A Hall of shame?

David S. Gunderson

The daytime television game show *Let's Make a Deal* first aired in December 1963 on NBC, hosted for many years by Monty Hall. One particular game on that show caused a great deal of controversy, in particular, among mathematicians. In "Ask Marilyn" [5], a column in the weekly newspaper insert magazine *Parade*, appeared a letter from a reader asking about what is now known as "Monty Hall's Problem":

> Suppose you're on a game show and you're given the choice of three doors: Behind one is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

One might guess that switching produces no advantage, since there are two doors remaining, and the chance of a car being behind one or the other is the same. Furthermore, one might find it hard to believe that an action taken after your initial choice would change your likelihood of success—retroactively!

"Marilyn" is Marilyn vos Savant, the person with the highest recorded IQ in the world. She replied:

> Yes, you should switch. The first door has a one-third chance of winning, but the second door has a two-thirds chance. Here's a good way to visualize what happened. Suppose there are a *million* doors, and you pick door No. 1. Then the host, who knows what's behind the doors and will always avoid the one with the prize, opens them all except door # 777,777. You'd switch to that door pretty fast, wouldn't you?

Thousands wrote Marilyn and chided her for being wrong. Among the many letters received, quite a few were from Ph.D. mathematicians! Here are partial quotations [6] from some, published a few months after the original article: "... As a professional mathematician, I'm very concerned with the general public's lack of mathematical skills. Please help by confessing your error and, in the future, being more careful." "... There is enough mathematical illiteracy in this country, and we don't need the world's highest IQ propogating more. Shame!" (Were they trying to get her goat?)

Marilyn then gave another proof for this curious result—spawning yet another round of letters—again, many from mathematicians at many prestigious universities and research centres! Two months after the first round of vitriol, she devoted yet another column on the problem. Again, she quoted more readers: "Maybe women look at math problems differently than men." "You are the goat!" "May I suggest that you obtain and refer to a standard textbook on probability before you try to answer of a question of this type again?" The controversy reached the *New York Times* [8], where it was mentioned that nearly 1000 Ph.D.'s wrote in to disagree with Marilyn. In all, over 10,000 letters were received, most of which disagreed with her. In [7], Marilyn wrote "But math answers aren't determined by votes" and held her ground.

Surprisingly, if you switch, your probability of winning the car goes up from 1/3 from 2/3. There are many ways to justify this, and here are a few—one just might convince you. In all of these solutions, we assume that you have picked door 1, and that Monty always offers a you the option of switching after a goat has been revealed. The first solution I give below is essentially Marilyn's first response. In [6] she also gave the reasoning in my second solution below. If neither of those convinces you, I give three more, using something called "conditional probability", a concept given in nearly every first course on statistics.

Notation: For each $i = 1, 2, 3$, let C_i denote the event that the car is behind door i, and let M_i denote the event that Monty opens door i. The notation "$E \wedge F$" denotes the joint event "E and F". The events E and F are said to be *independent* if and only if the probabilities satisfy $\Pr(E \wedge F) = \Pr(E) \cdot \Pr(F)$. The negation of an event E is denoted by $\neg E$.

Solution 1: *a priori*, $\Pr(C_1) = \frac{1}{3}$, and so $\Pr(\neg C_1) = \frac{2}{3}$. Thus the probability of "the car being behind either 2 or 3" is $\frac{2}{3}$. After Monty reveals what is behind one of these doors, this probability does not change.

One can extend this argument to 1000 doors. After your original choice (with probablity of success $\frac{1}{1000}$) Monty can open 998 more doors revealing goats. Then switch—with a probability of success being $\frac{999}{1000}$!!! $\square$

Solution 2: There are three cases; the car is behind one of

- 1—if you switch, you lose;
- 2—Monty opens 3; if you switch, you win;
- 3—Monty opens 2; if you switch, you win.

So, if you switch, you win in two of the three cases. $\square$

In the following solutions, we use the notation "$\Pr(F \mid E)$" to denote the probability of event F happening *given that E has already occurred*. Such a probability is called a *conditional probability*. Bayes theorem says that $\Pr(E \wedge F) = \Pr(E) \cdot \Pr(F \mid E)$.

Solution 3: Let E be the event that the car is behind one of 2 or 3, and let F be the event that Monty opens a remaining door. Then

$$\Pr(E \wedge F) = \Pr(E) \cdot \Pr(F \mid E) = \frac{2}{3} \cdot 1 = \frac{2}{3}.$$

$\square$

Solution 4:

$$\begin{aligned}
&\Pr(M_3 \wedge C_2) + \Pr(M_2 \wedge C_3) \\
&= \Pr(C_2) \cdot \Pr(M_3 \mid C_2) + \Pr(C_3) \cdot \Pr(M_2 \mid C_3) \\
&= \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1 = \frac{2}{3}.
\end{aligned}$$

$\square$

Solution 5: If the car is behind 1, it is reasonable to assume that Monty will open either of the remaining doors with equal probability $\frac{1}{2}$. If the car is not behind 1, then with equal probability, the car is behind one of 2 or 3. In any case, $\Pr(M_2) = \frac{1}{2}$. Then

$$\begin{aligned}
\Pr(C_3 \mid M_2) &= \frac{\Pr(C_3 \wedge M_2)}{\Pr(M_2)} \\
&= \frac{\Pr(C_3)\Pr(M_2 \mid C_3)}{\Pr(M_2)} \\
&= \frac{1/3 \cdot 1}{1/2} = \frac{2}{3}
\end{aligned}$$

Now repeating this argument with M_3 and C_2 gives $\Pr(C_2 \mid M_3) = \frac{2}{3}$. In any case, the probability of the car being behind the unpicked door is 2/3. $\square$

How is the game affected if Monty reserves the right to not offer the choice of a switch? When he does make the offer, should you then switch? In [8], Monty is quoted as then saying "My only advice to you is, if you can get me to offer you $5,000 not to open the door, take the money and go home."

The Monty Hall Problem existed long before Monty Hall and *Let's make a deal.* Martin Gardner wrote about this problem in 1959, but under the name "The three prisoners problem" [1], and called it "a wonderfully confusing little problem". He also included a discussion in [2]. It goes something like this: Three prisoners, A,B,C are condemned to death, but at the last minute, the prison governor decides to pardon one of them, but refuses to tell them who. Prisoner A then persuades the governor to inform him that C is one of those to die. Prisoner A then thinks that his odds of living just went up from 1/3 to 1/2. Is he correct? Martin Gardner could not recall where he got the problem from, however, according to *www.ftlmagazine.com/macaw/MR43.html*, an equivalent problem appears in J. Bertrand's *Calcul des Probabilitiés* of 1889, known as Bertrand's Box Paradox.

In 1976, the problem received its present name in an article with that title appearing in the journal *American Statistician.* It remains today a very popular topic; for example see [3] or a section called "Getting Monty's Goat" in [4]. The Monty Hall Problem has increased mathematical awareness in the general public, probably because so many can relate to games and television contests. Today, there are over 200,000 websites in which the Monty Hall problem is discussed, some of which have applets that allow you to play the game. (For example, try *www.ustat.toronto.edu/david/MH.html* or *www.stat.sc.edu/w̃est/javahtml/LetsMakeaDeal.html.*) Perhaps most convincing is to play the game with your classmates. Try it!

What lessons can be learned here? Should mathematicians be required to take a course in recreational mathematics? Should one run a few trials before jumping to a conclusion? Should one distrust intuition when it comes to probability? Is it okay to be wrong, as long as we learn from our mistakes? By the way, a few mathematicians did write back to Marilyn and apologized.

References

[1] Martin Gardner, Mathematical games, *Scientific American* **201**, Oct. 1959, 180–182 (and Nov. 1959, 188).

[2] Martin Gardner, *More mathematical puzzles and diversions from Scientific American*, Bell & Sons, London, 1963.

[3] Leonard Gillman, The car and the goats, *The American Mathematical Monthly* **99**, (Jan. 1992), MAA, 3–7.

[4] Robert M. Martin, *There are two errors in the the title of this book*, Broadview Press, Peterborough, Ontario, 1992.

[5] Marilyn vos Savant, "Ask Marilyn", *Parade*, 9 Sept. 1990.

[6] Marilyn vos Savant, "Ask Marilyn", *Parade* 2 Dec. 1990, p. 25.

[7] Marilyn vos Savant, "Ask Marilyn", *Parade* 17 Feb. 1991, P. 22.

[8] John Tierney, "Behind Monty Hall's 3 Doors: A Puzzle, a Debate and, Perhaps, an Answer, *New York Times*, 21 July 1991.