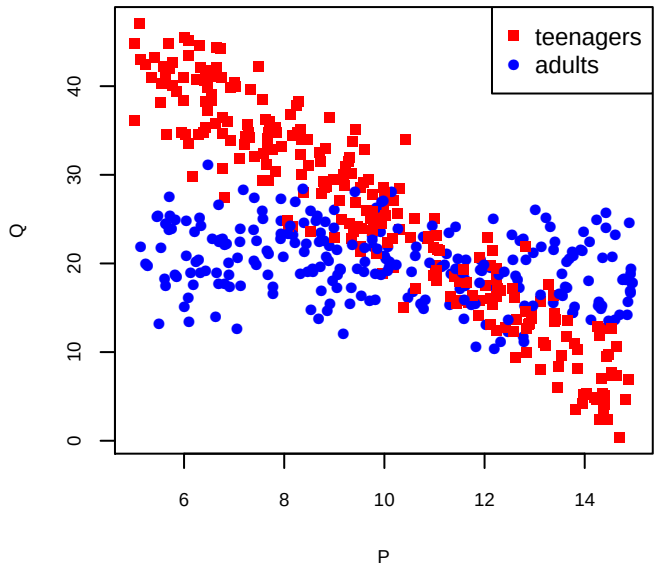
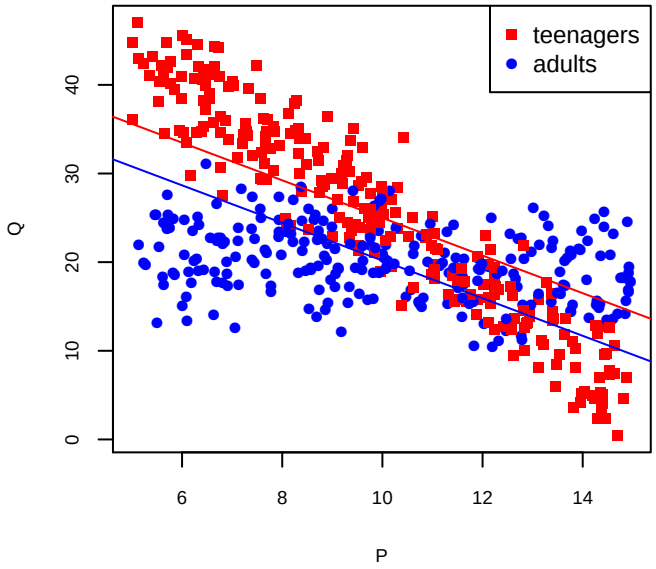


Figure 8.6: With the addition of the dummy variable, each group has a different intercept, but the same slope.




This still doesn't get us what we want. We need something new: an *interaction term*. This will allow for two separate marginal effects (slopes) for the two groups. The estimation is discussed later, but the results are shown graphically in figure 8.7.

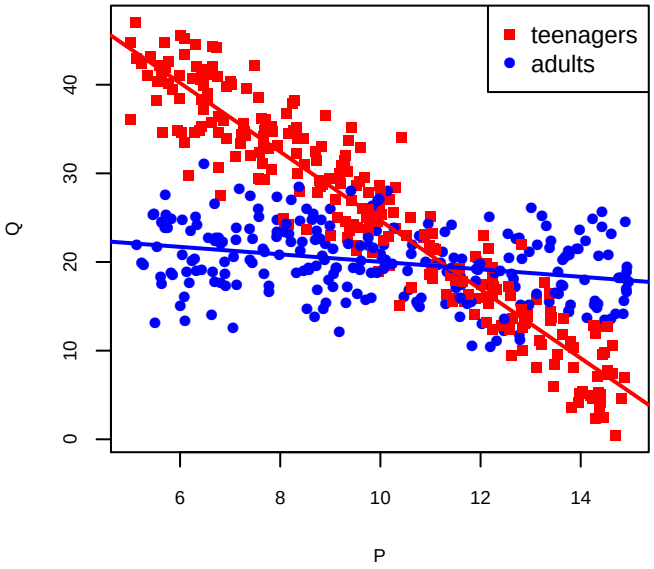
### 8.4.2 Dummy-continuous interaction

So how do we allow for two different marginal effects for the two different groups, and attain the type of estimated equation shown in figure 8.7? By using an interaction term. Specifically, this example uses a *dummy-continuous* interaction term. The population model that we want to estimate is:

$$Q = \beta_0 + \beta_1 P + \beta_2 adult + \beta_3 (adult \times P) + \epsilon \tag{8.4}$$

where $adult \times P$ is the interaction term, and is a new variable that is created by multiplying the other two variables together. To see how model 8.4 allows for two separate lines, consider what the population model is for teenagers ($adult = 0$), and for adults ($adult = 1$).

Figure 8.7: Two separate regression lines for the two different groups.




### Population model for teenagers

Let's substitute in the value $adult = 0$ into equation 8.4 and get the population model for teenagers:

$$\begin{aligned} Q &= \beta_0 + \beta_1 P + \beta_2(0) + \beta_3(0 \times P) + \epsilon \\ &= \beta_0 + \beta_1 P + \epsilon \end{aligned} \tag{8.5}$$

From equation 8.5, we can see that the intercept is $\beta_0$ and the slope is $\beta_1$.

### Population model for adults

Substituting in the value $adult = 1$ into equation 8.4, we get the population model for adults:

$$\begin{aligned} Q &= \beta_0 + \beta_1 P + \beta_2(1) + \beta_3(1 \times P) + \epsilon \\ &= (\beta_0 + \beta_2) + (\beta_1 + \beta_3)P + \epsilon \end{aligned} \tag{8.6}$$

For adults, the intercept is $\beta_0 + \beta_2$ and the slope is $\beta_1 + \beta_3$. The marginal effect of price on consumption differs by $\beta_3$ between the two groups.

### Estimation with an interaction term

To include a dummy-continuous interaction term in our regression, we simply create a new variable by multiplying the dummy variable ($adult$) and the continuous variable $P$ together:

```
adult_P <- adult*P
```

and include the new variable in the regression:

```
summary(lm(Q ~ P + adult + adult_P))
```

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  63.48944    0.85166   74.55   <2e-16 ***
P            -3.88168    0.08339  -46.55   <2e-16 ***
adult       -39.25222    1.21030  -32.43   <2e-16 ***
adult_P       3.45993    0.11695   29.58   <2e-16 ***
```

The estimated value of 3.46 (on the `adult_P` dummy-continuous interaction term) means that the decrease in consumption due to an increase in price of $1 is 3.46 grams/month less for adults than it is for teenagers. That is, the effect of price on quantity is -3.88 for teenagers, and (-3.88 + 3.46 = -0.42) for adults. The demand curve is much steeper for teenagers.

### 8.4.3 Dummy-dummy interaction: differences-in-differences

A dummy-dummy interaction is when two different dummy variables are multiplied together, creating a new variable. This new variable allows for an overlap of two differences. The two dummy variables give two different means, and the interaction term gives a "difference-in-difference".

As an example, consider the CPS data again. Instead of using the `education` variable which was continuous, we'll use a dummy variable `bach` which equals to 1 if the individual has a university (BA) degree, and 0 otherwise. First, we estimate the standard model without the interaction term, with $\log(wage)$ as the dependent variable:

```
summary(lm(log(wage) ~ female + bach))
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  2.07175    0.03108  66.657  < 2e-16 ***
female      -0.22886    0.04240  -5.397 1.02e-07 ***
bach         0.39177    0.04976   7.873 1.97e-14 ***
```

The interpretation of these results is that women make 23% less than men, and that individuals with a bachelors degree make 39% more than those without. However, this model does not allow for the possibility that education has a different effect for women than it does for men. There is a difference between men and women, and there is a difference between university degrees and high school degrees, but there is no difference within the difference.

To allow for education to have a different effect for men than for women (a difference-in-difference), we estimate the model:

$$\log(wage) = \beta_0 + \beta_1 female + \beta_2 bach + \beta_3 (female \times bach) + \epsilon$$

where $\beta_3$ is the additional percentage increase in wages for women with an education, versus men with an education. In R, we create the dummy-dummy interaction term by:

```
fem_bach <- female*bach
```

and include it in our regression:

```
summary(lm(log(wage) ~ female + bach + fem_bach))
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  2.08291    0.03292  63.280  < 2e-16 ***
female      -0.25309    0.04849  -5.219 2.58e-07 ***
bach         0.34500    0.06736   5.122 4.25e-07 ***
fem_bach     0.10292    0.09994   1.030    0.304
```

It is estimated that women make 25% less than men, that men with a BA degree make 35% more than men without a degree, and that women with a degree make (35% + 10% = 45%) more than women without a degree. There is a difference for men, a difference for women, and the difference between these two differences is $\beta_3$ (10%).

### 8.4.4 Hypothesis tests involving dummy interactions

An important use of dummy interaction terms is to test whether there is a different effect between two groups. In the marijuana example, the interaction term measures the difference in the slope of the demand curve between the two groups. To test the hypothesis that the sensitivity of marijuana consumption to changes in price is the same for teenagers as it is for adults, we could test the hypothesis:

$$H_0 : \beta_3 = 0$$
$$H_A : \beta_3 \neq 0$$

in the model:

$$Q = \beta_0 + \beta_1 P + \beta_2 adult + \beta_3 (adult \times P) + \epsilon$$

From the regression output from before, we see that the interaction term is highly significant, and we reject the null hypothesis. There is evidence that there is a different marginal effect for the two groups.

Similarly, testing $\beta_3 = 0$ in the model:

$$\log(wage) = \beta_0 + \beta_1 female + \beta_2 bach + \beta_3 (female \times bach) + \epsilon$$

is a test of whether there is a different effect of education for women than for men. From the regression output in the previous section, we see that the $p$-value for the estimated coefficient on `fem_bach` is 0.304. We fail to reject the null that there is no difference in the effect of education between men and women.

### 8.4.5 Some additional points

The third possibility, a continuous-continuous interaction term, was left out of the discussion. For example, the returns to education (measured in years as a continuous variable) may diminish as the worker ages (also a continuous variable). To capture this idea, we could multiply these two continuous variables together, and include the product in our regression.

The models presented in this section had dummy variable interaction terms that resulted in completely separate regression functions for the different groups. This complete separation was due to the simplicity of the models. That is, no other variables were included. We can include other variables in the regression as usual. For example, for the CPS data, we would probably want to include `age` and `experience` and possibly other variables as well. The interaction terms then have the interpretation of a difference between groups, *while controlling for other factors* (*ceteris paribus*).

Finally, the dummy interaction may involve *multiple variables*. This is particularly important when the polynomial regression model is used to capture a non-linear effect. For example, we might have used `education`$^2$ as a variable to capture a non-linear effect. Using a dummy interaction with education should then involve both of the variables (`education` and `education`$^2$). A test for no differences between groups would then require the $F$-test.

## 8.5 Review Questions

1. What is a polynomial regression model?

2. Why is it important to have a population model that can capture non-linear effects?

3. Use the following commands in R to load the data necessary for this question (there are only two commands that should be on two separate lines):

```
mydata <- read.csv("http://home.cc.umanitoba.ca/
        ~godwinrt/3040/data/chap8.csv")
attach(mydata)
```

   a) Plot the data. Which variable might have a non-linear relationship with $Y$?

   b) Estimate the population model: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + \beta_3 X_1^3 + \beta_4 X_1^4 + \beta_5 X_2 + \epsilon$.

   c) Determine the appropriate degree of the polynomial in $X_1$ (determine the right $r$).

d) What is the estimated effect of $X_1$ on $Y$?

4. Other than polynomials, what is another way to capture a non-linear effect in an OLS regression model?

5. What are the interpretations of the $\beta$s in population models that use logarithms?

6. Using the diamond data, estimate a linear-log, log-linear, and log-log model. Interpret your results in each case.

7. Describe the usefulness of interaction terms.

8. Using the CPS data, determine if there is a different effect of *education* on *wage*, between men and women.

## 8.6 Answers

1. A polynomial regression model is one that includes powers of one or more of the $X$ variables as additional regressors (e.g. $X_3^2$, $X_3^3$). This is done in order to approximate a non-linear relationship between the $X$ and $Y$ variables.

2. Many models in economics specify non-linear relationships between the variables. We want our econometric models to represent the features of the economic model. If non-linear relationships are ignored, the OLS estimator may be biased.

3. a) A plot of the data reveals that there is a possible non-linear relationship between $X_1$ and $Y$:

```
plot(X1,Y)
```

See figure 8.8. There is no apparent non-linear relationship between $X_2$ and $Y$.

b) We will need to create some new variables using $X_1$:

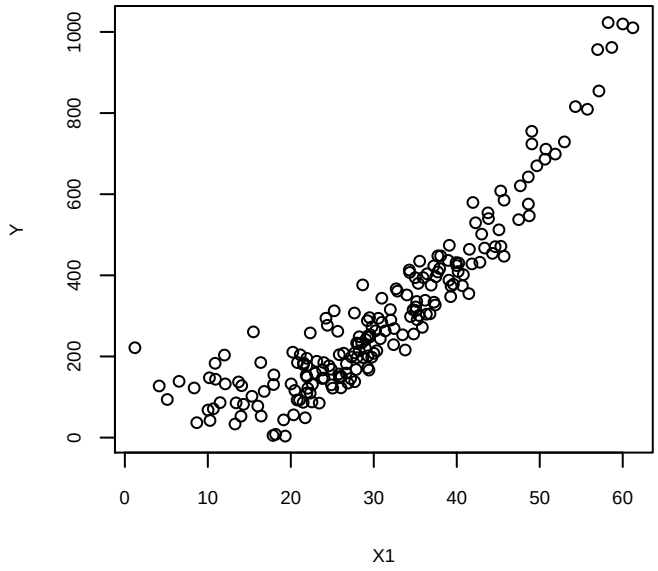
```
X12 <- X1^2
X13 <- X1^3
X14 <- X1^4
```

and include them in the model:

```
summary(lm(Y ~ X1 + X12 + X13 + X14 + X2))
```

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.901e+02  1.809e+01  10.509  < 2e-16 ***
X1          -1.059e+01  3.135e+00  -3.380 0.000878 ***
X12          5.076e-01  1.807e-01   2.810 0.005468 **
```

Figure 8.8: Question 3, part (a).




```
X13          -3.431e-03   4.132e-03   -0.831 0.407262
X14           3.141e-05   3.229e-05    0.973 0.331872
X2           -2.015e+00   6.118e-02  -32.944  < 2e-16 ***
```

c) In part (b), $X_1^2$, $X_1^3$, and $X_1^4$ were included in the regression, so that $r = 4$. We may not need to go as high as $X_1^4$ in order to adequately model the non-linear relationship between $X_1$ and $Y$. To determine the appropriate $r$, we can see if the highest power of $X_1$ is statistically significant. If not, we drop it from the model, and try again, stopping when the highest power *is* significant.

From the R output in part (b), we see that $X_1^4$ is "insignificant" (we fail to reject the null hypothesis that $\beta_4 = 0$). This indicates that $X_1^4$ is not needed in the polynomial, so we drop it from the model:

```
summary(lm(Y ~ X1 + X12 + X13 + X2))
```

```
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.775e+02  1.260e+01  14.081  < 2e-16 ***
X1          -7.870e+00  1.409e+00  -5.586 7.71e-08 ***
X12          3.382e-01  4.818e-02   7.020 3.60e-11 ***
X13          5.584e-04  4.985e-04   1.120    0.264
X2          -2.023e+00  6.070e-02 -33.326  < 2e-16 ***
```

Now, we test to see if $X_1^3$ is insignificant (from the output above, it is). Dropping it from the model we get:

```
summary(lm(Y ~ X1 + X12 + X2))
```

```
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) 188.355857   8.024835   23.47   <2e-16 ***
X1           -9.337920   0.517857  -18.03   <2e-16 ***
X12           0.391436   0.007933   49.34   <2e-16 ***
X2           -2.015532   0.060387  -33.38   <2e-16 ***
```

Finally, we see that the highest power of $X_1$ (now $X_1^2$) is statistically significant. We cannot drop it from the model. The appropriate degree of the polynomial in $X_1$ is $r = 2$.

d) In the estimated model

$$\hat{Y} = b_0 + b_1 X_1 + b_2 X_1^2 + b_3 X_2$$

one way to interpret the estimated effect of $X_1$ on $Y$ is to consider specific OLS predicted values. The difficulty in interpretation arises because the effect of $X_1$ on $Y$ now also depends on $X_1^2$, so that both $b_1$ and $b_2$ must be considered together.

The whole point of using the squared term ($X_1^2$) is to allow the change in $Y$ due to a change in $X_1$ to depend on the value of $X_1$ itself. So, let's consider a change in $X_1$ of 1 unit, for two different starting values of $X_1$: 20 and 40.

$$\begin{aligned}\hat{Y}|_{X_1=21} - \hat{Y}|_{X_1=20} &= (-9.338 \times 21 + 0.391 \times 21^2) \\ &\quad - (-9.338 \times 20 + 0.391 \times 20^2) = 6.693\end{aligned}$$

When $X_1 = 20$, the effect of a 1 unit increase in $X_1$ is to increase $Y$ by 6.693. Let's try for a larger value of $X_1$:

$$\begin{aligned}\hat{Y}|_{X_1=41} - \hat{Y}|_{X_1=40} &= (-9.338 \times 41 + 0.391 \times 41^2) \\ &\quad - (-9.338 \times 40 + 0.391 \times 40^2) = 22.333\end{aligned}$$

The estimated effect of $X_1$ on $Y$ is much larger, for larger values of $X_1$.

4. Besides polynomials, we can also take the logarithms of the $X$ and/or $Y$ variables. Exploiting a property of logarithms that small changes in $\log X$ (or $\log Y$) are approximately equal to percentage changes in $X$ (or $Y$. This leads the $\beta$s in the population regression model to have approximate percentage change interpretations of one variable on another. A percentage change is a non-linear change, since the actual amount of the change depends on the starting value.

5. See table 8.2 for the different population models using logs, and see the following discussion for the interpretations of the $\beta$s in the different models.

6. Load the diamond data (the first line of code is not needed if you have already installed the `Ecdat` package):

```
install.packages("Ecdat")
library(Ecdat)
data(Diamond)
attach(Diamond)
```

   The linear-log model:

```
summary(lm(price ~ log(carat)))
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   8397.4      133.7   62.78   <2e-16 ***
log(carat)    5833.8      172.2   33.87   <2e-16 ***
```

   The interpretation is that a 1% increase in *carats* is associated with an increase in *price* of $58.34.

   The log-linear model:

```
summary(lm(log(price) ~ carat))
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  6.44488    0.02938  219.40   <2e-16 ***
carat        2.84155    0.04264   66.64   <2e-16 ***
```

   The interpretation is that an increase in *carats* of 1 is associated with an increase in price of 284% (it may be more sensible to instead say that a 0.1 increase in *carats* is associated with a 28.4% increase in *price*).

   Finally, the log-log model:

```
summary(lm(log(price) ~ log(carat)))
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  9.12775    0.01440  633.99   <2e-16 ***
log(carat)   1.53726    0.01854   82.92   <2e-16 ***
```

   The interpretation is that a 1% increase in *carats* is associated with a 1.53% increase in *price*.

7. Interaction terms are useful when we want to allow the effect of $X$ on $Y$ to depend on a different $X$ variable. When one variable in the interaction term is a continuous variable, and the other is a dummy,

the interaction term allows for a different marginal effect for the two different groups (as defined by the dummy).

When both variables in the interaction term are dummies, we are able to estimate a "difference-in-difference". In both cases, interaction terms allow us to estimate, and test for, differences between groups.

8. Load the CPS data (you don't need the first line of code if you have already installed the `AER` package):

```
install.packages("AER")
library(AER)
data("CPS1985")
attach(CPS1985)
```

We'll introduce an interaction term into our population model:

$$\log wage = \beta_0 + \beta_1 education + \beta_2 female + \beta_3 age + \beta_4 experience + \beta_5 education \times female + \epsilon$$

To estimate this model in R, we can use the command (all on one line):

```
summary(lm(log(wage) ~ education + gender  + age
              + experience + gender * education))
```

```
Coefficients:
                        Estimate Std. Error t value Pr(>|t|)
(Intercept)              1.23263    0.69231   1.780 0.075576 .
education                0.14950    0.11402   1.311 0.190364
genderfemale            -0.69499    0.20315  -3.421 0.000672 ***
age                     -0.06472    0.11345  -0.570 0.568616
experience               0.07754    0.11355   0.683 0.494959
education:genderfemale   0.03362    0.01531   2.196 0.028545 *
```

The estimated difference is that an additional year of education increases wages by 3.36% more for women than for men (note that the dependent variable is $\log wage$. To test to see if this difference is *insignificant* we test the null hypothesis that the coefficient on the interaction term is equal to zero ($H_0 : \beta_5 = 0$). R has already performed this test for us: the associated $p$-value is 0.0286. We reject the null hypothesis that there are no differences in the effect of education on wages between men and women, at the 5% significance level.

# 9

# Heteroskedasticity

The estimators that we have used so far have good statistical properties provided that the following assumptions hold:

A1 The population model is linear in the $\beta$s.

A2 There is no perfect multicollinearity between the $X$ variables.

A3 The random error term, $\epsilon$, has mean zero.

A4 $\epsilon$ is identically and independently distributed.

A5 $\epsilon$ and $X$ are independent.

A6 $\epsilon$ is Normally distributed.

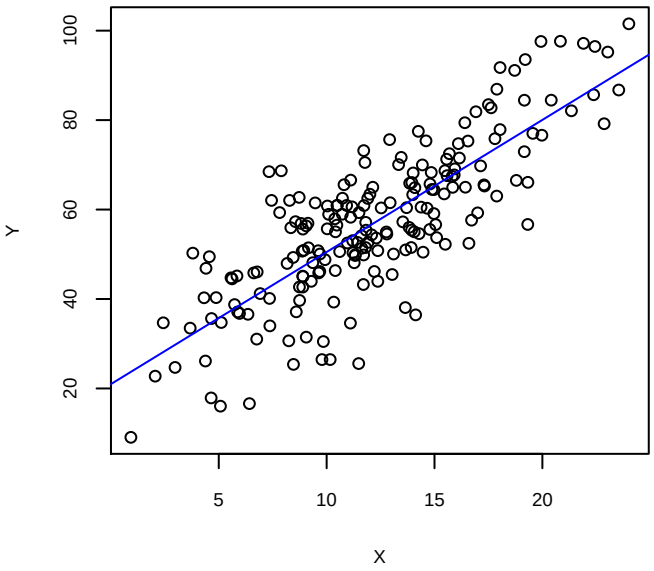
These assumptions assure that OLS is unbiased, efficient, and consistent, and that hypothesis testing is valid. A violation of one or more of these assumptions might lead us to estimators beyond OLS. OLS is simple, and easy to use, but is often thought of a starting point in econometric modelling since the above assumptions are often unreasonable.

In this section, we will consider that assumption A4 is violated in a particular way. Specifically, we consider what happens where the error term, $\epsilon$, is *not* identically distributed.

## 9.1 Homoskedasticity

If assumption A4 is satisfied, then $\epsilon$ is identically distributed. This means that all of the $\epsilon_i$ have the same variance. That is, all of the random effects that determine $Y$, outside of $X$, have the same dispersion. The term *homoskedasticity* (same dispersion) refers to this situation of identically distributed error terms.

Figure 9.1: Homoskedasticity. The average squared vertical distance from the data points to the OLS estimated line is the same, regardless of the value of $X$.




Stated mathematically, homoskedasticity means:

$$\text{Var}[\epsilon_i|X_i] = \sigma^2 \ , \ \forall i$$

The variance of $\epsilon$ is constant, even conditional on knowing the value of $X$.

Homoskedasticity means that the squared vertical distance of each data point from the (population or estimated) line is, on average, the same. The values of the $X$ variables do not influence this distance (the variance of the random unobservable effects are not determined by any of the values of $X$). See figure 9.1.

## 9.2 Heteroskedasticity

Heteroskedasticity refers to the situation where the variance of the error term $\epsilon$ is not equal for all observations. The term heteroskedasticity means *differing dispersion*. Mathematically:

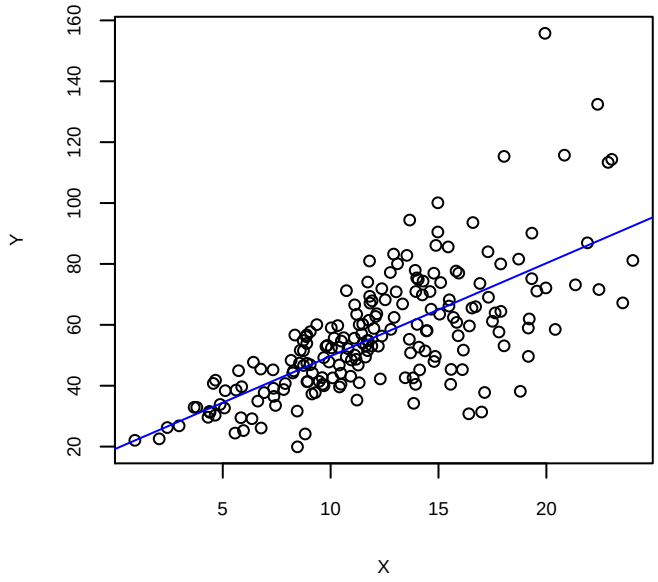
$$\text{Var}[\epsilon_i|X_i] \neq \sigma^2 \ , \ \forall i$$

or

$$\text{Var}[\epsilon_i|X_i] = \sigma_i^2$$

Each observation can have its own variance, and the value of $X$ may influence this variance.

Figure 9.2: Heteroskedasticity. The squared vertical distance of a data point from the OLS estimated line is influenced by $X$.




Heteroskedasticity means that the squared vertical distance of each data point from the estimated regression line is not the same on average, and may be influenced by one or more of the $X$ variables. See figure 9.2, where the larger the value of $X$ is, the larger the variance of $\epsilon$.

### 9.2.1 The implications of heteroskedasticity

Heteroskedasticity is a violation of A.4, since each $\epsilon_i$ is not identically distributed. Heteroskedasticity has two main implications for the estimation procedures we have been using in this book:

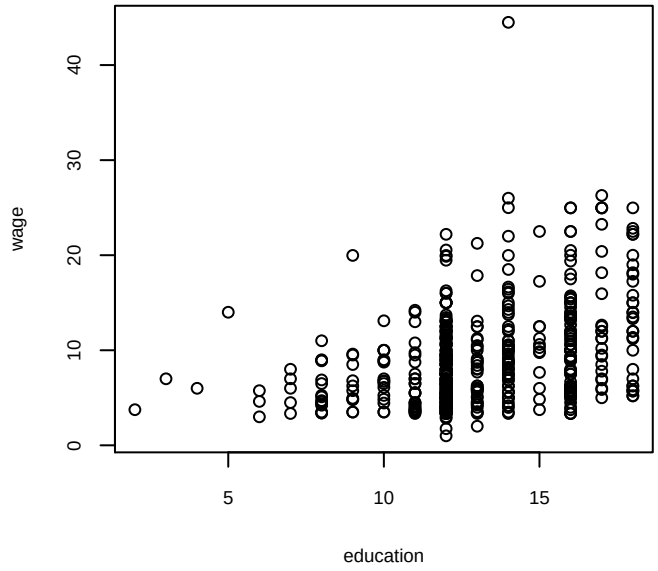
(i) The OLS estimator is no longer efficient.

(ii) The estimator for the variance of the OLS estimator is inconsistent.

The inefficency of OLS is arguably a smaller problem than the inconsistency of the variance estimator. (ii) means that the estimated standard errors in our regression output are wrong, leading to the incorrect $t$-statistics and confidence intervals. Hypothesis testing, in general, is invalid. The problem arises because the formula that is the basis for estimating the standard errors in OLS (equation 5.7):

$$\text{Var}\left[b_1\right] = \frac{\sigma_\epsilon^2}{\sum X_i^2 - \frac{\left(\sum X_i\right)^2}{n}},$$

Figure 9.3: Heteroskedasticity in the CPS data. The variance in `wage` may be increasing as `education` increases.




is only correct under homoskedasticity.

To fix problem (i), the inefficiency of OLS, we must use a different estimator, such as Generalized Least Squares (GLS). GLS is not discussed here. To fix (ii), the more important problem of the inconsistency of the standard errors, the formula for $\text{Var}[b_1]$ must be updated to take into account the possibility of heteroskedasticity.

Updating the formula to allow for heteroskedasticity in the estimation of the standard errors gives what is typically referred to as *robust standard errors*.

### 9.2.2 Heteroskedasticity in the CPS data

It may be the case that the variance in wages depends on education. The reasoning is that individuals who have not completed highschool (or university) are precluded from many high-paying jobs (doctors, lawyers, etc.). However, having many years of education does not preclude individuals from low-paying jobs. The spread in wages is higher for highly educated individuals. Figure 9.3 illustrates this point.

If heteroskedasticity is present in the CPS data, it means that all the hypothesis testing that we have done so far used the wrong standard errors, and our conclusions may have been incorrect. For example, in the regression:

```
summary(lm(wage ~ education + gender + age + experience))
```

```
Coefficients:
```

```
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   -1.9574     6.8350  -0.286    0.775
education      1.3073     1.1201   1.167    0.244
genderfemale  -2.3442     0.3889  -6.028 3.12e-09 ***
age           -0.3675     1.1195  -0.328    0.743
experience     0.4811     1.1205   0.429    0.668
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.458 on 529 degrees of freedom
Multiple R-squared:  0.2533,    Adjusted R-squared:  0.2477
F-statistic: 44.86 on 4 and 529 DF,  p-value: < 2.2e-16
```

the standard errors, $t$-statistics, and associated $p$-values are all wrong under heteroskedasticity. To estimate the *robust* standard errors (which will update the $t$-statistics and $p$-values as well), we can use the following commands in R:

```
results <- lm(wage ~ age + education + gender + experience)
coeftest(results, vcov = vcovHC(results, "HC1"))
```

```
t test of coefficients:

             Estimate Std. Error t value  Pr(>|t|)
(Intercept)  -1.95744    1.53006 -1.2793  0.201345
age          -0.36749    0.12384 -2.9675  0.003138 **
education     1.30727    0.12452 10.4983 < 2.2e-16 ***
genderfemale -2.34416    0.39543 -5.9282  5.53e-09 ***
experience    0.48107    0.13502  3.5629  0.000400 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Notice that the estimated $\beta$s have not changed, but that the standard errors have changed quite dramatically, leading to very different conclusions about the statistical significance of the $X$ variables.

Heteroskedastic errors have a pretty severe consequence; hypothesis testing may be invalid. The prevalence of heteroskedasticity in many economics data has led to the common practice of erring on the side of caution. Heteroskedastic robust standard errors are often used, if heteroskedasticity is suspected. Note that homoskedasticity is a special case of heteroskedasticity, so the downside of using the robust estimator when it is not needed, is small.

## 9.3 Review Questions

1. Explain the difference between homoskedasticity and heteroskedasticity.
2. Provide an example of heteroskedasticity using data from another chapter.
3. Describe the problem that heteroskedasticity brings to OLS estimation.

4. Briefly explain how to fix the inconsistency of the standard errors in OLS estimation, in the presence of heteroskedasticity.

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