EXPOSURE PROBLEM IN MULTI-UNIT AUCTIONS*

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Abstract

We characterize the optimal bidding of local and global bidders for two licenses in a multi-unit simultaneous ascending auction. The global bidders want to win both licenses to enjoy synergies. This gives them incentive to bid aggressively in the sense that they bid more than their stand alone valuation of a license. However, this exposes them to the risk of losing money, since they may win only one license. The existing literature assumes large synergies or equal stand alone valuations which guarantees that one global bidder will win all licenses. In this paper, we remove these assumptions and characterize the optimal bidding in the presence of exposure problem. We show that a global bidder may make a loss even if it wins all licenses.

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1 Introduction

In a typical American or Canadian spectrum license auction, hundreds of licenses are sold simultaneously. Each of these licences gives the spectrum usage right of a geographical area to the winning firm. In these auctions, some *'global'* firms are interested in winning all licenses in order to serve nationwide while some other *'local'* firms are interested in winning only specific licenses in order to serve in local markets.[1] In a model simplifying the American and the recent Canadian spectrum license auctions, we derive the optimal bidding of local and global firms in a simultaneous ascending auction. We mainly focus on how the global bidder may make a loss even if it wins all licenses!

In our model, two licenses are auctioned off to two type of bidders. Local bidders want to win only one of the licenses, while global bidders want to win both. Each global bidder has a stand alone valuation for each license. If she can win both, the global bidder enjoys a synergy on the top of her stand alone valuations. This gives the global bidder an incentive to bid aggressively in the sense that she bids more than her stand alone valuation. However, this kind of bidding exposes her to the risk of losing money since she may win only one license. This is known as the exposure problem.

Multi-unit auction papers such as Albano et. al. (2006), Kagel and Levin (2005), Rosenthal and Wang (1996), and Krishna and Rosenthal (1996) assume that global bidders have either very large synergies or equal stand alone valuations for each license. Such assumptions guarantee that one global

[1]In the recent Advanced Wireless Spectrum auction, firms such as Globalive and Rogers were interested in all licenses whereas firms such as Bragg Communication and Manitoba Telecom Services (MTS) were interested in East Coast and Manitoba licenses, respectively.

bidder wins all licenses and makes positive profits. In other words, these papers acknowledge the possibility of exposure problem but avoid analyzing it. In contrast, we allow for moderate synergies in which the global bidders should take exposure problem into account while bidding. In our model, exposure problem may arise ex-post.

Cramton et. al. (2006), writes that *"[exposure problem is] the problem of winning some-but not all-of a complementary collection of items in an auction without package bids. The bidder is 'exposed' to a possible loss if his bids include synergistic gains that might not be achieved."* Since the global bidder bids aggressively in the first auction, it may win the first license at a price that it makes a loss. If it does not win the second license, then it will end up with a loss (due to synergistic gains not being achieved). This is the exposure problem defined in Cramton et. al. (2006). However, we show that this bidder may find itself at a loss minimizing situation in the second license auction. Specifically, the bidder should agree to pay a high enough price for the second license such that its loss will be lower than the loss of winning only one license.

To give an example, assume that a global bidder values the first license at a price of \$5 and the second license at a price of \$10. It will enjoy a synergy of \$4 if it wins all licenses. Moreover assume that the global bidder finds that it is optimal to stay in the first license auction until the price reaches to \$8. Then it is possible that it may win the first license at a price of \$7 (a loss of \$2, if it ends up losing the second license). Then it should stay in the auction for the second license until the price reaches \$14. For example, it may win the second license at a price of \$13. In this example, the global bidder wins all licenses but makes a loss of \$1 ($loss = 5 + 10 + 4 - 7 - 13$).

This paper is organized as follows. We describe the two-license model and show our perfect Bayesian equilibrium for a special case of one global and two local bidders. Then, we analyze a more general case. All proofs are included in the Appendix.

2 The Two-License Model

There are 2 licenses, license A and B for sale. There are N global bidders who demand both licenses and M_j local bidders who demand only license $j = A, B$. Both local bidders and global bidders have a private stand alone valuation for a single license, v_{ij}, where i and j represent the bidder and the license, respectively. The valuations v_{ij} are drawn from the continuous distribution function $F(v_{ij})$ with support on $[0, 1]$ and probability density function $f(v_{ij})$. The type of bidders, global or local, is publicly known.

We assume that there are (heterogenous) positive synergies for global bidders, and denote this kind of synergies by $\alpha_i > 0$, $i = 1, 2, .., N$. Then, the global bidder i's total valuation, given that it wins two licenses is, $V_i = v_{iA} + v_{iB} + \alpha_i$. The valuation to a global bidder i who receives only one license j or the valuation of local bidder i who receives license j is $V_i = v_{ij}$.

We consider the case where the licenses auctioned off simultaneously through an ascending multi-unit auction. The auction proceeds in rounds. Prices start from zero for all licenses and increase simultaneously by a (very small) pre-determined increment. When only one bidder is left on a given license, that bidder wins that license at the price that the last bidder drops. At the same time, this price on the remaining licenses will continue to increase, if there are more than one bidder. Let us note that our equilibrium is still valid even if we have simultaneous closing.

The dropout is irreversible; once a bidder drops out of bidding for a given license, he cannot bid for this license again in the next round. The number of active bidders and the drop-out prices are publicly known.[2]

2.1 A Special Case: One single global bidder

We first start with a special case in which there is a single global bidder, called Firm 1, and two local bidders, called Firm 2 and Firm 3. The global bidder, Firm 1, is interested in both licenses A and B. Firm 1's total valuation of two licenses is given by $V_1 = v_{1A} + v_{1B} + \alpha_1$. His stand-alone valuation of license A or B is given by v_{1A} or v_{1B}, and we assume that $v_{1A} > v_{1B}$.[3] Firm 2 is only interested in license A and Firm 3 is only interested in license B. The stand-alone valuations are drawn from the uniform distribution function F with support on $[0, 1]$ and probability density function f.

We assume that the beginning prices and the price increments for both licenses are the same. We will show our symmetric perfect Bayesian equilibrium with the help of lemmas that follow. First, we describe the equilibrium strategy of the local bidder.

Lemma 1 *: Each local bidder has a weakly dominant strategy to stay in the auction until the price reaches his stand alone valuation.*

This is a weakly dominant strategy for a local bidder. If the local bidder

[2] To simplify the analysis, we also assume that there is no budget constraint for bidders.

[3] The global bidder will value Toronto license (license A) more than the Winnipeg license (license B), for example. We assume that two independent draws are made and the higher amount will be the valuation for license A.

drops out without winning the license, he earns zero profits. But if he stays in up to his valuation, he may win the license and hence, his expected profits are positive. Clearly, staying in the auction when the price is above the stand alone valuation will give negative expected profits. So it is optimal for the local bidder to be active on a given license until the price reaches his stand alone valuation for the given license.

Lemma 2 *: The global bidder stays in both license auctions at least until the price reaches the minimum of his/her stand alone valuations when his average valuation is no more than 1. Otherwise, it is optimal to stay in until the price reaches his average valuation.*

Proof. See the Appendix.

The result comes from comparing the expected profits of dropping out or not before the minimum of stand alone valuation. If a global bidder drops out before the minimum of its stand alone valuation, it loses the possibility of winning both licenses and enjoying the synergy.

When his average valuation exceeds 1; that is, the synergy is large enough, the global bidder will bid up to his average valuation where the global bidders shut out the local bidder since local bidder's stand alone valuation can be at most 1. Not surprisingly, only the global bidder stays in the subsequent rounds and the global bidder's strategy is equivalent to that in the single-unit auction. The global bidder with the higher average valuation $\frac{V_1}{2} > 1$ will win both licenses.

In order to make the analysis simple, we first give a special case. Assume that the global bidder value license A more than license B, that is, $v_{1A} \geq v_{1B}$.

We denote by p_1^* the optimal drop-out price for license B of Firm 1. We need to describe the two strategies at the optimal drop-out price: **Case 1**: at p_1^*, Firm 1 will drop out of the auction for license B without winning it and continue to stay in the auction for license A until v_{1A}. **Case 2**: at p_1^*, Firm 1 will win license B at the price equal to p_1^* and then continue to stay in the auction for license A until $v_{1A} + \alpha_1$. According to Lemma 2, $p_1^* \geq v_{1B}$.

By making the global bidder, Firm 1, indifferent between playing Case 1 and Case 2, we can find the optimal drop-out price p_1^*, in the case of Firm 1.

We denote the expected profit of Firm 1 when playing Case 1 by $E\Pi_1^1$ and the expected profit of Firm 1 when playing Case 2 by $E\Pi_1^2$, respectively. The superscript represents which strategy Firm 1 chooses to play and the subscript represents the global bidder, Firm 1. And p_1^* denotes the Firm 1's optimal drop-out price of license B. That is,

$$E\Pi_1^1 = \int_{p_1^*}^{v_{1A}} (v_{1A} - v_{2A}) f(v_{2A}|p_1^*) dv_{2A} \tag{1}$$

$$E\Pi_1^2 = \int_{p_1^*}^{v_{1A}+\alpha_1} (V_1 - p_1^* - v_{2A}) f(v_{2A}|p_1^*) dv_{2A} + \int_{v_{1A}+\alpha_1}^{1} (v_{1B} - p_1^*) f(v_{2A}|p_1^*) dv_{2A} \tag{2}$$

After the global bidder drops out of the auction for license B at p_1^*, he will continue to stay in the auction for license A until v_{1A}. If he wins, he will pay v_{2A}. In order to calculate his expected profit, he will be using $f(v_{2A}|p_1^*)$ which is the density of the local bidder's valuation for license A given p_1^*. This is the explanation of equation 1.

The first term of $E\Pi_1^2$ is Firm 1's expected profit of winning two licenses. If Firm 2's valuation v_{2A} is less than or equal to Firm 1's willingness to pay,

$v_{1A} + \alpha_1$, then Firm 1 wins license A and pays the price equal to v_{2A}. The second term of $E\Pi_1^2$ is Firm 1's expected profit of winning license B only which can happen only if If Firm 2's valuation $v_{2A} > v_{1A} + \alpha_1$. Note that the second term can be negative (which is the exposure problem).

By equating equation 1 and 2 and assuming that $f(.)$ is a uniform distribution, the optimal drop-out prices will be given in lemma 3,

Lemma 3

$$p_1^* = \begin{cases} \frac{1}{2}\{v_{1B} + \alpha_1 + 1 - (v_{1B}^2 + 1 - 2v_{1B} - \alpha_1^2 + 2v_{1B}\alpha_1 + 2\alpha_1 - 4v_{1A}\alpha_1)^{\frac{1}{2}}\}, \\ \qquad \textit{if } 0 \le v_{1A} \le 1 - \alpha_1 \textit{ and } 2(1 - v_{1A})(v_{1A} - v_{1B}) \ge \alpha_1^2; \\ \frac{1}{3}\{v_{1A} + v_{1B} + \alpha_1 + 1 - ((v_{1A} + v_{1B} + \alpha_1 + 1)^2 - 3(v_{1A} + \alpha_1)^2 - 6v_{1B})^{\frac{1}{2}}\}, \\ \qquad \textit{if } 0 \le v_{1A} \le 1 - \alpha_1 \textit{ and } 2(1 - v_{1A})(v_{1A} - v_{1B}) \le \alpha_1^2; \\ \frac{1}{2}\{v_{1B} + \alpha_1 + 1 - (v_{1B}^2 + 2v_{1A}^2 + \alpha_1^2 + 3 - 2v_{1B} - 2\alpha_1 + 2v_{1B}\alpha_1 - 4v_{1A})^{\frac{1}{2}}\}, \\ \qquad \textit{if } 1 - \alpha_1 \le v_{1A} \le 1. \end{cases} \tag{3}$$

In Lemma 3 below, we characterize how the global bidder will stay in the auctions for the two licenses and prove that they will drop out of the auction for one of licenses at the optimal drop-out price.

Lemma 4 *:*

A)Firm 1 will stay in the auctions for license A and license B until the optimal drop-out price p_1^*.

B)Firm 1 will drop out of the auction for its lower valuation license B at p_1^* *and continue to stay in the auction for license A until its stand-alone valuation* v_{1A}.

C) If Firm 1 wins license B at p_1^**, then it will continue to stay in the auction for license A until its stand-alone valuation* $v_{1A} + \alpha_1$.

Proof. See the Appendix.

In part c of lemma 4, we characterize what will happen if Firm 3 drops out before p_1^*. Here, without loss of generality, we assume that $v_{3B}^* = Min\{p_1^*, v_{3B}\}$. As price p increases, Firm 1 will choose to continue only if $v_{1A}+v_{1B}+\alpha_1-v_{3B}-p$ is greater than the drop-out payoff which is $v_{1B}-v_{3B}$. In other words, the firm will continue until price p becomes $v_{1A}+\alpha_1=p$.

We are ready to summarize our Perfect Bayesian equilibrium.

Proposition 5 *(Perfect Bayesian Equilibrium)*

Lemma 1,2, and 3 constitute a Perfect Bayesian Nash Equilibrium in two-license case with two local bidders and single global bidder.[4]

At the beginning of the game, each firm calculates its optimal drop-out price p_i^*. When the price reaches the minimum of these prices, one firm drops out of license B auction. If, for example, Firm 3 dropped out before Firm 1 at price v_{3B}, Firm 1, will continue to stay in the auction for license A until the price reaches $v_{1A}+\alpha_1$. In equilibrium, it is optimal for a global bidder to stay in the auctions for both licenses up to his optimal drop-out price when his average valuation is below 1 or his average valuation when his average valuation exceeds 1.

Now we can discuss the exposure problem. If the global bidder drops out of the auction on one of two licenses without winning it, he will continue to stay in the auction on the other license until the price reaches his/her new stand alone valuation of that license. If he wins one license, take license B as an example, at the price p (a price above the stand alone valuation), he

[4]Out-of-equilibrium-path beliefs: if the global bidder, Firm 1, drops out of license A before license B then the local bidder, Firm 3 believes that Firm l will act like a local bidder and bid at most 1 on license A.

will continue to bid for the remaining license A until the price reaches his new valuation of license A, i.e., $v_{1A} + \alpha_1$. But, if he cannot win the other license, given that he wins the first one at the price higher than his stand-alone valuation for it, then this global firm will take a risk of earning negative expected profit. This is one type of exposure problem. If he wins the other license, given that he wins the first one at the price higher than his stand-alone valuation, then this global firm also may take a risk of earning negative expected profit when the price of license A is higher than $v_{1A} + v_{1B} + \alpha_1 - p$. In other words, the second type of exposure problem will occur. So we try to find the optimal drop out prices for the global bidder to alleviate the expected loss caused by the exposure problem.

Corollary 6 : *Firm 1's drop-out price p_1^* will increase as α_1, v_{1B} and/or v_{1A} increases.*

As the total valuation increases, the drop out price increases.

2.2 A General Case

In this section, we assume that there are two global bidders and $2m$ local bidders. Half of the local bidders are interested in receiving license A and the other half are interested in receiving license B. Their valuations are given as v_{ji} where $j = 3, 4, .., m+2$ denote the local bidder firms and $i = A, B$ denote the license they are interested in. We will use firm 1 and firm 2 for the global bidders. We still keep the assumption that, for global bidders, $v_{1A} \geq v_{1B}$ and $v_{2A} \geq v_{2B}$. We also look for the moderate synergy cases in which $0 < \alpha_i < 1$.

We assume that the synergies are common knowledge.[5]

Since the local bidders will bid until their valuation, we concentrate on finding the global bidders' optimal strategy. As in the previous section, we have to compare their expected profit, $E\Pi_1^1$, in Case 1 (dropping out without winning license B) and the expected profit, $E\Pi_1^2$, in Case 2 (winning license B). The equations below show these expected profits. In the equations, p_A denotes (to be determined) price of license A, and $g(p_A|p_1^*)$ denotes the density function of p_A when the drop out price of license B is p_1^*.

$$E\Pi_1^1 = \int_{p_1^*}^{v_{1A}} (v_{1A} - p_A) g(p_A|p_1^*) d(p_A) \tag{4}$$

$$E\Pi_1^2 = \int_{p_1^*}^{v_{1A}+\alpha_1} (V_1 - p_1^* - p_A) g(p_A|p_1^*) dp_A + \int_{v_{1A}+\alpha_1}^{1} (v_{1B} - p_1^*) g(p_A|p_1^*) dp_A \tag{5}$$

We have $p_A = \max\{B_2^A, v_{3A}, ..., v_{(2+m)A}\}$, where B_2^A represents Firm 2's (i.e., the other global bidder's) valuation of license A. If firm 2 cannot win license B, then $B_2^A = v_{2A}$. If it wins license B, then $B_2^A = v_{2A} + \alpha_2$.

If $B_2^A = v_{2A}$, the distribution function $G(p_A|p_1^*) = (F(p_A|p_1^*))^{m+1} = (\frac{p_A - p_1^*}{1-p_1^*})^{m+1}$ and the density function $g(p_A|p_1^*) = (m+1)(F(p_A|p_1^*)^m f(p_A|p_1^*) = (\frac{m+1}{1-p_1^*})(\frac{p_A-p_1^*}{1-p_1^*})^m$ since v_{2A} is uniformly distributed on $[0,1]$.

If $B_2^A = v_{2A} + \alpha_2$, then B_2^A has the uniform density functions on the interval $[\alpha_2, 1+\alpha_2]$. In this case, the corresponding density and distribution functions will be:

[5] The case in which $\alpha_i > 1$ is analyzed by Albano et. al (2006). They also assume the same synergy for each bidder. In other words, they assume common knowledge synergies like us.

$$G(p_A|p_1^*) = \begin{cases} 0, & \text{if } p_1^* \le p_A < \alpha_2; \\ (\frac{p_A-\alpha_2}{1-\alpha_2})^{m+1}, & \text{if } \alpha_2 \le p_A \le 1; \\ \frac{p_A-1}{\alpha_2}, & \text{if } 1 < p_A \le \alpha_2+1; \\ 1, & \text{if } p_A \ge \alpha_2+1. \end{cases} \tag{6}$$

$$g(p_A|p_1^*) = \begin{cases} \frac{m+1}{1-\alpha_2}(\frac{p_A-\alpha_2}{1-\alpha_2})^{m}, & \text{if } \alpha_2 \le p_A \le 1; \\ \frac{1}{\alpha_2}, & \text{if } 1 < p_A \le \alpha_2+1; \\ 0, & \text{Otherwise.} \end{cases} \tag{7}$$

First, we calculate the optimal drop-out price from the following equation,

$$E\Pi_1^1 = E\Pi_1^2 \tag{8}$$

At the beginning of the auction, the global bidder 1 should take into account the possibility that the other bidder may win license B. It will use the following equation to determine its optimal drop out price as long as the other global bidder does not drop out from license B auction. Note that the optimal drop out price is revised as other local bidders drop out.

$$\int_{p_1^*}^{v_{1A}} (v_{1A} - p_A)(\frac{m+1}{1-\alpha_2})(\frac{p_A-\alpha_2}{1-\alpha_2})^m dp_A = \int_{p_1^*}^{v_{1A}+\alpha_1} (V_1 - p_1^* - p_A)(\frac{m+1}{1-p_1^*})(\frac{p_A-p_1^*}{1-p_1^*})^m dp_A$$
$$+ \int_{v_{1A}+\alpha_1}^{1} (v_{1B} - p_1^*)(\frac{m+1}{1-p_1^*})(\frac{p_A-p_1^*}{1-p_1^*})^m dp_A \tag{9}$$

Once the other global bidder drops out of license B auction, global bidder 1 will use the following equation to calculate its drop out price from license B.

$$\int_{p_1^*}^{v_{1A}} (v_{1A} - p_A)(\frac{m+1}{1-p_1^*})(\frac{p_A - p_1^*}{1-p_1^*})^m dp_A = \int_{p_1^*}^{v_{1A}+\alpha_1} (V_1 - p_1^* - p_A)(\frac{m+1}{1-p_1^*})(\frac{p_A - p_1^*}{1-p_1^*})^m dp_A$$
$$+ \int_{v_{1A}+\alpha_1}^{1} (v_{1B} - p_1^*)(\frac{m+1}{1-p_1^*})(\frac{p_A - p_1^*}{1-p_1^*})^m dp_A \qquad (10)$$

Lemma 7 *:*

A) Given the other global bidder is active in the license B auction, Firm 1's optimal drop out price will be calculated by using equation 9.

B) Given the other global bidder drops out of license B, Firm 1's optimal drop out price will be calculated by using equation 10.

C) If Firm 1 drops out of the auction for license B, it will continue to stay in the auction for license A until the price reaches v_{1A}.

If the other global bidder is still active, there is a chance that it may win license B and bid up to $v_{2A} + \alpha_2$. Therefore, the global bidders should use Equation 9 that takes this case into account. If the other global bidder drops out of license B auction, then the other global bidder will bid up to v_{2A}. Therefore, the global bidders should use equation 10 that analyzes this case.

Proposition 8 *(Perfect Bayesian Nash Equilibrium)*

a) Out-of-equilibrium-path beliefs: If a global bidder drops out of license A before license B at the price p, all other bidders will believe that this global bidder's valuation of B, v_{1B}*, is uniformly distributed on* $[0, 1]$.

b) Lemma 1,2,7 and the out of equilibrium path beliefs constitute a Perfect Bayesian Nash Equilibrium.

At the beginning of the game, each firm calculates its optimal drop-out price p_i^*. When the price reaches the minimum of these prices, one firm drops out of license B auction. The rest will update their optimal drop out prices p_i^* and they will continue to stay in the auction for both licenses until their new p_i^*. Note that a global bidder will drop out of the auction for one license with lower valuation at his optimal drop-out price or his average valuation in order to win the other higher valuation license. In equilibrium, it is optimal for a global bidder to stay in the auctions for both licenses up to his optimal drop-out price when his average valuation is below 1 or his average valuation when his average valuation exceeds 1. When he wins license B before or at his optimal drop-out price p_i^* and then continues to stay in the auction for license A until his higher valuation for license A, $v_{iA} + \alpha_i$.

3 Conclusion and Discussion

We show the optimal strategies of global bidders when there is moderate synergies. We analyze the cases in which the global bidder may win all licenses but make a loss. Especially, this part of exposure problem has not been studied in the literature previously.

4 Appendix

Proof of Lemma 2: Let p be the drop-out price before the global bidder's lowest stand-alone value v_{1B}, that is, $p < v_{1B}$. When the global bidder drops out of bidding for the single license at p without winning it, then he will keep bidding for license A until the price reaches v_{1A}. The expected profits in this case are given by,

$$E\Pi_1(p) = \int_p^{v_{1A}} (v_{1A} - p_A) g(p_A|p) dp_A \tag{11}$$

When the global bidder keeps bidding and drops out of bidding for the license B until v_{1B}, then the expected profits are given by,

$$\begin{aligned} E\Pi_1(V_{1B}) = & \int_p^{v_{1A}+\alpha_1} \int_p^{v_{1B}} (V_1 - p_A - p_B) g(p_B|p) g(p_A|p) dp_B dp_A \\ & + \int_{v_{1A}+\alpha_1}^{1} \int_p^{v_{1B}} (v_{1B} - p_B) g(p_B|p) g(p_A|p) dp_B dp_A \\ & + \int_p^{v_{1A}} \int_{v_{1B}}^{1} (v_{1A} - p_A) g(p_B|p) g(p_A|p) dp_B dp_A \end{aligned} \tag{12}$$

Where p_A and p_B denote the prices of the given license A and B respectively, and are defined by $p_A = \max\{B_2^A, B_3^A\}$, and $p_B = B_2^B$, where B_i^j denote the bidder i's bid for license j. $g(p_A|p)$ denotes the probability density function of the highest bid for license A between global bidder 2 and local bidders given the current price p. Moreover, we assume that B_i^j is independently distributed on $[0,1]$ with the distribution function $F(B_i^j)$ and the corresponding density function $f(B_i^j)$. $g(p_j|p)$ denotes the density of the highest bid for license A among two global bidders and m local bidders' bids equal to p_j given that the current price is p. Moreover, $g(p_j|p) = (m+1)(F(p_j|p))^m f(p_j|p) = (\frac{m+1}{1-p})(\frac{p_j - p}{1-p})^m$. In particular, let $p_B \leq v_{1B}$ when the global bidder wins license B. Then,

$E\Pi_1(V_{1B}) = \int_p^{v_{1A}+\alpha_1} \int_p^{v_{1B}} (V_1 - p_A - p_B) g(p_B|p) g(p_A|p) dp_B dp_A$
$+ \int_{v_{1A}+\alpha_1}^{1} \int_p^{v_{1B}} (v_{1B} - p_B) g(p_B|p) g(p_A|p) dp_B dp_A + \int_p^{v_{1A}} \int_{v_{1B}}^{1} (v_{1A} - p_A) g(p_B|p) g(p_A|p) dp_B dp_A$
$\geq \int_p^{v_{1A}} \int_p^{v_{1B}} (v_{1A} - p_A) g(p_B|p) g(p_A|p) dp_B dp_A + \int_p^{v_{1A}} \int_{v_{1B}}^{1} (v_{1A} - p_A) g(p_B|p) g(p_A|p) dp_B dp_A$

$+\int_{v_{1A}}^{1}\int_{p}^{v_{1B}}(v_{1B}-p_B)g(p_B|p)g(p_A|p)dp_Bdp_A+\int_{p}^{v_{1A}}\int_{p}^{v_{1B}}(v_{1B}-p_B)g(p_B|p)g(p_A|p)dp_Bdp_A$
$=\int_{p}^{v_{1A}}\int_{p}^{1}(v_{1A}-p_A)g(p_B|p)g(p_A|p)dp_Bdp_A+\int_{p}^{1}\int_{p}^{v_{1B}}(v_{1B}-p_B)g(p_B|p)g(p_A|p)dp_Bdp_A$
$\geq\int_{p}^{v_{1A}}\int_{p}^{1}(v_{1A}-p_A)\frac{(m+1)(p_B-p)^m}{(1-p)^{m+1}}g(p_A|p)dp_Bdp_A$
$=\int_{p}^{v_{1A}}(v_{1A}-p_A)g(p_A|p)dp_A=E\Pi_1(p)$

So we can conclude that,

$$E\Pi_1(V_{1B})\geq E\Pi_1(p) \tag{13}$$

Thus, it is optimal for the global bidder to stay in both auctions until his lowest stand-alone valuation for a single license. Therefore, the expected profits from continuing to stay in the auction at least up to $\min\{v_{1A},v_{1B}\}$ are greater than or equal to that from dropping out before $\min\{v_{1A},v_{1B}\}$. So the global bidder prefers to stay in the auctions until the price reaches $\min\{v_{1A},v_{1B}\}$.

When $\frac{V_1}{2}>1$, the global bidder, Firm 1, can stay in the auction until $\frac{V_1}{2}>1$ to shut out all the local bidders and then competes with the other global bidder only in the following rounds. In this case, Firm 1's behavior is similar to a local bidder's bidding strategy. According to Lemma 1, it is optimal for the global bidder to bid until his average valuation to win both or none.

■

Proof of Lemma 3: When $0\leq v_{1A}\leq 1-\alpha_1$ and $2(1-v_{1A})(v_{1A}-v_{1B})\geq\alpha_1^2$, which means that $v_{1B}<p_1^*\leq v_{1A}$, by solving $E\Pi_1^1=E\Pi_1^2$, we have $2(p_1^*)^2-2p_1^*(1+v_{1B}+\alpha_1)+\alpha_1^2+2v_{1A}\alpha_1+2v_{1B}=0$, and the two solutions to this equation are as follows,

$$p_1^* = \frac{1}{2}\{v_{1B} + \alpha_1 + 1 - (v_{1B}^2 + 1 - 2v_{1B} - \alpha_1^2 + 2v_{1B}\alpha_1 + 2\alpha_1 - 4v_{1A}\alpha_1)^{\frac{1}{2}}\} \tag{14}$$

$$p_+^* = \frac{1}{2}\{v_{1B} + \alpha_1 + 1 + (v_{1B}^2 + 1 - 2v_{1B} - \alpha_1^2 + 2v_{1B}\alpha_1 + 2\alpha_1 - 4v_{1A}\alpha_1)^{\frac{1}{2}}\} \tag{15}$$

The optimal drop-out price cannot exceed the global bidder's average valuation. However, in equation 15, we have $p_+^* \geq \frac{V_1}{2}$ since $v_{1B} + \alpha_1 + 1 +$ *some positive constant* is greater than $V_1 = v_{1B} + \alpha_1 + v_{1A}$. Note that v_{1A} can be at most 1. Hence, we rule out this root. Later, we will show that the other roots exist.

When $0 \leq v_{1A} \leq 1 - \alpha_1$ and $2(1 - v_{1A})(v_{1A} - v_{1B}) \leq \alpha_1^2$, which means that $v_{1B} < p_1^*$ and $v_{1A} < p_1^*$, by solving,

$0 = E\Pi_1^1 = E\Pi_1^2 = \int_{p_1^*}^{v_{1A}+\alpha_1}(V_1 - p_1^* - v_{2A})f(v_{2A}|p_1^*)dv_{2A} + \int_{v_{1A}+\alpha_1}^{1}(v_{1B} - p_1^*)f(v_{2A}|p_1^*)dv_{2A}$,

we have $3(p_1^*)^2 - 2p_1^*(1 + v_{1A} + v_{1B} + \alpha_1) + (v_{1A} + \alpha_1)^2 + 2v_{1B} = 0$, and the two solutions to this equation but the root that is not greater than $\frac{V_1}{2}$ is as follows,

$$p_{1(2)}^* = \frac{1}{3}\{v_{1A} + v_{1B} + \alpha_1 + 1 - ((v_{1A} + v_{1B} + \alpha_1 + 1)^2 - 3(v_{1A} + \alpha_1)^2 - 6v_{1B})^{\frac{1}{2}}\} \tag{16}$$

Finally, when $1 - \alpha_1 \leq v_{1A} \leq 1$, $p_1^* < v_{1A}$, by solving equations:

$E\Pi_1^1 = E\Pi_1^2$

$\Leftrightarrow \int_{p_1^*}^{v_{1A}}(v_{1A} - v_{2A})f(v_{2A}|p_1^*)dv_{2A} = \int_{p_1^*}^{1}(v_{1A} + \alpha_1 - v_{2A})f(v_{2A}|p_1^*)dv_{2A} + (v_{1B} - p_1^*)$

we get two roots but the root that is not greater than $\frac{V_1}{2}$ is:

$$p_{1(3)}^* = \frac{1}{2}\{v_{1B} + \alpha_1 + 1 - (v_{1B}^2 + 2v_{1A}^2 + \alpha_1^2 + 3 - 2v_{1B} - 2\alpha_1 + 2v_{1B}\alpha_1 - 4v_{1A})^{\frac{1}{2}}\} \tag{17}$$

Note that when $1-\alpha_1 \leq v_{1A} \leq 1$, we have $p_1^* < v_{1A}$.

■

Proof of corollary 6:

We take partial derivative of p_1^* from equation 14 with respect to α_1, when $0 \leq v_{1A} \leq 1-\alpha_1$,and $2(1-v_{1A})(v_{1A}-v_{1B}) \geq \alpha_1^2$, we have

$\frac{\partial p_1^*}{\partial \alpha_1} = \frac{1}{2}\{1 + \frac{\alpha_1+2v_{1A}-1-v_{1B}}{\sqrt{v_{1B}^2+1-2v_{1B}-\alpha_1^2-4v_{1A}\alpha_1+2\alpha_1+2v_{1B}\alpha_1}}\} > 0$

by eliminating $\frac{1}{2}$ and taking the fraction to the left hand side and multiplying each side with the denominator, we get

$\iff \sqrt{v_{1B}^2+1-2v_{1B}-\alpha_1^2-4v_{1A}\alpha_1+2\alpha_1+2v_{1B}\alpha_1}$

$> 1+v_{1B}-\alpha_1-2v_{1A}$

by squaring both sides and with some algebra that we skip, we get:

$\iff (2v_{1B}-2v_{1A}-\alpha_1)(1-\alpha_1-v_{1A})+\alpha_1(v_{1A}-1) < 0$

The term above is negative since the first parenthesis is negative by the fact that $v_{1B} < v_{1A}$; the second parenthesis is non-negative by the fact that $v_{1A} \leq 1-\alpha_1$, and the third term is negative by the fact that $v_{1A} < 1$.

Thus, $\frac{\partial p_1^*}{\partial \alpha_1} > 0$ when $0 \leq v_{1A} \leq 1-\alpha_1$.

Next, we show that $\frac{\partial p_1^*}{\partial v_{1B}} > 0$.

$\frac{\partial p_1^*}{\partial v_{1B}} = \frac{1}{2}\{1 + \frac{1-\alpha_1-v_{1B}}{\sqrt{v_{1B}^2+1-2v_{1B}-\alpha_1^2-4v_{1A}\alpha_1+2\alpha_1+2v_{1B}\alpha_1}}\} > 0.$

This is positive since the numerator is positive by the assumption that $v_{1B} < v_{1A}$ and $0 \leq v_{1A} \leq 1-\alpha_1$. The denominator is always positive since it is a square root.

Now we show that $\frac{\partial p_1^*}{\partial v_{1A}} = \frac{\alpha_1}{\sqrt{v_{1B}^2+1-2v_{1B}-\alpha_1^2-4v_{1A}\alpha_1+2\alpha_1+2v_{1B}\alpha_1}} > 0.$

Then we take partial derivative of p_1^* from equation 16 with respect to v_{1A}, v_{1B}, and α_1, respectively, when $0 \leq v_{1A} \leq 1-\alpha_1$ and $2(1-v_{1A})(v_{1A}-v_{1B}) \leq \alpha_1^2$, we have,

$\frac{\partial p_1^*}{\partial \alpha_1} = \frac{\partial p_1^*}{\partial v_{1A}} = \frac{1}{3}\{1 - \frac{1+v_{1B}-2\alpha_1-2v_{1A}}{\sqrt{(v_{1A}+v_{1B}+\alpha_1+1)^2-3(v_{1A}+\alpha_1)^2-6v_{1B}}}\} > 0$

$\Longleftrightarrow \sqrt{(v_{1A}+v_{1B}+\alpha_1+1)^2 - 3(v_{1A}+\alpha_1)^2 - 6v_{1B}} > 1 + v_{1B} - 2\alpha_1 - 2v_{1A}$

By squaring both sides and with some algebra that we skip, we get:

$\Longleftrightarrow (v_{1A}+\alpha_1)(1+v_{1B}-\alpha_1-v_{1A}) - v_{1B} > 0$

Since $v_{1A} \leq 1-\alpha_1$, then $(1+v_{1B}-\alpha_1-v_{1A}) > v_{1B}$, we have,

$\Longleftrightarrow (v_{1A}+\alpha_1)(1+v_{1B}-\alpha_1-v_{1A}) - v_{1B} > (v_{1A}+\alpha_1)v_{1B} - v_{1B} > 0$

$\Longleftrightarrow (v_{1A}+\alpha_1)(1+v_{1B}-\alpha_1-v_{1A}) - v_{1B} > (v_{1A}+\alpha_1-1)v_{1B} > 0$

Thus, $\frac{\partial p_1^*}{\partial \alpha_1} = \frac{\partial p_1^*}{\partial v_{1A}} > 0$.

And $\frac{\partial p_1^*}{\partial v_{1B}} = \frac{1}{3}\{1 - \frac{2-v_{1A}-v_{1B}-\alpha_1}{\sqrt{(v_{1A}+v_{1B}+\alpha_1+1)^2-3(v_{1A}+\alpha_1)^2-6v_{1B}}}\} > 0$ since $2 - v_{1A} - v_{1B} - \alpha_1 > 0$ when $0 \leq v_{1A} \leq 1-\alpha_1$ and $2(1-v_{1A})(v_{1A}-v_{1B}) \leq \alpha_1^2$.

Finally, we take partial derivative of p_1^* from equation 17 with respect to v_{1A}, v_{1B}, and α_1, respectively, when $1-\alpha_1 \leq v_{1A} \leq 1$, and we have,

$\frac{\partial p_1^*}{\partial \alpha_1} = \frac{\partial p_1^*}{\partial v_{1B}} = \frac{1}{2}\{1 - \frac{\alpha_1-1+v_{1B}}{\sqrt{v_{1B}^2+2v_{1A}^2+\alpha_1^2+3-2v_{1B}-2\alpha_1-4v_{1A}+2v_{1B}\alpha_1}}\} > 0$

$\Longleftrightarrow \sqrt{v_{1B}^2+2v_{1A}^2+\alpha_1^2+3-2v_{1B}-2\alpha_1-4v_{1A}+2v_{1B}\alpha_1}$

$> \alpha_1 - 1 + v_{1B}$

By squaring both sides and with some algebra that we skip, we get:

$\Longleftrightarrow (v_{1A}-1)^2 > 0$

And $\frac{\partial p_1^*}{\partial v_{1A}} = \frac{1-v_{1A}}{\sqrt{v_{1B}^2+2v_{1A}^2+\alpha_1^2+3-2v_{1B}-2\alpha_1-4v_{1A}+2v_{1B}\alpha_1}} > 0$ unless v_{1A} is equal to 1.

■

Proof of Lemma 7:

First, assuming that the other global bidder is active in the license B auction. On the one hand, we calculate the optimal drop-out price from equation 9, when $0 \leq v_{1A} \leq 1-\alpha_1$, $p_1^* < v_{1A}$, and when $\alpha_2 \leq p_A \leq 1$.

$E\Pi_1^1 = E\Pi_1^2 \Leftrightarrow \int_{p_1^*}^{v_{1A}} (v_{1A}-p_A)(\frac{m+1}{1-\alpha_2})(\frac{p_A-\alpha_2}{1-\alpha_2})^m dp_A$

$= \int_{p_1^*}^{v_{1A}+\alpha_1}(V_1-p_1^*-p_A)(\frac{m+1}{1-p_1^*})(\frac{p_A-p_1^*}{1-p_1^*})^m dp_A+\int_{v_{1A}+\alpha_1}^{1}(v_{1B}-p_1^*)(\frac{m+1}{1-p_1^*})(\frac{p_A-p_1^*}{1-p_1^*})^m dp_A$

$\Longleftrightarrow \frac{1}{(1-\alpha_2)^{m+1}}\int_{p_1^*}^{v_{1A}}(v_{1A}-p_A)(p_A-\alpha_2)^m dp_A$

$= \frac{1}{(1-p_1^*)^{m+1}}\int_{p_1^*}^{v_{1A}+\alpha_1}(v_{1A}+\alpha_1-p_A)(p_A-p_1^*)^m dp_A+\int_{p_1^*}^{1}(v_{1B}-p_1^*)(p_A-p_1^*)^m dp_A$

$\Longleftrightarrow \frac{(v_{1A}-\alpha_2)^{m+2}-(v_{1A}-p_1^*)(p_1^*-\alpha_2)^{m+1}}{(1-\alpha_2)^{m+1}(m+1)(m+2)} = \frac{(v_{1A}+\alpha_1-p_1^*)^{m+2}}{(1-p_1^*)^{m+1}(m+2)}+(v_{1B}-p_1^*)$

By removing the integral and rearranging the equation, we find that p_1^* is the qualified solution to the following equation.

$$\begin{aligned}&[(v_{1A}-\alpha_2)^{m+2}-(v_{1A}-p_1^*)(p_1^*-\alpha_2)^{m+1}-(v_{1B}-p_1^*)(m+2)(1-\alpha_2)^{m+1}](1-p_1^*)^{m+1}\\&\quad=(v_{1A}+\alpha_1-p_1^*)^{m+2}(1-\alpha_2)^{m+1}\end{aligned} \tag{18}$$

If $p_1^* > v_{1A}$, then we calculate the optimal drop out price from the following equation,

$0 = E\Pi_1^1 = E\Pi_1^2 = \int_{p_1^*}^{v_{1A}+\alpha_1}(V_1-p_1^*-p_A)(\frac{m+1}{1-p_1^*})(\frac{p_A-p_1^*}{1-p_1^*})^m dp_A+\int_{v_{1A}+\alpha_1}^{1}(v_{1B}-p_1^*)(\frac{m+1}{1-p_1^*})(\frac{p_A-p_1^*}{1-p_1^*})^m dp_A$

By removing the integral and rearranging the equation, we find that p_1^* is the qualified solution to the following equation.

$$(v_{1A}+\alpha_1-p_1^*)^{m+2}+(m+2)(v_{1B}-p_1^*)(1-p_1^*)^{m+1}=0 \tag{19}$$

On the other hand, when $1-\alpha_1 \leq v_{1A} \leq 1$ and $\alpha_2 \leq p_A \leq 1$, the optimal drop out price is derived from the following equation,

$E\Pi_1^1 = E\Pi_1^2$

$\Leftrightarrow \int_{p_1^*}^{v_{1A}}(v_{1A}-p_A)g(p_A|p_1^*)dp_A = \int_{p_1^*}^{1}(v_{1A}+\alpha_1-p_A)g(p_A|p_1^*)dp_A+(v_{1B}-p_1^*)$

$\Longleftrightarrow \frac{1}{(1-\alpha_2)^{m+1}}\int_{p_1^*}^{v_{1A}}(v_{1A}-p_A)(p_A-\alpha_2)^m dp_A$

$= \frac{1}{(1-p_1^*)^{m+1}}\int_{p_1^*}^{1}(v_{1A}+\alpha_1-p_A)(p_A-p_1^*)^m dp_A+(v_{1B}-p_1^*)$

$\Longleftrightarrow \frac{(v_{1A}-\alpha_2)^{m+2}-(v_{1A}-p_1^*)(p_1^*-\alpha_2)^{m+1}}{(1-\alpha_2)^{m+1}(m+2)} = \frac{(v_{1A}+\alpha_1-p_1^*)(m+2)-(1-p_1^*)(m+1)}{(m+2)}+v_{1B}-p_1^*$

By removing the integral and rearranging the equation, we find that p_1^* is the qualified solution to the following equation.

$$(v_{1A}-\alpha_2)^{m+2}-(v_{1A}-p_1^*)(p_1^*-\alpha_2)^{m+1} = (1-\alpha_2)^{m+1}[(v_{1A}+\alpha_1-p_1^*)(m+2)-(1-p_1^*)(m+1)+(v_{1B}-p_1^*)] \tag{20}$$

Next, assuming that a local bidder will win license B. On the one hand, we calculate the optimal drop-out price from equation 10, when $0 \leq v_{1A} \leq 1-\alpha_1$.

$E\Pi_1^1 = E\Pi_1^2$

$\Leftrightarrow \int_{p_1^*}^{v_{1A}}(v_{1A}-p_A)(\frac{m+1}{1-p_1^*})(\frac{p_A-p_1^*}{1-p_1^*})^m dp_A = \int_{p_1^*}^{v_{1A}+\alpha_1}(V_1-p_1^*-p_A)(\frac{m+1}{1-p_1^*})(\frac{p_A-p_1^*}{1-p_1^*})^m dp_A + \int_{v_{1A}+\alpha_1}^{1}(v_{1B}-p_1^*)(\frac{m+1}{1-p_1^*})(\frac{p_A-p_1^*}{1-p_1^*})^m dp_A$

$\Longleftrightarrow \int_{p_1^*}^{v_{1A}}(v_{1A}-p_A)(p_A-p_1^*)^m dp_A$

$= \int_{p_1^*}^{v_{1A}+\alpha_1}(V_1-p_1^*-p_A)(p_A-p_1^*)^m dp_A + \int_{v_{1A}+\alpha_1}^{1}(v_{1B}-p_1^*)(p_A-p_1^*)^m dp_A$

By removing the integral and rearranging the equation, we find that p_1^* is the qualified solution to the following equation.

$$(v_{1A}-p_1^*)^{m+2} = (v_{1A}+\alpha_1-p_1^*)^{m+2} + (m+2)(v_{1B}-p_1^*)(1-p_1^*)^{m+1} \tag{21}$$

As we have shown above, when $p_1^* > v_{1A}$, then the optimal drop out price is from equation 19.

On the other hand, when $1-\alpha_1 \leq v_{1A} \leq 1$ and then $p_1^* < v_{1A}$ the optimal drop out price is derived from the following equation.

$E\Pi_1^1 = E\Pi_1^2 \Leftrightarrow \int_{p_1^*}^{v_{1A}}(v_{1A}-p_A)g(p_A|p_1^*)dp_A$

$= \int_{p_1^*}^{1}(v_{1A}+\alpha_1-p_A)g(p_A|p_1^*)dp_A + (v_{1B}-p_1^*)$

$\Longleftrightarrow \int_{p_1^*}^{v_{1A}}(v_{1A}-p_A)(p_A-p_1^*)^m dp_A = \int_{p_1^*}^{1}(v_{1A}+\alpha_1-p_A)(p_A-p_1^*)^m dp_A + \frac{(v_{1B}-p_1^*)(1-p_1^*)^{m+1}}{m+1}$

By removing the integral and rearranging the equation, we find that p_1^* is the qualified solution to the following equation.

$$(v_{1A}-p_1^*)^{m+2} = (m+2)(v_{1B}-p_1^*)(1-p_1^*)^{m+1}+[(m+2)(v_{1A}+\alpha_1-1)+1-p_1^*](1-p_1^*)^{m+1} \tag{22}$$

If $p \leq p_1^*$, then $E\Pi_1^1 \leq E\Pi_1^2$, Firm 1 should choose Case 2 to continue to stay in both licenses auctions unless the price reaches the level to make two expected profits equal to each other. If $p \geq p_1^*$, then $E\Pi_1^1 \geq E\Pi_1^2$, Firm 1 should choose Strategy 1 to drop out of the auction for license B and continue to stay in the auction for license A until the price reaches his valuation v_{1A}.

■

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