On a Ramsey-type problem and the Turán numbers

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Abstract

For each n and k, we examine bounds on the largest number m so that for any k-coloring of the edges of K_n there exists a copy of K_m whose edges receive at most $k-1$ colors. We show that for $k \geq \sqrt{n} + \Omega(n^{1/3})$, the largest value of m is asymptotically equal to the Turán number $t(n, \lfloor \binom{n}{2}/k \rfloor)$, while for any constant $\epsilon > 0$, if $k \leq (1-\epsilon)\sqrt{n}$ then m can be asymptotically larger than that Turán number.

1 Introduction

For any finite set S and positive integer k, we use the notation $[S]^k = \{T \subset S : |T| = k\}$. A graph G is an ordered pair $(V, E) = (V(G), E(G))$ where V is a finite set and $E \subset [V]^2$. Elements of V are called vertices and elements of E are called edges. If G is a graph with $|V(G)| = n$ and $E(G) = [V]^2$, then we say that G is a complete graph on n vertices, denoted by K_n. It will be convenient to use $V(K_n) = [n] = \{1, \ldots, n\}$ and $E(K_n) = [n]^2$. The complement of G will be denoted by $\overline{G} = (V(G), [V(G)]^2 \backslash E(G))$; so, for example, $\overline{K_n}$ is an independent set of n vertices.

The standard Ramsey arrow notation $n \to (m)^2_k$ means that for every k-coloring of the edges of K_n, there exists a copy of K_m all of whose edges

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¶The remaining authors would like to dedicate this paper to the memory of Paul Erdős.

receive the same color. In two early papers on Ramsey theory ([3], [5]) it was shown that

$$n \to \left(\frac{\log n}{2\log 2}\right)^2_2 \quad \text{and} \quad n \not\to \left(\frac{2\log n}{\log 2}\right)^2_2.$$

When more than two colors are used, techniques from these two papers show that there are universal constants c_1 and c_2, so that for each $k \geq 2$,

$$n \to \left(\frac{c_1 \log n}{k\log k}\right)^2_k \quad \text{and} \quad n \not\to \left(\frac{c_2\log n}{\log k}\right)^2_k.$$

Erdős, Hajnal, and Rado [4] introduced another Ramsey arrow notation, $n \to [m]^2_k$, meaning that for any k-coloring of $E(K_n)$, there exists a copy of K_m whose edges receive at most $k-1$ colors. (When $k=2$, this is the standard Ramsey arrow.) This concept was also examined in [8]. It was shown in [6] that there exists a universal constant c_1 so that for every n and $k \leq n$,

$$n \to \left[\frac{c_1 k}{\log k}\log n\right]^2_k, \tag{1}$$

and techniques from [3] give a universal constant c_2 so that

$$n \not\to [c_2 k \log n]^2_k. \tag{2}$$

Rather than being given an m and k and finding bounds on n, we shall be concerned with bounds on m given n and k.

Definition 1.1 *Let $f(n,k)$ be the largest integer so that $n \to [f(n,k)]^2_k$.*

Thus for a given n, under any k-coloring of $E(K_n)$, there is always a clique of size at least $f(n,k)$ which is not *completely multi-colored*. Restating equations (1) and (2), there are universal constants c_1 and c_2 so that for $k \leq n$:

$$\frac{c_1 k}{\log k}\log n \leq f(n,k) < c_2 k \log n. \tag{3}$$

In particular, for k constant, $f(n,k) = \Theta(\log n)$. In this paper, we are interested in the asymptotic order of $f(n,k)$ when k is a function of n.

2 Preliminaries

Lemma 2.1 *If $2 \leq k < n$, then $f(n,k) \leq f(n+1,k)$ and $f(n,k) \leq f(n,k+1)$.*

Proof: That $f(n,k) \leq f(n+1,k)$ follows immediately from the fact that K_{n+1} has K_n as a subgraph.

To prove that $f(n,k) \leq f(n,k+1)$, let $f(n,k) = m$ and let a $(k+1)$-coloring of $E(K_n)$ be given. Examine an auxiliary k-coloring defined by identifying two color classes. By the choice of m, there exist m vertices which induce at most $k-1$ colors from the auxiliary coloring, hence at most k colors of the original $(k+1)$-coloring. Hence, $f(n,k+1) \geq m$. $\square$

It is not difficult to check that $f(6,3) < 4$ (by giving a 3-coloring of $E(K_6)$ under which every K_4 is 3-colored—one such coloring uses three paths of length 5). It easily follows that $f(6,3) = 3$. Since $f(6,2) = 3$ as well, the inequality $f(n,k) \leq f(n,k+1)$ is not always a strict inequality.

Let $t(n,m)$ denote the maximum t so that every graph with n vertices and m edges has an independent set of size at least t. This function is known precisely for all n and m, by Turán's Theorem. It is the number of cliques in the appropriate Turán graph, which we denote here by $T(n,m)$. This is the disjoint union of the minimum possible number of nearly equal cliques on n vertices whose total number of edges is at most m. We define

$$g(n,k) = t(n, \lfloor \binom{n}{2}/k \rfloor).$$

This yields an easy lower bound for $f(n,k)$, which was also observed in [8]:

Lemma 2.2 $f(n,k) \geq g(n,k)$.

Proof: Fix any k-coloring $E(K_n) = E_1 \cup \cdots \cup E_k$. Some color class, say E_1, has at most $\lfloor \binom{n}{2}/k \rfloor$ edges. By Turán's theorem the subgraph formed by E_1 contains an independent set of t vertices, and this independent set is a t-clique in G that is at most $(k-1)$-colored. $\square$

The main focus of this paper will be to determine for which values of n,k, equality holds, or at least nearly holds, in Lemma 2.2. Note that if there is a coloring of $E(K_n)$ in which each color class is a copy of the appropriate Turán graph, then equality holds. Similarly, for equality to nearly hold, we must find a coloring in which each color class is very close to the Turán graph.

Here we show that $g(n,k)$ is the precise value of $f(n,k)$ for all $k \geq n$, (as well as in several other cases), that for every $k \geq (1+\epsilon)\sqrt{n}$

$$f(n,k) = (1+o(1))g(n,k),$$

where the $o(1)$ term tends to 0 as n (and hence k) tend to infinity, and that for every $\epsilon > 0$ there exist n_0 and $\delta > 0$ such that if $k < (1-\epsilon)\sqrt{n}, n > n_0$

then

$$f(n,k) > (1+\delta)g(n,k).$$

It will be helpful to observe that if both k and n/k tend to infinity then $g(n,k) = (1+o(1))k$.

3 Case 1: $k \geq n$

In Theorem 3.1 we shall employ a result about *balanced* edge-colorings. A *proper* edge-coloring of a graph is one where no two incident edges have the same color. An edge-coloring of a graph G using k colors is called *balanced* if each color class has either $\lfloor |E(G)|/k \rfloor$ or $\lceil |E(G)|/k \rceil$ edges. The following fact is well known:

Fact: If a graph G has a k-edge coloring, then it has a balanced proper k-edge coloring.

Theorem 3.1 *For each $k \geq n$, $f(n,k) = g(n,k)$.*

Proof: We use a balanced proper k-coloring of $E(K_n)$ and observe that each color class contains the appropriate Turán graph, which in this case is simply a matching of size $\lfloor \binom{n}{2}/k \rfloor$ or $\lceil \binom{n}{2}/k \rceil$. $\square$

4 Case 2: $k = n/\alpha$ for a fixed real $\alpha \geq 1$

We need the following well known result of Wilson [12] on graph decomposition. We say that a graph G has an H-decomposition if the set of its edges can be colored such that each color class forms a copy of H. These colored copies are called the members of the decomposition.

Theorem 4.1 (Wilson) *Let $H = (V,E)$ be a graph with q edges and let g denote the greatest common divisor of the degrees of H. Then there is an $n_0 = n_0(H)$ such that for every $n > n_0$ for which $\binom{n}{2}$ is divisible by q and $n-1$ is divisible by g, K_n has an H-decomposition.*

We need to show that we can find complete graphs which have H-decompositions with additional regularity properties.

Lemma 4.2 *For every graph H with h vertices and q edges, there is a $2q$-regular graph H_1 on h^4 vertices which has an H-decomposition, such that each vertex of H_1 is incident with precisely h members of the decomposition.*

Proof: We begin with the following:

Claim There exist $g_1, ..., g_h \in \{0, ..., h^4 - 1\}$ such that the $h^2 - h$ differences $g_i - g_j \pmod{h^4}$ are pairwise distinct.

To see this, we take a uniform random choice of $g_1, ..., g_h$. The probability that $(g_{i_1}, g_{j_1}), (g_{i_2}, g_{j_2})$ with, say, $j_2 \neq i_1, j_1$, have the same difference is at most $\frac{1}{h^4-3}$ since after fixing $g_{i_1}, g_{j_1}, g_{i_2}$, there are still at least $h^4 - 3$ choices for g_{j_2}, at most one of which yields the same difference. The case $j_1 = i_2, j_2 = i_1$ (for even h) has an even lower probability. Therefore, the expected number of pairs which have the same difference is at most $\binom{h^2-h}{2}\frac{1}{h^4-3} < 1$ and so there is at least one choice for which this number is zero.

Now we simply take the copy of H formed by mapping its vertex number i to g_i, and let H_1 consist of this copy and all the $h^4 - 1$ cyclic shifts of it. $\square$

Corollary 4.3 *For every graph H there is a complete graph K_m which has an H-decomposition so that each vertex of K_m is incident with the same number of members of the decomposition.*

Proof: This follows from applying Wilson's Theorem to H_1 from Lemma 4.2. $\square$

Corollary 4.4 *For every graph H there are $c = c(H)$ and $n_0 = n_0(H)$ such that for every $n > n_0(H)$ there is some n' satisfying $n \leq n' \leq n + c$ for which $K_{n'}$ has an H-decomposition in which each vertex is incident with the same number of members of the decomposition, and any two vertices lie in at most c common members of the decomposition.*

Proof: First apply Wilson's Theorem to K_m from Corollary 4.3 to decompose $K_{n'}$ to copies of K_m, for an appropriate value of $n \leq n' \leq n + m(m-1)$. Then decompose each such K_m copy into copies of H. Note that each pair of vertices can lie only in copies of H that lie in the same K_m. $\square$

This yields the main theorem of this section:

Theorem 4.5 *If $k = n/\alpha$ and $\alpha \geq 1$ is a fixed real, then $f(n, k) = (1 + o(1))g(n, k)$.*

Proof: It is useful to note that for this case, $g(n, k) = \Theta(n)$.

Consider any $\epsilon > 0$. We will prove that $f(n', k') \leq (1 + \epsilon)g(n, k)$ for a suitable $n' \geq n$ and $k' \geq k$. Our result then follows from Lemma 2.1. Our goal is to find a k'-colouring of $E(K_{n'})$ such that each colour class contains a Turán graph on n' vertices with independence number $t \leq (1 + \frac{\epsilon}{2})g(n,k)$. Suppose that $a < \alpha+1 \leq a+1$ for an integer a. A straightforward calculation shows that we can choose $(1 + \frac{\epsilon}{4})g(n,k) \leq t \leq (1 + \frac{\epsilon}{2})g(n,k)$ so that the corresponding Turán graph will contain rn' disjoint a-cliques and sn' disjoint $(a+1)$-cliques, so long as rn', sn' are integers, where r, s are rationals with denominators bounded by some constant function of α, ϵ.

Now we define H to be a collection of rh disjoint a-cliques and sh disjoint $(a+1)$-cliques, where h is the LCM of the denominators of r, s. Take some n' as in Corollary 4.4 along with the corresponding decomposition of $K_{n'}$. Define $\mathcal{H}$ to be the $|H|$-uniform hypergraph whose vertices are the vertices of $K_{n'}$, and whose edges are the vertex sets of the copies of H. This hypergraph is κ-regular for some $\kappa \geq (1+\delta)k$ where $\delta = \delta(\epsilon)$, and any pair of vertices lies in at most $c = c(\alpha)$ edges. Therefore, by the main theorem of [9], it has a proper edge-colouring C using $c = \kappa(1 + o(1))$ colours.

We use C to find our edge coloring of $K_{n'}$ as follows. Since $\mathcal{H}$ is $|H|$-uniform and κ-regular, $|E(\mathcal{H})| = \frac{\kappa n'}{|H|}$ and no colour class contains more than $\frac{n'}{|H|}$ hyperedges. We remove from C all color classes which contain fewer than $\frac{n'}{|H|}(1-\gamma)$ hyperedges (i.e., we uncolor all hyperedges belonging to one of those classes), where γ is a positive constant such that $\gamma n' < \frac{\epsilon}{4}g(n,k)$. A simple calculation yields that we remove $o(\kappa)$ colour classes, and so the number of remaining colour classes is $k' = \kappa - o(\kappa) > k$. Furthermore, the subgraph induced by any remaining colour class has independence number at most $t + \gamma\frac{n'}{|H|} \times |H| < t + \frac{\epsilon}{4}g(n,k) < (1+\epsilon)g(n,k)$. We complete our edge colouring of K_n by assigning to any uncoloured edge, an arbitrary colour from amongst the remaining colour classes of C. As this won't increase the independence number of a colour class, we obtain our desired colouring. □

5 Case 3: $k = o(n), k^2 > (1 + o(1))n$

For an integer $q \geq 2$, Lemma 2.2 yields $f(q^2, q+1) \geq q$. The next lemma shows that this bound is tight when q is a power of a prime.

Lemma 5.1 *If q is a power of a prime, then $f(q^2, q+1) = g(q^2, q+1) = q$.*

Proof: The coloring is given by the affine plane. For completeness we describe it in details. Let F_q denote the field of order q. The affine plane of order q is the geometry on the q^2 points $\{(x,y) : x, y \in F_q\}$, where lines are solutions to equations of the form $ax + by = c$, where a, b, and c are constants from the field with a and b both not zero (so there are $q^2 + q$ lines, each with q points). Let $n = q^2$ and define a $(q+1)$-coloring of the complete graph on the points of the affine plane of order q by assigning to each edge the slope of the line containing its endpoints. Each line of the plane induces a monochromatic clique of size q and every color class is determined by a parallel class of lines, so is formed by a union of q cliques each of size q. So each color class determines a subgraph of K_{q^2} with no independent set of size $q + 1$. Thus every set of $q + 1$ points induces every color, that is, $f(q^2, q + 1) < q + 1$, and so, by the preceding discussion, $f(q^2, q + 1) = q$. □

This, together with Lemma 2.1 and well known results about the distributions of primes implies the following

Corollary 5.2 *For every* $k \geq \sqrt{n} + \Omega(n^{1/3})$, $f(n, k) \leq (1 + o(1))k$. *Therefore, if, in addition,* $k = o(n)$, *then* $f(n, k) = (1 + o(1))g(n, k) = (1 + o(1))k$.

Remark: The assertion of Lemma 5.1 can be easily generalized. In fact, any resolvable balanced incomplete block design supplies an example in which $f(n, k) = g(n, k)$ precisely. In particular, one can use the lines in an affine geometry of dimension d to prove the following

Claim 1: For any prime power q and any integer $d > 1$

$$f(q^d, q^{d-1} + q^{d-2} + \ldots + 1) = g(q^d, q^{d-1} + q^{d-2} + \ldots + 1) = q^{d-1}.$$

Similarly, the existence of Kirkman's triple systems (c.f., e.g., [2]) gives the following

Claim 2: For every $n \equiv 3 (mod\ 6)$,

$$f(n, (n-1)/2) = g(n, (n-1)/2) = n/3.$$

The existence of resolvable Steiner systems of the form $S(2, 4, n)$ for all $n \equiv 4(\ mod\ 12)$ (c.f., e.g., [2]) implies

Claim 3: For every $n \equiv 4 (mod\ 12)$,

$$f(n, (n-1)/3) = g(n, (n-1)/3) = n/4.$$

6 Smaller values of k

We will need the following lemma:

Lemma 6.1 *Let $d_1, d_2, \ldots, d_n$ be n non-negative reals whose average value is d, and suppose that at least γn of them are at least $(1+\gamma)d$. Then*

$$\sum_{i=1}^{n} \frac{1}{d_i+1} \geq \frac{n}{(d+1)}(1+\Omega(\gamma^3)).$$

Proof: By the convexity of $f(z) = 1/z$, the minimum of the above sum, subject to $\sum d_i = dn$, is obtained when $d_i = d$ for all i. As long as there is some $d_i > (1+\gamma)d$ there is some other $d_j \leq d$, and replacing each of them by their average decreases the sum by $\frac{1}{d_i+1} + \frac{1}{d_j+1} - \frac{4}{d_i+d_j+2} = \Omega(\gamma^2/(d+1))$. Since we can perform at least γn steps of this form the desired estimate follows. $\Box$

Let $\alpha(G)$ denote the maximum size of an independent set in G.

Lemma 6.2 *For every $\epsilon > 0$ there is a $\delta > 0$ and d_0 such that for any graph $G = (V, E)$ on n vertices with average degree at most d, $d > d_0$, and maximum clique size at most $(1-\epsilon)d$,*

$$\alpha(G) \geq (1+\delta)\frac{n}{d+1}.$$

Proof: Let $d_1, \ldots, d_n$ be the degrees of the vertices of G. If there are at least $\frac{\epsilon}{4}n$ degrees which exceed $(1+\frac{\epsilon}{4})d$, then the result, with $\delta = \Omega(\epsilon^3)$, follows from Lemma 6.1 together with the well known fact that

$$\alpha(G) \geq \sum_{i=1}^{n} \frac{1}{d_i+1},$$

(see, e.g., [1], page 81, for a short proof.)

Otherwise, omit all vertices of degree that exceeds $(1+\frac{\epsilon}{4})d$. The remaining graph has at least $(1-\frac{\epsilon}{4})n$ vertices, has maximum degree $\Delta \leq (1+\frac{\epsilon}{4})d$ and has maximum clique size $\omega \leq (1-\epsilon)d$. Fajtlowicz[7] has shown that for every graph G,

$$\alpha(G) \geq \frac{2|V(G)|}{\omega(G)+\Delta(G)+1},$$

(see also [10]). The result now follows with $1+\delta = (1-\frac{\epsilon}{4})/(1-\frac{3\epsilon}{8})$. $\Box$

Theorem 6.3 *For every $\epsilon > 0$ there is a $\delta > 0$ and n_0 such that for every $n > n_0$ and $2 \leq k \leq (1-\epsilon)\sqrt{n}$,*

$$f(n,k) \geq (1+\delta)g(n,k).$$

Proof: In the above range $g(n,k) = \Theta(k)$. Since for every $k \geq 2$, $f(n,k) = \Omega(\log n)$ the assertion is trivial for $k = o(\log n)$, and we thus may and will assume that $k \geq \Omega(\log n)$. Thus $g(n,k) = (1+o(1))k$. Consider a coloring of $E(K_n)$, $n > n_0$, by k colors, and let G be the graph consisting of all edges of the least popular color. Then the average degree of G is at most $(n-1)/k < n/k$. Fix a $\gamma = \gamma(\epsilon) > 0$ such that $(1-\gamma)\frac{n}{k} > (1+\gamma)k$. If G contains a clique of size at least $(1-\gamma)n/k$, then the induced subgraph on the vertices of this clique misses a color (in fact, misses all colors but one) and is of size at least $(1+\gamma)k > (1+\gamma/2)g(n,k)$ for $n > n_0$. Otherwise, by Lemma 6.2, $\alpha(G) \geq (1+\delta)k$ for some $\delta = \delta(\gamma) > 0$, and so G has a large independent set corresponding to a complete subgraph of size $(1+\delta)k$ which does not contain the least popular color. $\square$

7 Concluding remarks

Turán's theorem implies lower bounds on $f(n,k)$, and we have found that these lower bounds are tight (or nearly tight) whenever k is slightly bigger than $\sqrt{n}$, and are not nearly tight whenever k is slightly smaller than $\sqrt{n}$. The problem of finding an asymptotic formula for $f(n,k)$ for smaller values of k is more difficult (and in particular that of finding an asymptotic formula for $f(n,2)$ is a well known, difficult open problem in Ramsey theory). It would be interesting to find an estimate, up to a constant factor, for $f(n,k)$, for smaller values of k. Thus, for example, when $k \sim \sqrt{n}/\log n$, $f(n,k)$ is at least $\frac{\sqrt{n}}{\log n}$ and at most $O(\sqrt{n})$, and it would be interesting to find a more accurate estimate.

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