Different number of bidders in sequential auctions.

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Abstract

We analyze a second-price sequential auction for two heterogenous synergistic goods with local and global bidders. We prove that as the number of local bidders in the second auction approaches to infinity, the global bidder bids truthfully, and hence, the outcome is always efficient. However, as the number of local bidders in the first auction approaches to infinity, the global bidder does not bid truthfully, and the outcome is inefficient with a positive probability. These results arise from the fact that the number of local bidders in the first auction does not affect the global bidder's equilibrium bidding.

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1 Introduction

There are some auction settings in which global and local bidders bid for synergistic goods. Some examples are highway procurement auctions (De Silva (2005) and spectrum license auctions (Gunay and Meng (2017b)). In this paper, we analyze the effect of the number of local bidders on the efficiency of the sequential auctions -in the limit-.

The global bidder overbids his stand-alone valuation in the first auction due to the hope of winning the second auction and enjoying the synergy. The equilibrium bid is the highest price she is willing to pay for the first good that equates the expected payoff from winning and losing the first auction. This bid is not affected by the number of local bidders in the first auction as the equilibrium incentives are conditional on winning the first auction. However, the number of local bidders in the second auction affects the bid price as it affects the probability of winning the second auction and the synergy. As the number of local bidders in the second auction approaches to infinity, the global bidder bids truthfully in the limit, and the result is an efficient outcome. However, when the number of local bidders in the first auction approaches to infinity, the global bidder still overbids, and the outcome is inefficient with a positive probability.

In the auction literature with synergy, it has been assumed that the goods are either equivalent (Krishna and Rosenthal, 1996; Branco, 1997), or the second good becomes more valuable to the winner of the first good (Jeitschko and Wolfstetter, 2002; Leufkens et. al., 2010). In these papers, it is naturally assumed that the number of bidders are equal in each auctions. Krishna and Rosenthal (1996) studies the effect of varying the (finite) number of local bidders for simultaneous auctions. We study the effect of number of local bidders in the first and the second auction on efficiency in the limit unlike the aforementioned papers. We show that the number of local bidders in the first auction has no effect on the global bidder's equilibrium bidding in the first auction, which derives our efficiency/inefficiency results.

2 The Model

Consider two goods, A and B, that has zero value to the seller, who sells them with a second-price sequential auction. There is one risk-neutral global bidder, G.[1] If she wins both goods, she enjoys a synergy of $\theta > 0$.[2] There are also $N_i > 0$ risk neutral local bidders bidding for good $i = A, B$. $N_i + 1$ independent draws from the distribution function F_i determines the private valuation, v_{ki}, for each bidder, $k = G, 1, 2.., N_i$, and $i = A, B$. The distribution function F_i, has a twice differentiable density function $f_i > 0$ on the interval $(0, 1]$ with $f_i(0) \geq 0$.

In describing the model below, we closely follow Gunay and Meng, 2017a.

We use symmetric subgame perfect Bayesian equilibrium in weakly undominated strategies. It is well-known that local bidders bid their valuations truthfully in both auctions. The global bidder's second-auction equilibrium strategy is bidding the marginal valuation for the second good truthfully as this is the last stage of the game. That is, bidding $v_{Gj} + \theta$ if won good i in the first auction, and v_{Gj} otherwise, where $i, j = A, B$ and $i \neq j$.

To derive the global bidder's equilibrium strategy in the first auction for good i, we maximize her payoff given the sequential rationality. Let $p_i = max\{v_{ki}\}, k = 1, 2.., N_i$ denote the maximum valuation of local bidders for good $i = A, B$. This p_i is the price that the global bidder pays if he wins good i. The distribution function for p_i is $G_i(.) = [F_i(.)]^{N_i}$. The expected payoff for the global bidder when bidding p is

$$Max_p \int_0^p (v_{Gi} - p_i) dG_i(p_i) + G_i(p) \int_0^{min\{v_{Gj}+\theta,1\}} (v_{Gj} + \theta - p_j) dG_j(p_j)$$
$$+(1 - G_i(p)) \int_0^{v_{Gj}} (v_{Gj} - p_j) dG_j(p_j) \tag{1}$$

The first integral is the expected profit from winning i in the first auction, second is the expected profit from winning j after winning i, and the third is the expected profit from

[1] Multiple global bidders' strategies have not been defined in the literature yet when types are multi-dimensional for moderate synergy levels (see Meng and Gunay, 2017b and the references therein). Hence, assuming one global bidder is not uncommon in the literature for such cases.

[2] We assume θ to be a public information but it will be clear from the proofs that assuming that it is private has no effect on any of the results.

winning j after losing i. Note that, $Pr(p > p_i) = G_i(p)$, is the probability of the global bidder winning auction i.

Equation 2 is the first order condition, and gives the equilibrium bidding price, p_{ij}, when good i is auctioned first, and j second.

$$\frac{dG_i}{dp_{ij}} \underbrace{[(v_{Gi} - p_{ij}) + \int_0^{min\{v_{Gj}+\theta,1\}} (v_{Gj} + \theta - p_j) dG_j(p_j)]}_{\text{Expected profit from winning the first auction at } p_{ij}} = \frac{dG_i}{dp_{ij}} \underbrace{[\int_0^{v_{Gj}} (v_{Gj} - p_j) dG_j(p_j)]}_{\text{and losing the first auction}} \tag{2}$$

Note that the only term dependent on the number of bidders in the first auction, $\frac{dG_i}{dp_{ij}}$, cancels out from both sides. By using integration by parts and equation 2, we derive the global bidder's equilibrium bid.

Proposition 1 *The global bidder's first-auction equilibrium bid, p_{ij}, for good i is*

$$a)\ \textit{If } v_{Gj} + \theta < 1, \textit{then } \ p_{ij}(v_{Gi}, v_{Gj}, N_j) \ = \ v_{Gi} + \int_{v_{Gj}}^{v_{Gj}+\theta} G_j(p_j, N_j) dp_j$$

$$b)\ \textit{If } v_{Gj} + \theta \geq 1, \textit{then } \ p_{ij}(v_{Gi}, v_{Gj}, N_j) \ = \ v_{Gi} + (v_{Gj} + \theta - 1) + \int_{v_{Gj}}^{1} G_j(p_j, N_j) dp_j$$

Proof is in the Appendix. The global bidder's bid p_{ij} is the highest price he is willing to pay to win good i. In other words, the global bidder's equilibrium incentives are conditional on winning the first auction; hence, only the number of local bidders in the second auction matters as this affects the probability of winning the second good and the synergy. This is explained in the first part of the corollary below.

Corollary 2 *i) The number of local bidders in the first auction has no effect on the bidding price.*

ii) As the number of local bidders in the second auction approaches to infinity, the global bidder's bid is

$p_{ij} \to v_{Gi}$, *if* $v_{Gj} + \theta < 1$.

$p_{ij} \to v_{Gi} + v_{Gj} + \theta - 1$ *if* $v_{Gj} + \theta > 1$.

As the number of local bidders in the second auction, N_j, approaches to infinity, $G_j = F_j[.]^{N_j}$ approaches to zero, and hence the integrals in proposition 1 approaches to zero. When $v_{Gj} + \theta < 1$, the global bidder bids his valuation, v_{Gi} for the first good. As he knows that he cannot win the second good given that the maximum of local bidders valuation approaches to 1. When $v_{Gj} + \theta > 1$, the global bidder wins the second auction for sure if he wins the first good. Hence, he bids truthfully for the first good, $v_{Gi} + v_{Gj} + \theta - 1$, which is his total valuation minus the price of second good which is 1. In short, the global bidder bids truthfully when the number of local bidders in the second auction approaches to infinity.

The corollary implies that the ordering of auctions has implications for efficiency. When the number of local bidders in the first auction approaches to infinity the outcome might be inefficient as the global bidder does not bid truthfully. However, as the number of bidders in the second auction approaches to infinity, the outcome is always efficient, as all bidders bid truthfully.

Proposition 3 *i) As the number of local bidders in the first auction approaches to infinity, the outcome of the auction might be inefficient.*

ii) As the number of local bidders in the second auction approaches to infinity, the outcome of the sequential auction is efficient.

When the number of local bidders in the first auction approaches to infinity but finite in the second auction, the global bidder might win the first auction by bidding over 1 as there is a chance of winning the second auction. However, if the global bidder loses the second auction or wins with a high enough price, there is an ex-post loss. This is an example of an inefficient outcome and might happen with a positive probability. The proof is in the appendix.

3 Appendix

Proof of Proposition 1.

From equation 2, we have, $p_{ij} = v_{Gi} + \int_0^{min\{v_{Gj}+\theta,1\}} (v_{Gj} + \theta - p_j) dG_j(p_j, N_j) - \int_0^{v_{Gj}} (v_{Gj} -$

$p_j)dG_j(p_j, N_j)$ By using integration by parts, we find the global bidder's first-auction equilibrium bid, p_{ij}, for good i when $v_{Gj} + \theta < 1$ is,

$p_{ij} = v_{Gi} + \int_0^{v_{Gj}+\theta} (v_{Gj} + \theta - p_j)dG_j(p_j, N_j) - \int_0^{v_{Gj}} (v_{Gj} - p_j)dG_j(p_j, N_j)$
$= v_{Gi} + (v_{Gj} + \theta - p_j)G_j(p_j, N_j) \mid_0^{v_{Gj}+\theta} - \int_0^{v_{Gj}+\theta} G_j(p_j, N_j)d(v_{Gj} + \theta - p_j)$
$- (v_{Gj} - p_j)G_j(p_j, N_j) \mid_0^{v_{Gj}} + \int_0^{v_{Gj}} G_j(p_j, N_j)d(v_{Gj} - p_j)$
$= v_{Gi} + \int_0^{v_{Gj}+\theta} G_j(p_j, N_j)dp_j - \int_0^{v_{Gj}} G_j(p_j, N_j)dp_j = v_{Gi} + \int_{v_{Gj}}^{v_{Gj}+\theta} G_j(p_j, N_j)dp_j$

And the global bidder's first-auction equilibrium bid, p_{ij}, for good i when $v_{Gj} + \theta > 1$ is,

$p_{ij} = v_{Gi} + \int_0^{1} (v_{Gj} + \theta - p_j)dG_j(p_j, N_j) - \int_0^{v_{Gj}} (v_{Gj} - p_j)dG_j(p_j, N_j)$
$= v_{Gi} + (v_{Gj} + \theta - p_j)G_j(p_j, N_j) \mid_0^{1} - \int_0^{v_{Gj}+\theta} G_j(p_j, N_j)d(v_{Gj} + \theta - p_j)$
$- (v_{Gj} - p_j)G_j(p_j, N_j) \mid_0^{v_{Gj}} + \int_0^{v_{Gj}} G_j(p_j, N_j)d(v_{Gj} - p_j)$
$= v_{Gi} + (v_{Gj} + \theta - 1) + \int_0^{1} G_j(p_j, N_j)dp_j - \int_0^{v_{Gj}} G_j(p_j, N_j)dp_j = v_{Gi} + (v_{Gj} + \theta - 1) + \int_{v_{Gj}}^{1} G_j(p_j, N_j)dp_j$

Given the FOC equation 2, and $G_i(.) = [F_i(.)]^{N_i}$, we show that the SOC is satisfied. Since the FOC is met, then

$$\frac{dG_i}{dp_{ij}} \underbrace{[(v_{Gi} - p_{ij}) + \int_0^{min\{v_{Gj}+\theta,1\}} (v_{Gj} + \theta - p_j)dG_j(p_j)] - [\int_0^{v_{Gj}} (v_{Gj} - p_j)dG_j(p_j)]}_{\text{Equal to zero}} = 0$$

Taking the derivative of the FOC with respect to dp_{ij} and the SOC, calculated at the optimum p_{ij} is,

$$\frac{d^2G_i}{dp_{ij}^2} \underbrace{[(v_{Gi} - p_{ij}) + \int_0^{min\{v_{Gj}+\theta,1\}} (v_{Gj} + \theta - p_j)dG_j(p_j)] - [\int_0^{v_{Gj}} (v_{Gj} - p_j)dG_j(p_j)]}_{\text{Equal to zero at } p_{ij}} - \frac{dG_i}{dp_{ij}} =$$

$-\frac{dG_i(p_{ij},N_i)}{dp_{ij}} = -\frac{dF_i(p_{ij},N_i)^{N_i}}{dp_{ij}} = N_i F_i(p_{ij}, N_i)^{N_i-1} f_i(p_{ij}, N_i) \leq 0$ since N_i is finite, f is positive by assumption except at zero and F is positive except at zero. Since there is a unique p_{ij} satisfying the FOC and SOC, we get our maximizer. Note that the corner solution of bidding zero cannot be the solution as the global bidder will always lose. This ends the proof.

Proof of Proposition 3. i) We show that the global bidder, with a positive probability, might win only the first license with a loss. This is one of the inefficient outcomes.[3] This will be sufficient to prove that the outcome might be inefficient.

[3]There are other inefficient outcomes (see Meng and Gunay 2017a) but only showing this case is sufficient to prove that the outcome might be inefficient

By Proposition 1, the global bidder's bid even when N_i approaches to infinity is

$$p_{ij} \;=\; v_{Gi} + \int_{v_{Gj}}^{v_{Gj}+\theta} G_j(p_j, N_j)dp_j \quad \text{if} \quad v_{Gj} + \theta < 1$$

As the global bidder is bidding over his stand alone valuation, he can win the first good with a (potential) loss if

$$\underbrace{Pr(v_{Gj} < 1-\theta)Pr(v_{Gi} < p_i < v_{Gi} + \int_{v_{Gj}}^{v_{Gj}+\theta} G_j(p_j, N_j)dp_j)}_{\textbf{Probability of winning the first license over the stand-alone value when } v_{Gj}+\theta<1} \tag{3}$$

Note that the above has a positive probability as we assume $v_{Gj} + \theta < 1$ and as the integral is positive as G_j is continuous and N_j is finite. The global bidder loses the second good if $Pr(v_{Gj} + \theta < p_j)$, as p_j is the maximum of the local bidders. This also has a positive probability as $v_{Gj} + \theta < 1$ and N_j is finite. The probability of winning the first good with a loss is the product of equation 3 and $Pr(v_{Gj}+\theta < p_j)$. Hence, the probability of an inefficient outcome is positive even if the number of local bidders in the first auction approaches to infinity.

ii)All bidders bid truthfully in this case by our corollary and the explanations in the text. When all bidders bid truthfully, the outcome is efficient.

The global bidder bids $p_{AB} \to v_{GA}$, when $v_{GB} + \theta < 1$. This is truthful bidding since the global bidder should win only A in this case if $v_{LA} < v_{GA}$ and lose B as the maximum bid is 1 by a local bidder. This ends up in an efficient outcome. The global bidder loses both licenses if $v_{LA} > v_{GA}$, which is the efficient outcome.

The global bidder bids $p_{AB} \to v_{GA} + v_{GB} + \theta - 1$ when $v_{GB} + \theta > 1$. The global bidder wins both license A and B if $v_{GA} + v_{GB} + \theta - 1 > v_{LA}$. But the inequality can be written as $v_{GA} + v_{GB} + \theta > 1 + v_{LA}$ which shows that the outcome is efficient. Conversely, the global bidder loses license A (and B) if $v_{GA} + v_{GB} + \theta - 1 < v_{LA}$. Re-writing the last inequality, $v_{GA} + v_{GB} + \theta < 1 + v_{LA}$. shows that the outcome is efficent.

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