# Independent Deuber sets in graphs on the natural numbers

David S. Gunderson[*]     Imre Leader[†]     Hans Jürgen Prömel[‡]     Vojtěch Rödl [§]

final revised version for JCTA submission, 28 April 2003

## Abstract

We show that for any $k, m, p, c$, if $G$ is a $K_k$-free graph on $\mathbb{N}$ then there is an independent set of vertices in $G$ that contains an $(m, p, c)$-set. Hence if $G$ is a $K_k$-free graph on $\mathbb{N}$, then one can solve any partition regular system of equations in an independent set. This is a common generalization of partition regularity theorems of Rado (who characterized systems of linear equations $A\mathbf{x} = \mathbf{0}$ a solution of which can be found monochromatic under any finite coloring of $\mathbb{N}$) and Deuber (who provided another characterization in terms of $(m, p, c)$-sets and a partition theorem for them), and of Ramsey's theorem itself.

## 1   Introduction and statement of results

In this paper we are interested in graphs whose vertices are natural numbers, and in arithmetic properties of independent sets in such graphs. We use $\mathbb{N}$ to denote the set of natural numbers—not including 0—and $\omega = \mathbb{N} \cup \{0\}$. We write $[a, b] = \{c \in \mathbb{Z} : a \leq c \leq b\}$ to denote an interval of integers.

Let $A$ be a finite matrix with integer entries. The system of linear equations $A\mathbf{x} = \mathbf{0}$ is called *partition regular* (over $\mathbb{N}$) if for every partition of $\mathbb{N}$ into finitely many classes there exists a solution completely contained in one class.

Schur's theorem [17] says that for any positive integer $r$, there exists $n$ so that for every coloring $\rho : [1, n] \to [1, r]$ there exist $x, y \in [1, n]$ with $\rho(x) = \rho(y) = \rho(x + y)$. The equation $x + y - z = 0$ describes these Schur triples, and so is partition regular. Van der Waerden's theorem [19] states that for any positive integers $r$, $\ell$, there exists $n$ so that for any coloring $\rho : [1, n] \to [1, r]$ there is a monochromatic $\ell$-term arithmetic progression. Solutions to equations $x - 2y + z = 0$ are 3-term arithmetic progressions or are constant and so this system is also partition regular. Similarly, systems of equations describing any longer arithmetic progressions

form partition regular systems. An example of a simple system which is not partition regular is $x + y = 3z$. (See, e.g., [6] or [7] for more details.)

A characterization of partition regular systems of equations was first given by Rado [15] in terms of something (which is not relevant to our use here) called the "columns property". Deuber [2] later gave another characterization of partition regular systems using structures called "$(m, p, c)$-sets", which we now define.

**Definition 1.1** *Let $p, c \in \mathbb{N}$ with $c \leq p$, and let $m \in \omega$. A set of integers $S$ is an $(m, p, c)$-set if $S \subset \mathbb{N}$ and there exist positive integers (generators) $x_0, x_1, \ldots, x_m$ so that $S = R_0(S) \cup R_1(S) \cup \cdots \cup R_m(S)$, where*

$$
\begin{aligned}
R_0(S) \quad &= \quad \{cx_0 + \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m : \quad \lambda_1, \ldots, \lambda_m \in [-p, p]\}, \\
R_1(S) \quad &= \quad \{cx_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m : \quad \lambda_2, \ldots, \lambda_m \in [-p, p]\}, \\
&\vdots \\
R_{m-1}(S) \quad &= \quad \{cx_{m-1} + \lambda_m x_m : \quad \lambda_m \in [-p, p]\}, \\
R_m(S) \quad &= \quad \{cx_m\}.
\end{aligned}
$$

*In this case we write $S = (x_0, x_1, \ldots, x_m)_{p,c}$ and we say that $R_k(S)$ is the $(k+1)$-st row of $S$.*

We note that the condition $c \leq p$ is for convenience only; nothing would be lost without this condition because for any $p' > p$, every $(m, p', c)$-set trivially contains an $(m, p, c)$-set.

In honor of Deuber's contributions to the field, if a set $S$ is an $(m, p, c)$-set for some $m, p, c$, then we might simply say that $S$ is a *Deuber set* without specifying the parameters $m, p, c$.

**Theorem 1.2 (Deuber [2])** *A linear system $A\boldsymbol{x} = \boldsymbol{0}$ is partition regular if and only if there exist positive integers $m, p, c$ such that every $(m, p, c)$-set contains a solution of $A\boldsymbol{x} = \boldsymbol{0}$.*

In proving a conjecture by Rado regarding partition regular systems, Deuber used the following partition theorem.

**Theorem 1.3 (Deuber [2])** *For every $m \in \omega$, every $p, c \in \mathbb{N}$ with $c \leq p$ and every $k \in \mathbb{N}$, there exists $n, q, d \in \mathbb{N}$ with $d \leq q$ so that for every $(n, q, d)$-set $X$ and every coloring $\rho : X \to [1, k]$, there exists a monochromatic $(m, p, c)$-set contained in $X$.*

To state our results, we adopt standard notation. For a set $S$ and $n \in \omega$, let $[S]^n = \{F \subseteq S : |F| = n\}$. Let $G = (V, E)$ denote a (simple) graph on vertex set $V = V(G)$ with edge set $E = E(G) \subseteq [V]^2$. A set $Y \subset V(G)$ is called *independent* in $G$ if $[Y]^2 \cap E(G) = \emptyset$. When $E(G) = [V(G)]^2$, we say that $G$ is complete, and the complete graph on $n$ vertices is denoted by $K_n$. A graph $G = (V, E)$ is *k-partite* if $V$ can be partitioned into $k$ sets, $V = V_1 \cup \cdots \cup V_k$, each $V_i$ containing no edges, and is a *complete k-partite* graph if for each $i \neq j$, whenever $x \in V_i$ and $y \in V_j$ then $\{x, y\} \in E$. A complete bipartite graph on sets of size $m$ and $n$ will be denoted by $K_{m,n}$.

The main result in this paper is the following.

**Theorem 1.4** *Given $k, p, c \in \mathbb{N}$ with $c \leq p$ and $m \in \omega$, there exist $n, q, d \in \mathbb{N}$ so that any $K_k$-free graph on an $(n, q, d)$-set contains an independent $(m, p, c)$-set.*

**Remarks: 1.** Theorem 1.4 generalizes Theorem 1.3 by the following reasoning: Fix $k, m, p, c$, let $n, q, d$ be guaranteed by Theorem 1.4, and fix an $(n, q, d)$-set $X$. Color $X$ with $r = k-1$ colors and form the complete $(k-1)$-partite graph $G$ whose partite sets are color classes. Since $G$ is $K_k$-free, by Theorem 1.4 some $(m, p, c)$-set is independent in $G$, and hence must be contained in one partite set, i.e., a single color class. Hence there is a monochromatic $(m, p, c)$-set. Since $(m, p, c)$-sets contain sum-sets and arithmetic progressions, Theorem 1.4 also implies theorems of van der Waerden, Schur, and others.

**2.** Theorem 1.4 also generalizes Ramsey's theorem for graphs, because under any red-blue coloring of the pairs of a large set, rather than guaranteeing either a red $K_k$ or a large blue clique, we guarantee either a red $K_k$, or a large blue clique on an $(m, p, c)$-set.

Since any $(n, q, d)$-set sits in some initial interval of the positive integers, Theorem 1.4 immediately implies the following statement:

**Corollary 1.5** *Given $k, p, c \in \mathbb{N}$, $m \in \omega$, and any $K_k$-free graph $G$ with vertex set $\mathbb{N}$, there exists an $(m, p, c)$-set which is independent in $G$.*

Corollary 1.5 can be formulated in terms of partition regular systems by using the $m, p, c$ guaranteed in Theorem 1.2:

**Corollary 1.6** *For any $k \geq 2$ and any $K_k$-free graph on $\mathbb{N}$, one can solve any partition regular system in an independent set.*

We do not know if there is a hypergraph version of Theorem 2.1. For example, is there an analogous condition on a family of triples of $\mathbb{N}$ that would imply that there is an $(m, p, c)$-set not containing any triple? If so, it is not so simple, as the following example indicates. Let $H$ be the 3-uniform hypergraph on $\mathbb{N}$ defined with hyperedges of the form $\{x, x+d, x+3d\}$. Then $H$ is $K_4^{(3)}$-free, yet every arithmetic progression of length 4 contains a hyperedge.

## 2    Earlier work

Ramsey's theorem for graphs [16] says that for any positive integers $r$, and $m$, there exists $n$ so that for any coloring $\rho : [1, n]^2 \to [1, r]$, there exists $M \in [1, n]^m$ so that $[M]^2$ is monochromatic. Erdős [4] asked whether the following natural generalization of both Ramsey's theorem and Schur's theorem holds: If $G$ is a triangle-free graph on vertex set $\mathbb{N}$, does there always exist an independent Schur triple, that is, do there exist $x, y, x \neq y$ so that $FS(x, y) = \{x, y, x+y\}$ is independent in $G$? The answer is yes, as proved in [13] where it was shown that in fact, for fixed $k$ and $d$, if $G$ is a $K_k$-free graph on $\mathbb{N}$, then there exist distinct integers $a_1, a_2, ..., a_d$, so that the finite sum set $FS(\{a_1, \ldots, a_d\})$ is an independent set in $G$. Harborth *et al.* (see, e.g., [1], [11]) have given some sharp lower bounds on $n$ so that if $G$ is a graph on $[1, n]$, these results hold (except the $a_i$ need not be distinct).

Related progress was also made for an infinite version of Erdős' question. Given a set $\{x_i\}_{i \in I}$ of distinct positive integers, let $FS(\{x_i\}_{i \in I}) = \left\{ \sum_{j \in J} x_j : \emptyset \neq J \subseteq I, |J| < \infty \right\}$ denote the finite sums (with no repetitions) from the set. When $I$ is infinite, we say that $FS(\{x_i\}_{i \in I})$ is a *Hindman set*. In 1995, Hajnal asked the following (see [5]): If $G$ is a triangle-free graph on $\mathbb{N}$, does there always exist a Hindman set independent in $G$? Hajnal's question has been

answered in the negative in [3]. Variants of Hajnal's question have been shown to indeed have a positive answer; for example, if the condition "triangle-free" is replaced by "$K_{k,k}$-free" (see [3], [9], and [13]).

A common generalization of Ramsey's theorem and van der Waerden's theorem was also found in [9]: For fixed $k$ and $\ell$, if $G$ is a $K_k$-free graph on $\mathbb{N}$, then there exists an $\ell$-term arithmetic progression which spans an independent set in $G$.

Coloring theorems for arithmetic progressions or finite sums have abstract analogues (the Hales-Jewett theorem and the Graham-Rothschild theorem, respectively; see, e.g., [7] or [14]), from which they can be deduced instantly. In contrast, Deuber's theorem for partitioning $(m, p, c)$-sets can not be accomplished by any one application of such a theorem; several iterations are required. As with Deuber's theorem, one would not expect to be able to prove our main result for $(m, p, c)$-sets with any single application of an abstract theorem. Indeed, the first situation where a single process does not seem to work is for an arithmetic progression together with its difference—which is the simplest kind of subset of an $(m, p, c)$-set not necessarily contained in any one row. In [9] it was proved that for any $k, \ell \geq 3$, in any $K_k$-free graph, there exists an $\ell$-term arithmetic progression together with its difference, all contained in an independent set. This proof used a form of the Gallai-Witt theorem applied iteratively; it does not seem to follow from one application of any of the major abstract theorems (like the Hales-Jewett or Graham-Rothschild theorems).

## 3 Preliminary results

We now briefly describe one of our main tools, the Hales-Jewett theorem.

Let $A$ denote a finite alphabet; write $A^s = \{(x_1, \ldots, x_s) : x_i \in A\}$. Let $[1, s] = F \cup M_1 \cup \cdots \cup M_t$ be a partition with $|M_j| > 0$ for $j = 1, 2, \ldots, t$ and let $(g_i)_{i \in F} \in A^F$ be a fixed $|F|$-tuple. A *t-dimensional subcube* of $A^s$ (associated with $(g_j)_{j \in F}$ and the partition $F \cup M_1 \cup \cdots \cup M_t$) is a set of the form

$$\mathrm{HJC}(F, M_1, \ldots, M_t, (g_j)_{j \in F}) = \{(x_1, \ldots, x_s) \quad : \quad x_j = g_j \text{ for } j \in F \text{ and}$$
$$x_j = x_{j'} \text{ if } j, j' \in M_\alpha \text{ for some } \alpha\}.$$

We now state the central theorem regarding parameter sets.

**Theorem 3.1 (Hales-Jewett)** *For every $t, r \in \mathbb{N}$ and every finite alphabet $A$, there exists $s = HJ(t, r, |A|)$ so that for every coloring $\rho : A^s \to [1, r]$, there exists a monochromatic t-dimensional subcube of $A^s$.*

The original version in [10] yields a 1-dimensional subcube. That version easily implies the current version (see [7, p. 40]). See [14] for a survey of results, applications, and notation for parameter words, another language to describe the Hales-Jewett theorem.

**Definition 3.2** *Let $p, c, q, d \in \mathbb{N}$ with $c \leq p$ and $q \leq d$, let $U = (x_0, x_1, \ldots, x_m)_{p,c}$ be an $(m, p, c)$-set, and let $V = (y_0, y_1, \ldots, y_n)_{q,d}$ be an $(n, q, d)$-set. We say that $U$ is naturally contained in $V$, written $U \preceq V$ if and only if there is a strictly increasing function*

$$\psi : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$$

*such that for each $i \in \{0, 1, \ldots, m\}$, $R_i(U) \subseteq R_{\psi(i)}(V)$.*

Notice that natural containment is trivially transitive.

**Example 3.3** *The $(1,3,1)$-set $A = (10,1)_{3,1}$ is contained in the $(2,2,1)$-set $B = (20,5,1)_{2,1}$ but $A$ is not naturally contained in $B$. In fact, $R_0(A)$ is not contained in any row of $B$.*

We now present some simple results that guarantee that all inclusions with which we shall be concerned are natural.

For any $p, m \in \mathbb{N}$, we use

$$[a_0, a_1 \ldots, a_m]_p = \{\lambda_0 a_0 + \lambda_1 a_1 + \ldots + \lambda_m a_m : \lambda_0, \lambda_1, \ldots, \lambda_m \in [-p, p]\},$$

to denote the span of $a_0, a_1, \ldots, a_n m$.

**Lemma 3.4** *Let $m, p, x_0, x_1, \ldots, x_m \in \mathbb{N}$. If for each $i \in \{0, 1, \ldots, m-1\}$, $x_i > 2p \sum_{j=i+1}^m x_j$, then expressions in $[x_0, x_1, \ldots, x_m]_p$ are unique. That is, if*

$$\lambda_0, \lambda_1, \ldots, \lambda_m, \mu_0, \mu_1, \ldots, \mu_m \in [-p, p] \ and \ \sum_{i=1}^m \lambda_i x_i = \sum_{i=1}^m \mu_i x_i,$$

*then for each $i \in \{0, 1, \ldots, m\}$, $\lambda_i = \mu_i$.*

**Proof**: Assume that $\lambda_0, \lambda_1, \ldots, \lambda_m, \mu_0, \mu_1, \ldots, \mu_m \in [-p, p]$ and $\sum_{i=1}^m \lambda_i x_i = \sum_{i=1}^m \mu_i x_i$. Suppose that there is some $i \in \{0, 1, \ldots, m\}$ such that $\lambda_i \neq \mu_i$ and pick the first such $i$. Assume without loss of generality that $\lambda_i > \mu_i$. If $\lambda_m x_m = \mu_m x_m$, then $\lambda_m = \mu_m$, so we have that $i < m$. Then

$$\sum_{j=i+1}^m (\mu_j - \lambda_j)x_j = (\lambda_i - \mu_i)x_i \geq x_i > 2p \sum_{j=i+1}^m x_j \geq \sum_{j=i+1}^m (\mu_j - \lambda_j)x_j,$$

a contradiction. $\qquad\qquad\square$

With a little more work one can show that $6pc$ can be replaced by $5pc$ in the following lemma. (Here and later, if $i = m$, then we set $\sum_{j=i+1}^m x_j = 0$.)

**Lemma 3.5** *Let $m, p, c \in \mathbb{N}$ with $c \leq p$ and assume that $x_0, x_1, \ldots, x_m \in \mathbb{N}$ and $(x_0, x_1, \ldots, x_m)_{6pc,c} \subseteq \mathbb{N}$. Let $U = (x_0, x_1, \ldots, x_m)_{p,c}$. For each $i \in \{0, 1, \ldots, m-1\}$, $x_i > 6p \sum_{j=i+1}^m x_j$ and $\min R_i(U) > \max R_{i+1}(U)$. Also, any length 3 arithmetic progression in $U$ is contained in some row of $U$.*

**Proof**: Let $i \in \{0, 1, \ldots, m-1\}$. Then $cx_i - \sum_{j=i+1}^m 6pcx_j \in \mathbb{N}$ so $x_i > 6p \sum_{j=i+1}^m x_j$. Thus

$$\min R_i(U) = cx_i - \sum_{j=i+1}^m px_j \geq x_i - \sum_{j=i+1}^m px_j > \sum_{j=i+1}^m px_j \geq cx_{i+1} + \sum_{j=i+2}^m px_j = \max R_{i+1}(U).$$

Now assume we have $d > 0$ such that $\{a, a+d, a+2d\} \subseteq U$. Pick $i, k \in \{0, 1, \ldots, m\}$ and $\lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_m, \mu_{k+1}, \mu_{k+2}, \ldots, \mu_m \in [-p, p]$ such that

$$\begin{aligned} a &= cx_i + \sum_{j=i+1}^m \lambda_j x_j \ and \\ a+d &= cx_k + \sum_{j=k+1}^m \mu_j x_j. \end{aligned}$$

Since $a + d > a$, we have that $k \leq i$. Suppose that $k < i$. Then

$$
\begin{aligned}
d &= cx_k + \sum_{j=k+1}^{i-1} \mu_j x_j + (\mu_i - c)x_i + \sum_{j=i+1}^{m}(\mu_j - \lambda_j)x_j \text{ and so} \\
a + 2d &= 2cx_k + \sum_{j=k+1}^{i-1} 2\mu_j x_j + (2\mu_i - c)x_i + \sum_{j=i+1}^{m}(2\mu_j - \lambda_j)x_j\,.
\end{aligned}
$$

Since the absolute value of each coefficient in the expansion of $a + 2d$ is at most $3p$, we have by Lemma 3.4 that $a + 2d \notin U$.

Thus $k = i$ and

$$
\begin{aligned}
d &= \sum_{j=i+1}^{m}(\mu_j - \lambda_j)x_j \text{ and so} \\
a + 2d &= cx_i + \sum_{j=i+1}^{m}(2\mu_j - \lambda_j)x_j\,.
\end{aligned}
$$

Again by Lemma 3.4 we have that $a + 2d \in R_i(U)$. $\qquad\square$

**Lemma 3.6** *Let* $n, q, d \in \mathbb{N}$ *with* $d \leq q$ *and assume that* $x_0, x_1, \ldots, x_n \in \mathbb{N}$ *and* $(x_0, x_1, \ldots, x_n)_{6qd,d} \subseteq \mathbb{N}$. *If* $m \in \omega$, $p, c \in \mathbb{N}$ *with* $c \leq p$, $y_0, y_1, \ldots, y_m \in \mathbb{N}$, *and* $Y = (y_0, y_1, \ldots, y_m)_{p,c} \subseteq (x_0, x_1, \ldots, x_n)_{q,d} = X$, *then for each* $i \in \{0, 1, \ldots, m\}$ *there exists* $j \in \{0, 1, \ldots, n\}$ *such that* $R_i(Y) \subseteq R_j(X)$.

**Proof**: We proceed by induction on $m$. The case $m = 0$ is trivial, so assume that $m \in \mathbb{N}$ and the assertion is true for $m-1$. Let $i \in \{0, 1, \ldots, m\}$. If $i > 0$, then $R_i(Y) = R_{i-1}((y_1, y_2, \ldots, y_m)_{p,c})$ so the conclusion holds by the induction hypothesis. So assume that $i = 0$. Let $D = R_0((y_0, y_1, \ldots, y_{m-1})_{p,c})$ and pick by the induction hypothesis some $j \in \{0, 1, \ldots, n\}$ such that $D \subseteq R_j(X)$. Now let $\lambda_1, \lambda_2, \ldots, \lambda_m \in [-p, p]$. If $\lambda_m = 0$, then $cy_0 + \sum_{l=1}^{m} \lambda_l y_l \in D \subseteq R_j(X)$, so assume $\lambda_m \neq 0$. Then $E = \{cy_0 + \sum_{l=1}^{m-1} \lambda_l y_l - \lambda_m y_m, cy_0 + \sum_{l=1}^{m-1} \lambda_l y_l, cy_0 + \sum_{l=1}^{m-1} \lambda_l y_l + \lambda_m y_m\}$ is a three term arithmetic progression in $X$ so is contained $R_k(X)$ for some $k$ by Lemma 3.5. Since $D \cap E \neq \emptyset$, $k = j$. $\qquad\square$

**Lemma 3.7** *Let* $n, q, d \in \mathbb{N}$ *with* $d \leq q$ *and assume that* $x_0, x_1, \ldots, x_n \in \mathbb{N}$ *and* $(x_0, x_1, \ldots, x_n)_{6qd,d} \subseteq \mathbb{N}$. *If* $m \in \omega$, $p, c \in \mathbb{N}$ *with* $c \leq p$, $y_0, y_1, \ldots, y_m \in \mathbb{N}$, *and* $Y = (y_0, y_1, \ldots, y_m)_{p,c} \subseteq (x_0, x_1, \ldots, x_n)_{q,d} = X$, *then* $Y \preceq X$.

**Proof**: Let $i \in \{0, 1, \ldots, m-1\}$ and pick by Lemma 3.6 $j, k \in \{0, 1, \ldots, n\}$ such that $R_i(Y) \subseteq R_j(X)$ and $R_{i+1}(Y) \subseteq R_k(X)$. We show that $j < k$. Pick $\lambda_{j+1}, \lambda_{j+2}, \ldots, \lambda_n, \mu_{j+1}, \mu_{j+2}, \ldots, \mu_n,$ $\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_n \in [-q, q]$ such that

$$
\begin{aligned}
cy_i &= dx_j + \sum_{l=j+1}^{n} \lambda_l x_l\,, \\
cy_i + cy_{i+1} &= dx_j + \sum_{l=j+1}^{n} \mu_l x_l, \text{ and} \\
cy_{i+1} &= dx_k + \sum_{l=k+1}^{n} \gamma_l x_l\,.
\end{aligned}
$$

Then $cy_{i+1} = \sum_{l=j+1}^{n}(\mu_l - \lambda_l)x_l$, so by Lemma 3.4 $k \geq j + 1$. $\qquad\square$

We shall refer later to the conclusion of the following theorem by stating that "all inclusions in $(x_0, x_1, \ldots, x_n)_{q,d}$ are natural".

**Theorem 3.8** *Let* $n, q, d \in \mathbb{N}$ *with* $d \leq q$ *and assume that* $x_0, x_1, \ldots, x_n \in \mathbb{N}$ *and* $(x_0, x_1, \ldots, x_n)_{6qd,d} \subseteq \mathbb{N}$. *If* $m, M \in \omega$, $p, c, P, C \in \mathbb{N}$ *with* $c \leq p$ *and* $C \leq P$, $y_0, y_1, \ldots, y_m, z_0,$ $z_1, \ldots, z_M \in \mathbb{N}$, $Y = (y_0, y_1, \ldots, y_m)_{p,c} \subseteq (x_0, x_1, \ldots, x_n)_{q,d} = X$, $Z = (z_0, z_1, \ldots, z_M)_{P,C} \subseteq X$, *and* $Y \subseteq Z$, *then* $Y \preceq Z$.

**Proof**: By Lemma 3.7 we have that $Y \preceq X$ and $Z \preceq X$. We show first that for each $i \in \{0, 1, \ldots, m\}$ there exists $j \in \{0, 1, \ldots, M\}$ such that $R_i(Y) \subseteq R_j(Z)$. Suppose instead that one has $i \in \{0, 1, \ldots, m\}$, $a, b \in R_i(Y)$, and $j < k$ in $\{0, 1, \ldots, M\}$ such that $a \in R_j(Z)$ and $b \in R_k(Z)$. Since $Z \preceq X$, we have $u < v$ in $\{0, 1, \ldots, n\}$ such that $a \in R_u(X)$ and $b \in R_v(X)$. By Lemma 3.5, $R_u(X) \cap R_v(X) = \emptyset$ and by Lemma 3.6, $R_i(Y) \subseteq R_u(X)$, a contradiction.

Now let $i \in \{0, 1, \ldots, m-1\}$ and pick $j, k \in \{0, 1, \ldots, M\}$ such that $R_i(Y) \subseteq R_j(Z)$ and $R_{i+1}(Y) \subseteq R_k(Z)$. Pick $u < v$ in $\{0, 1, \ldots, n\}$ such that $R_i(Y) \subseteq R_u(X)$ and $R_{i+1}(Y) \subseteq R_v(X)$. Then $R_j(Z) \cap R_u(X) \neq \emptyset$ so $R_j(Z) \subseteq R_u(X)$. Likewise $R_k(Z) \subseteq R_v(X)$, and therefore $j < k$. $\square$

We shall need a slightly strengthened version of Deuber's Theorem (Theorem 1.3).

**Theorem 3.9** *For every $m \in \omega$, every $p, c \in \mathbb{N}$ with $c \leq p$ and every $k \in \mathbb{N}$, there exists $n, q, d \in \mathbb{N}$ with $d \leq q$ so that for every $(n, q, d)$-set $X$ and every coloring $\rho : X \to [1, k]$, there exists a monochromatic $(m, p, c)$-set naturally contained in $X$.*

**Proof**: Pick $n', q', d'$ as guaranteed by Theorem 1.3. Let $n = n'$, $q = 6q'd'$, and $d = d'$. Let $X = (x_0, x_1, \ldots, x_n)_{q,d}$ be $k$-colored. Then $Y = (x_0, x_1, \ldots, x_{n'})_{q',d'}$ is naturally contained in $X$ and by Theorem 1.3 $Y$ contains a monochromatic $(m, p, c)$-set. By Lemma 3.7, this inclusion is natural. $\square$

The following technical lemma completes our preliminaries.

**Lemma 3.10** *Let $m, p, c, M, P, C \in \mathbb{N}$ with $c \leq p$ and $C \leq P$. Let $w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_M \in \mathbb{N}$. If $(w_1, w_2, \ldots, w_m)_{p,c} \subseteq (cv_1, cv_2, \ldots, cv_M)_{P,C}$, then $[w_1, w_2, \ldots, w_m]_p \subseteq [v_1, v_2, \ldots, v_M]_{(2c+p)P}$.*

**Proof**: First consider any $\sum_{j=2}^m \lambda_j w_j$ with each $\lambda_j \in [-p, p]$. Then

$$cw_1 + \sum_{j=2}^m \lambda_j w_j \in (w_1, w_2, \ldots, w_m)_{p,c} \subseteq (cv_1, cv_2, \ldots, cv_M)_{P,C} \subseteq [cv_1, cv_2, \ldots, cv_M]_P$$

and $cw_1 \in [cv_1, cv_2, \ldots, cv_M]_P$, so $\sum_{j=2}^m \lambda_j w_j \in [cv_1, cv_2, \ldots, cv_M]_{2P} \subseteq [v_1, v_2, \ldots, v_M]_{2cP}$. Also $cw_1 \in [cv_1, cv_2, \ldots, cv_M]_P$ so $w_1 \in [v_1, v_2, \ldots, v_M]_P$, so for any $\lambda_1 \in [-p, p]$ one has $\lambda_1 w_1 \in [v_1, v_2, \ldots, v_M]_{pP}$ and thus $\sum_{j=1}^m \lambda_j w_j \in [v_1, v_2, \ldots, v_M]_{pP+2cP}$. $\square$

# 4 Main proof: existence of independent $(m, p, c)$-sets

In the proof of Theorem 1.4, we use the following earlier result.

**Theorem 4.1** *For every $k, n, q, d \in \mathbb{N}$ there exist $n', q', d' \in \mathbb{N}$ so that for any $(n', q', d')$-set $X$ and any $K_k$-free graph $G$ with vertex set $X$, there exists an $(n, q, d)$-set $S$ naturally contained in $X$, each of whose rows is an independent set in $G$.*

The proof of Theorem 4.1 (see [8]) is accomplished by repeating the standard parameter sets proof of Deuber's partition theorem (see, e.g., [12]), once one knows that in a $K_k$-free graph on a large-dimensional Hales-Jewett cube there is always a line (or, more generally, a $d$-dimensional subspace) that is independent— this latter fact is proved in [9].

In view of Theorem 4.1, it is sufficient to prove Theorem 1.4 under the assumption that the graph $G$ on an $(n, q, d)$-set $S$ has all rows as independent sets. Rather than prove Theorem 1.4 with this additional assumption, we will prove a stronger statement, Theorem 4.3, below. Since a large complete $k$-partite graph contains many copies of $K_k$, Theorem 1.4 will clearly follow. This somewhat stronger theorem turns out to be easier to prove.

**Definition 4.2** *Let $k, p, c, t \in \mathbb{N}$ with $c \leq p$ and let $m \in \omega$. Then $\varphi(k, m, p, c, t)$ is the statement "there exist $n, q, d \in \mathbb{N}$ such that whenever $S$ is an $(n, q, d)$-set and $G$ is a graph on $S$ such that the rows of $S$ are independent, there exist either*

*(a) an independent $(m, p, c)$-set contained in $S$ or*

*(b) $z_1, z_2, \ldots, z_k, a_0, a_1, \ldots, a_t \in \mathbb{N}$ such that $(a_0, a_1, \ldots, a_t)_{p,c} \subseteq S$ and the sets $\langle cz_i + [a_0, a_1, \ldots, a_t]_p \rangle_{i=1}^{k}$ form a complete $k$-partite subgraph of $G$."*

**Theorem 4.3** *For all $k, p, c, t \in \mathbb{N}$ with $c \leq p$ and all $m \in \omega$, the statement $\varphi(k, m, p, c, t)$ holds.*

**Proof**: The proof is by induction on $m$ and $k$.

For the base cases, note that for all $m', p, c, t$, part (b) of $\varphi(1, m', p, c, t)$ holds, and for all $k', p, c, t$, part (a) of $\varphi(k', 0, p, c, t)$ holds.

So assume that $k \geq 2$ and $m \geq 1$, and for the induction hypotheses, suppose that for all $m', p, c, t$, statement $\varphi(k - 1, m', p, c, t)$ holds, and for all $p, c, t$, statement $\varphi(k, m - 1, p, c, t)$ holds. We need to show that $\varphi(k, m, p, c, t)$ holds for all $p, c, t$. Pick $p, c, t$ such that part (a) of $\varphi(k, m, p, c, t)$ fails; we show that part (b) holds.

(1) Let $(M, P, C)$ be the $(n, q, d)$ guaranteed by $\varphi(k, m - 1, p, c, t)$.

(2) Let $(N, Q, D)$ be such that whenever an $(N - 1, Q, D)$-set is $(k - 1)$-colored, it naturally contains a monochromatic $(M, (p+2c)P, cC)$-set. (Such $(N, Q, D)$ exist by Theorem 3.9.) We may assume that $N > m$.

(3) Set $Q' = (2cC + (2c + p)P)Q$, $D' = cCD$, and let
$$T = \mathrm{HJ}(t + 1, (2Q' + 1)^{Nk}, (4pQ' + 1)^{N+1}) + N.$$

(4) Put $Q'' = 2cp(Q')^2(N+1)$, $D'' = c^2(D')^2$, and let $(n', q', d')$ satisfy $\varphi(k-1, N, Q'', D'', T)$.

(5) Pick $(n, q, d)$ such that any $(n, q, d)$-set $S'$ naturally contains an $(n', q', d')$-set for which all inclusions are natural. (Such $(n, q, d)$ exists by Theorem 3.8.)

Since part (a) of $\varphi(k, m, p, c, t)$ fails, pick an $(n, q, d)$-set $S'$ and a graph $G'$ on $S'$ for which the rows are independent but $S'$ does not contain an independent $(m, p, c)$-set.

Pick an $(n', q', d')$-set $S$ which is naturally contained in $S'$ and such that all inclusions within $S$ are natural. Let $G$ be the subgraph of $G'$ induced on $S$. Note that $S$ does not contain an independent $(m, p, c)$-set.

**Claim 1** *If $U = (v_0, v_1, \ldots, v_{M+1})_{(p+2c)P, cC} \subseteq S$, then there is an edge of $G$ between a point in the first row of $U$ and a point in some later row of $U$.*

**Proof**: We have $(cv_1, cv_2, \ldots, cv_{M+1})_{P,C} \subseteq U \subseteq S$ so by the choice of $(M, P, C)$, pick an independent $(m-1, p, c)$-set $(w_1, w_2, \ldots, w_m)_{p,c} \subseteq (cv_1, cv_2, \ldots, cv_{M+1})_{P,C}$. (If clause (b) of the definition of $\varphi(k, m-1, p, c, t)$ applied, one would have $\varphi(k, m, p, c, t)$.)

Let $V = (Cv_0, w_1, w_2, \ldots, w_m)_{p,c}$. One has immediately that

$$\bigcup_{i=1}^m R_i(V) = (w_1, w_2, \ldots, w_m)_{p,c} \subseteq (cv_1, cv_2, \ldots, cv_{M+1})_{P,C} \subseteq \bigcup_{i=1}^{M+1} R_i(U) \subseteq S \,.$$

Also, $R_0(V) = cCv_0 + [w_1, w_2, \ldots, w_m]_p \subseteq cCv_0 + [v_1, v_2, \ldots, v_m]_{(2c+p)P} = R_0(U)$, where the inclusion holds by Lemma 3.10.

Now $R_0(V)$ is contained in a row of $S$, so is independent, and

$$\bigcup_{i=1}^m R_i(V) = (w_1, w_2, \ldots, w_m)_{p,c} \,,$$

which is independent, so, since $V$ is not independent, there must be an edge between a point of $R_0(V)$ and a later row of $V$ and hence between a point of $R_0(U)$ and a later row of $U$. $\quad\square$

**Claim 2** *Let $w_1, w_2, \ldots, w_N \in \mathbb{N}$. Recall that $Q' = (2cC + (2c+p)P)Q$ and $D' = cCD$. Assume that for each $i \in \{1, 2, \ldots, k-1\}$, $\beta_i \in \mathbb{N}$ and $(\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'} \subseteq S$. Then there is some $x \in (cCw_1, cCw_2, \ldots, cCw_N)_{Q,D}$ (and therefore some $x \in (w_1, w_2, \ldots, w_N)_{Q',D'}$) such that for each $i \in \{1, 2, \ldots, k-1\}$ there is an edge from $x$ to a point in the first row of $(\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'}$.*

**Proof**: Suppose not and color $x \in (cCw_1, cCw_2, \ldots, cCw_N)_{Q,D}$ by the first $i \in \{1, 2, \ldots, k-1\}$ such that there is no edge from $x$ to a point in the first row of $(\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'}$. By the choice of $(N, Q, D)$ pick $i \in \{1, 2, \ldots, k-1\}$ and $v_0, v_1, \ldots, v_m$ such that $(v_0, v_1, \ldots, v_M)_{(p+2c)P,cC} \subseteq (cCw_1, cCw_2, \ldots, cCw_N)_{Q,D}$ and for each $x \in (v_0, v_1, \ldots, v_M)_{(p+2c)P,cC}$ there is no edge from $x$ to any point in the first row of $(\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'}$.

Let $U = (D\beta_i, v_0, v_1, \ldots, v_M)_{(p+2c)P,cC}$. Now

$$(v_0, v_1, \ldots, v_M)_{(p+2c)P,cC} \subseteq (w_1, w_2, \ldots, w_N)_{cCQ,cCD} \,.$$

We claim that $R_0(U) \subseteq R_0((\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'})$. To see this, let $y = cCD\beta_i + \sum_{l=0}^M \lambda_l v_l$ where each $\lambda_l \in [-(p+2c)P, (p+2c)P]$. By Lemma 3.10,

$$\sum_{l=0}^M \lambda_l v_l \in [w_1, w_2, \ldots, w_N]_{(2cC+(p+2c)P)Q} \,,$$

so $y \in R_0((\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'})$ as claimed.

Thus by Claim 1, there is an edge between a point $y \in R_0(U)$ and some point $x$ in a later row of $U$. But then $x \in (cCw_1, cCw_2, \ldots, cCw_N)_{Q,D}$ and $y \in R_0((\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'})$, a contradiction. $\quad\square$

We now observe that there is no independent $(N, Q'', D'')$-set in $S$. Indeed, assume one has $(x_0, x_1, \ldots, x_N)_{Q'',D''} \subseteq S$. Then, since $m < N$ and $pc(D')^2 < Q''$, one has that

$$(c(D')^2 x_0, c(D')^2 x_1, \ldots, c(D')^2 x_m)_{p,c} \subseteq (x_0, x_1, \ldots, x_N)_{Q'',D''} \,.$$

Since $(c(D')^2 x_0, c(D')^2 x_1, \ldots, c(D')^2 x_m)_{p,c}$ is not independent, neither is $(x_0, x_1, \ldots, x_N)_{Q'',D''}$.

By the choice of $(n', q', d')$, since there is no independent $(N, Q'', D'')$-set in $S$, pick $b_0, b_1, \ldots,$ $b_T, z_1', z_2', \ldots, z_{k-1}'$ in $\mathbb{N}$ such that $(b_0, b_1, \ldots, b_T)_{Q'', D''} \subseteq S$ and the sets

$$\langle D'' z_i' + [b_0, b_1, \ldots, b_T]_{Q''} \rangle_{i=1}^{k-1}$$

form a complete $(k-1)$-partite graph.

Now let $y_0, y_1, \ldots, y_N \in [b_{N+1}, b_{N+2}, \ldots, b_T]_{2pQ'}$ and $i \in \{1, 2, \ldots, k-1\}$, and for $j \in \{0, 1, \ldots, N\}$, pick $\lambda_{j,N+1}, \lambda_{j,N+2}, \ldots, \lambda_{j,T} \in [-2pQ', 2pQ']$ such that $y_j = \sum_{l=N+1}^{T} \lambda_{j,l} b_l$. We claim that the first row of $(c^2 D' z_i' + c^2 D' b_0 + c y_0, c^2 D' b_1 + c y_1, c^2 D' b_2 + c y_2, \ldots, c^2 D' b_N + c y_N)_{Q', D'}$ is contained in $D'' z_i' + [b_0, b_1, \ldots, b_T]_{Q''}$. To see this, let $\mu_1, \mu_2, \ldots, \mu_N \in [-Q', Q']$, so that

$$w = c^2 (D')^2 z_i' + c^2 (D')^2 b_0 + c D' y_0 + \sum_{j=1}^{N} c^2 D' \mu_j b_j + \sum_{j=1}^{N} c \mu_j y_j$$

is a typical member of the first row of $(c^2 D' z_i' + c^2 D' b_0 + c y_0, c^2 D' b_1 + c y_1, c^2 D' b_2 + c y_2, \ldots, c^2 D' b_N + c y_N)_{Q', D'}$. For each $j \in \{0, 1, \ldots, N\}$, the absolute value of the coefficient of $b_j$ in the given expansion of $w$ is at most $c^2 D' Q' < Q''$. And for $l \in \{N+1, N+2, \ldots, T\}$, the absolute value of the coefficient of $b_l$ in the given expansion of $w$ is

$$\left| c D' \lambda_{0,l} + \sum_{j=1}^{N} c \mu_j \lambda_{j,l} \right| \leq c D' 2pQ' + 2pc(Q')^2 N < Q''.$$

Next we claim that $(c^2 D' b_1 + c y_1, c^2 D' b_2 + c y_2, \ldots, c^2 D' b_N + c y_N)_{Q', D'}$ is contained in $(b_0, b_1, \ldots, b_T)_{Q'', D''}$. To see this, let $\mu_1, \mu_2, \ldots, \mu_N \in [-Q', Q']$ such that,

$$\text{if } r = \min\{j \in \{1, 2, \ldots, N\} : \mu_j \neq 0\}, \text{ then } \mu_r = D',$$

and let

$$w = \sum_{j=1}^{N} \mu_j (c^2 D' b_j + c y_j) = \sum_{j=1}^{N} \mu_j c^2 D' b_j + \sum_{l=N+1}^{T} \sum_{j=1}^{N} c \mu_j \lambda_{j,l} b_l.$$

Then $\mu_r c^2 D' = D''$, and for $j \in \{r+1, r+2, \ldots, N\}$, if any, the absolute value of the coefficient of $b_j$ in the given expansion of $w$ is at most $c^2 D' Q' < Q''$. Also, for $l \in \{N+1, N+2, \ldots, T\}$, the absolute value of the coefficient of $b_l$ in the given expansion of $w$ is at most $\sum_{j=r}^{N} c 2p(Q')^2 < Q''$.

In particular, we have established that

$$(c^2 D' z_i' + c^2 D' b_0 + c y_0, c^2 D' b_1 + c y_1, c^2 D' b_2 + c y_2, \ldots, c^2 D' b_N + c y_N)_{Q', D'} \subseteq S,$$

so we may apply Claim 2.

We define $\tau : ([-2pQ', 2pQ']^{N+1})^{T-N} \to ([-Q', Q']^N)^k$ as follows: Let

$$\overline{\lambda} = ((\lambda_{0,N+1}, \lambda_{1,N+1}, \ldots, \lambda_{N,N+1}), (\lambda_{0,N+2}, \lambda_{1,N+2}, \ldots, \lambda_{N,N+2}),$$

$$\ldots, (\lambda_{0,T}, \lambda_{1,T}, \ldots, \lambda_{N,T})) \in ([-2pQ', 2pQ']^{N+1})^{T-N}.$$

For $j \in \{0, 1, \ldots, N\}$, let $y_j = \sum_{l=N+1}^{T} \lambda_{j,l} b_l$. Then by Claim 2 applied to

$$\langle (c^2 D' z_i' + c^2 D' b_0 + c y_0, c^2 D' b_1 + c y_1, c^2 D' b_2 + c y_2, \ldots, c^2 D' b_N + c y_N)_{Q', D'} \rangle_{i=1}^{k-1}$$

there is some point in $(c^2 D'b_1 + cy_1, c^2 D'b_2 + cy_2, \ldots, c^2 D'b_N + cy_N)_{Q',D'}$ with an edge to a point in the first row of each $(c^2 D'z_i' + c^2 D'b_0 + cy_0, c^2 D'b_1 + cy_1, c^2 D'b_2 + cy_2, \ldots, c^2 D'b_N + cy_N)_{Q',D'}$. That is there is some

$$\overline{\gamma} = ((\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{1,N}), (\gamma_{2,1}, \gamma_{2,2}, \ldots, \gamma_{2,N}), \ldots, (\gamma_{k,1}, \gamma_{k,2}, \ldots, \gamma_{k,N})) \in ([-Q,Q]^N)^k$$

such that, if $r = \min\{j \in \{1, 2, \ldots, n\} : \gamma_{k,j} \neq 0\}$, then $\gamma_{k,r} = D'$ and for each $i \in \{1, 2, \ldots, k-1\}$, there is an edge between $\sum_{j=1}^{N} \gamma_{k,j}(c^2 D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l} b_l)$ and

$$c^2(D')^2 z_i' + c^2(D')^2 b_0 + \sum_{l=N+1}^{T} D'c\lambda_{0,l} b_l + \sum_{j=1}^{N} \gamma_{i,j}(c^2 D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l} b_l).$$

Define $\tau(\overline{\lambda}) = \overline{\gamma}$.

Now since $T = HJ(t + 1, (2Q' + 1)^{Nk}, (4pQ' + 1)^{N+1}) + N$, Pick $F, M_0, M_1, \ldots, M_t$, $\langle (\nu_{0,l}, \nu_{1,l}, \ldots, \nu_{N,l}) \rangle_{l \in F}$,

$$\overline{\eta} = ((\eta_{1,1}, \eta_{1,2}, \ldots, \eta_{1,N}), (\eta_{2,1}, \eta_{2,2}, \ldots, \eta_{2,N}), \ldots, (\eta_{k,1}, \eta_{k,2}, \ldots, \eta_{k,N})) \in ([-Q,Q]^N)^k,$$

and $r \in \{1, 2, \ldots, N\}$ such that

(1) $F, M_0, M_1, \ldots, M_t$ are pairwise disjoint;

(2) $F \cup M_0 \cup M_1 \cup \ldots \cup M_t = \{N + 1, N + 2, \ldots, T\}$;

(3) each $M_s \neq \emptyset$ and $\min M_0 < \min M_1 < \ldots < \min M_t$;

(4) for each $l \in F$, $(\nu_{0,l}, \nu_{1,l}, \ldots, \nu_{N,l}) \in [-2pQ', 2pQ']^{N+1}$;

(5) $r = \min\{j \in \{1, 2, \ldots, N\} : \eta_{k,j} \neq 0\}$ and $\eta_{k,r} = D'$; and,

(6) whenever

$$\overline{\lambda} = ((\lambda_{0,N+1}, \lambda_{1,N+1}, \ldots, \lambda_{N,N+1}), (\lambda_{0,N+2}, \lambda_{1,N+2}, \ldots, \lambda_{N,N+2}),$$

$$\ldots, (\lambda_{0,T}, \lambda_{1,T}, \ldots, \lambda_{N,T})) \in ([-2pQ', 2pQ']^{N+1})^{T-N}$$

satisfies

(a) for each $l \in F$, $(\lambda_{0,l}, \lambda_{1,l}, \ldots, \lambda_{N,l}) = (\nu_{0,l}, \nu_{1,l}, \ldots, \nu_{N,l})$ and

(b) for each $s \in \{0, 1, \ldots, t\}$ and each $l, v \in M_s$,
$(\lambda_{0,l}, \lambda_{1,l}, \ldots, \lambda_{N,l}) = (\lambda_{0,v}, \lambda_{1,v}, \ldots, \lambda_{N,v})$

one has $\tau(\overline{\lambda}) = \overline{\eta}$, and consequently, for each $i \in \{1, 2, \ldots, k-1\}$, there is an edge between $\sum_{j=1}^{N} \eta_{k,j}(c^2 D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l} b_l)$ and

$$c^2(D')^2 z_i' + c^2(D')^2 b_0 + \sum_{l=N+1}^{T} D'c\lambda_{0,l} b_l + \sum_{j=1}^{N} \eta_{i,j}(c^2 D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l} b_l).$$

Now for each $s \in \{0, 1, \ldots, t\}$, let $a_s = \sum_{l \in M_s} (D')^2 c b_l$. For $i \in \{1, 2, \ldots, k-1\}$, let

$$z_i = (D')^2 c z_i' + (D')^2 c b_0 + \sum_{l \in F} D' \nu_{0,l} b_l + \sum_{j=1}^{N} D' c \eta_{i,j} b_j + \sum_{l \in F} \sum_{j=1}^{N} \eta_{i,j} \nu_{j,l} b_l,$$

and let

$$z_k = \sum_{j=1}^{N} D' c \eta_{k,j} b_j + \sum_{l \in F} \sum_{j=1}^{N} \eta_{k,j} \nu_{j,l} b_l.$$

We shall show that $(a_0, a_1, \ldots, a_t)_{p,c} \subseteq S$ and that the sets $\langle c z_i + [a_0, a_1, \ldots, a_t]_p \rangle_{i=1}^{k}$ form a complete $k$-partite subgraph of $G$, completing the proof that part (b) of $\varphi(k, m, p, c, t)$ holds.

We show first that $(a_0, a_1, \ldots, a_t)_{p,c} \subseteq (b_0, b_1, \ldots, b_T)_{Q'', D''}$. So let $x \in (a_0, a_1, \ldots, a_t)_{p,c}$, and pick $\beta \in \{0, 1, \ldots, t\}$ and $\mu_{\beta+1}, \mu_{\beta+2}, \ldots, \mu_t \in [-p, p]$ such that

$$x = c a_\beta + \sum_{s=\beta+1}^{t} \mu_s a_s.$$

Then

$$x = \sum_{l \in M_\beta} c^2 (D')^2 b_l + \sum_{s=\beta+1}^{t} \sum_{l \in M_s} \mu_s c (D')^2 b_l.$$

Since $\min M_\beta < \min \bigcup_{s=\beta+1}^{t} M_s$, we have that the leading coefficient in this expansion is $D''$, while all other coefficients are at most $p c (D')^2 < Q''$.

Next we show that for each $i \in \{1, 2, \ldots, k-1\}$,

$$c z_i + [a_0, a_1, \ldots, a_t]_p \subseteq D'' z_i' + [b_0, b_1, \ldots, b_T]_{Q''},$$

and consequently the sets $\langle c z_i + [a_0, a_1, \ldots, a_t]_p \rangle_{i=1}^{k-1}$ form a complete $(k-1)$-partite graph. To this end, let $\mu_0, \mu_1, \ldots, \mu_t \in [-p, p]$. Then

$$c z_i + \sum_{s=0}^{t} \mu_s a_s =$$

$$(D')^2 c^2 z_i' + (D')^2 c^2 b_0 + \sum_{l \in F} c D \nu_{0,l} b_l + \sum_{j=1}^{N} D' c^2 \eta_{i,j} b_j + \sum_{l \in F} \sum_{j=1}^{N} c \eta_{i,j} \nu_{j,l} b_l + \sum_{s=0}^{t} \sum_{l \in M_s} \mu_s (D')^2 c b_l.$$

The coefficient on $z_i'$ in this expansion is $D''$ while the coefficients on the $b_l$'s have absolute value at most $2 p c (Q')^2 < Q''$.

Finally we let $i \in \{1, 2, \ldots, k-1\}$, let $w \in c z_i + [a_0, a_1, \ldots, a_t]_p$, let $x \in c z_k + [a_0, a_1, \ldots, a_t]_p$, and show that there is an edge between $w$ and $x$. Pick $\alpha_0, \alpha_1, \ldots, \alpha_s, \delta_0, \delta_1, \ldots, \delta_s \in [-p, p]$ such that $w = c z_i + \sum_{s=0}^{t} \alpha_s a_s$ and $x = c z_k + \sum_{s=0}^{t} \delta_s a_s$. Then

$$w = (D')^2 c^2 z_i' + (D')^2 c^2 b_0 + \sum_{l \in F} c D' \nu_{0,l} b_l + \sum_{j=1}^{N} D' c^2 \eta_{i,j} b_j + \sum_{l \in F} \sum_{j=1}^{N} c \eta_{i,j} \nu_{j,l} b_l + \sum_{s=0}^{t} \sum_{l \in M_s} \alpha_s (D')^2 c b_l,$$

and

$$x = \sum_{j=1}^{N} D' c^2 \eta_{k,j} b_j + \sum_{l \in F} \sum_{j=1}^{N} c\eta_{k,j}\nu_{j,l}b_l + \sum_{s=0}^{t} \sum_{l \in M_s} \delta_s (D')^2 cb_l.$$

For $l \in F$ and $j \in \{0, 1, \ldots, N\}$, let $\lambda_{j,l} = \nu_{j,l}$. For $s \in \{0, 1, \ldots, t\}$ and $l \in M_s$, let $\lambda_{r,l} = D'\delta_s$, let $\lambda_{0,l} = D'\alpha_s - \eta_{i,r}\delta_s$, and for $j \in \{1, 2, \ldots, N\} \setminus \{r\}$, let $\lambda_{j,l} = 0$. Note that each $|\lambda_{j,l}| \leq 2pQ'$. Note also that for $s \in \{0, 1, \ldots, t\}$ and $l \in M_s$,

$$\sum_{j=1}^{N} \eta_{k,j}\lambda_{j,l} = (D')^2 \delta_s$$

and

$$D'c\lambda_{0,l} + \sum_{j=1}^{N} \eta_{i,j}\lambda_{j,l}c = (D')^2 c\alpha_s.$$

Then

$$\overline{\lambda} = ((\lambda_{0,N+1}, \ldots, \lambda_{N,N+1}), (\lambda_{0,N+2}, \ldots, \lambda_{N,N+2}), \ldots, (\lambda_{0,T}, \ldots, \lambda_{N,T})) \in ([-2pQ', 2pQ']^{N+1})^{T-N}$$

satisfies

(a) for each $l \in F$, $(\lambda_{0,l}, \lambda_{1,l}, \ldots, \lambda_{N,l}) = (\nu_{0,l}, \nu_{1,l}, \ldots, \nu_{N,l})$ and

(b) for each $s \in \{0, 1, \ldots, t\}$ and each $l, v \in M_s$, $(\lambda_{0,l}, \lambda_{1,l}, \ldots, \lambda_{N,l}) = (\lambda_{0,v}, \lambda_{1,v}, \ldots, \lambda_{N,v})$.

Consequently, there is an edge between $\sum_{j=1}^{N} \eta_{k,j}(c^2 D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l}b_l)$ and

$$c^2(D')^2 z_i' + c^2(D')^2 b_0 + \sum_{l=N+1}^{T} D'c\lambda_{0,l}b_l + \sum_{j=1}^{N} \eta_{i,j}(c^2 D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l}b_l).$$

Now

$$\sum_{j=1}^{N} \eta_{k,j}(c^2 D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l}b_l)$$

$$= \sum_{j=1}^{N} \eta_{k,j}c^2 D'b_j + \sum_{l \in F} \sum_{j=1}^{N} \eta_{k,j}\nu_{j,l}cb_l + \sum_{s=0}^{t} \sum_{l \in M_s} \sum_{j=1}^{N} \eta_{k,j}\lambda_{j,l}cb_l$$

$$= \sum_{j=1}^{N} \eta_{k,j}c^2 D'b_j + \sum_{l \in F} \sum_{j=1}^{N} \eta_{k,j}\nu_{j,l}cb_l + \sum_{s=0}^{t} \sum_{l \in M_s} c(D')^2 \delta_s b_l$$

$$= x.$$

Also,

$$c^2(D')^2 z_i' + c^2(D')^2 b_0 + \sum_{l=N+1}^{T} D'c\lambda_{0,l}b_l + \sum_{j=1}^{N} \eta_{i,j}(c^2 D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l}b_l)$$

$$= c^2(D')^2 z_i' + c^2(D')^2 b_0 + \sum_{j=1}^{N} \eta_{i,j} c^2 D' b_j + \sum_{l \in F} (D' c \lambda_{0,l} + \sum_{j=1}^{N} \eta_{i,j} \lambda_{j,l} c) b_l$$

$$+ \sum_{s=0}^{t} \sum_{l \in M_s} (D' c \lambda_{0,l} + \sum_{j=1}^{N} \eta_{i,j} \lambda_{j,l} c) b_l$$

$$= c^2(D')^2 z_i' + c^2(D')^2 b_0 + \sum_{j=1}^{N} \eta_{i,j} c^2 D' b_j + \sum_{l \in F} (D' c \nu_{0,l} + \sum_{j=1}^{N} \eta_{i,j} \nu_{j,l} c) b_l + \sum_{s=0}^{t} \sum_{l \in M_s} (D')^2 c \alpha_s b_l$$

$$= w.$$

$\square$

# 5 Independent arithmetic progressions, revisited

The following (in a more general form) was originally proved in [9] by application of the Hales-Jewett theorem. To illustrate a different approach in a special case, we now give a different proof, this time using Szemerédi's density theorem for arithmetic progressions [18] (which says that for any $m$ and $\epsilon > 0$ there exists an $n$ so that any set of $\epsilon n$ elements from $[1, n]$ contains an $m$-term arithmetic progression).

**Theorem 5.1 ([9])** *Fix $k$ and $\ell$. If $G$ is a $K_k$-free graph on $\mathbb{N}$, then there exists an $\ell$-term arithmetic progression which spans an independent set in $G$.*

**Proof:** Denote by $S(k, \ell)$ the following statement: There exists an integer $n = n(k, \ell)$ such that for every $K_k$-free graph $G$ whose vertex set is an arithmetic progression of length $n$, there exists an arithmetic progression of length $\ell$ which is an independent set in $G$.

If for every $k$ and $\ell$, $S(k, \ell)$ holds, then Theorem 5.1 follows. For each fixed $\ell$ we will prove $S(k, \ell)$ by induction on $k$.

Observe that $S(2, \ell)$ is trivially true with $n(2, \ell) = \ell$. Suppose, therefore, that $S(k-1, \ell)$ holds and set $n^* = n(k-1, \ell)$. Let $n = n(k, \ell)$ be very large and consider a $K_k$-free graph $G$ with vertex set $\{a, a+d, \ldots, a+(n-1)d\}$. Assume that $G$ contains no independent set which is an arithmetic progression of length $\ell$. Also, observe that $V(G)$ contains

$$(n - (\ell - 1)) + (n - 2(\ell - 1)) + \ldots + \left( n - \left\lfloor \frac{n-1}{\ell - 1} \right\rfloor (\ell - 1) \right) \geq \frac{n^2}{3\ell}$$

arithmetic progressions of length $\ell$. Since each of these arithmetic progressions contains an edge and each edge is contained in at most $\binom{\ell}{2}$ arithmetic progressions of length $\ell$ the graph $G$ contains at least

$$\frac{n^2}{3\ell} \binom{\ell}{2}^{-1} \geq \frac{2n^2}{3\ell^3}$$

edges. This means that there exists a vertex $x$ joined to at least $\frac{4n}{3\ell^3}$ other vertices. Now if $n$ is sufficiently large compared to $n^*$, we may infer by Szemerédi's theorem that the neighborhood of $x$ contains an arithmetic progression $Y$ of length $n^*$. Since $\{x, y\} \in E(G)$ for every $y \in Y$ the subgraph $G[Y]$ of $G$ induced by $Y$ does not contain $K_{k-1}$. Thus, by the induction assumption

$S(k - 1, \ell)$, the set $Y$ contains an arithmetic progression of length $\ell$ which is an independent set in $G[Y]$ and hence also in $G$. □

We note that since an early draft of this paper, J. Solymosi (personal communication) has independently observed a similar proof of Theorem 5.1.

# 6  Concluding remarks

# References

[1] J. P. Bode, H. D. Gronau, and H. Harborth, *Some Ramsey Schur numbers*, manuscript, 2002.

[2] W. Deuber, Partitionen und lineare Gleichungssysteme, *Math. Z.* **133** (1973), 109–123.

[3] W. Deuber, D. S. Gunderson, N. Hindman, and D. Strauss, Independent finite sums for $K_m$-free graphs, *J. Combin. Th. Ser. A* **78** (1997), 171–198.

[4] P. Erdős, personal communication with DSG, CERN, Luminy, Marseilles, France, September 4, 1995.

[5] P. Erdős, A. Hajnal, and J. Pach, On a metric generalization of Ramsey's theorem, *Israel J. Math* **101** (1997), 283–295.

[6] P. Frankl, R. L. Graham, and V. Rödl, Quantitative theorems for regular systems of equations, *J. Combin. Th. Ser. A* **47** (1988), 246–261.

[7] R. L. Graham, B. L. Rothschild, and J. Spencer, *Ramsey theory*, 2nd ed., J. Wiley & Sons, New York, 1990.

[8] D. S. Gunderson, On Deuber's partition theorem for $(m, p, c)$-sets, *Ars Combin.* **63** (2002), 15–31.

[9] D. S. Gunderson, I. Leader, H. J. Prömel, and V. Rödl, Independent arithmetic progressions in clique-free graphs on the natural numbers, *J. Combin. Th. Ser. A*, **93** (2001), 1–17.

[10] A. W. Hales and R. I. Jewett, Regularity and positional games, *Trans. Amer. Math. Soc.* **106** (1963), 222–229.

[11] H. Harborth, Triangle-free labelled graphs and a Schur property, in "Paul Erdős and mathematics (Budapest, July 4–11, 1999)", 102–103, *Janos Bolyai Math. Soc.*, Budapest, 1999.

[12] K. Leeb, A full Ramsey theorem for the Deuber category, in *Infinite and finite sets II* (A. Hajnal, R. Rado, V.Sós, eds.) (*Colloq. Math. Soc. János Bolyai* **10**), North Holland, Amsterdam, 1975, 1043-1049.

[13] T. Łuczak, V. Rödl, and T. Schoen, Independent finite sums in graphs defined on the natural numbers, *Discrete Math.* **181** (1998), 289–294.

[14] H. J. Prömel and B. Voigt, Graham-Rothschild parameter sets, in *Mathematics of Ramsey Theory*, (J. Nešetřil, V. Rödl, eds.) Springer-Verlag, 1990, 113–149.

[15] R. Rado, Studien zur Kombinatorik, *Math. Z.* **36** (1933), 424–480.

[16] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* **30** (1930), 264–286.

[17] I. Schur, Über die Kongruenz $x^m + y^m \equiv z^m \pmod{p}$, *Jber. Deutsch. Math. Verein* **25** (1916), 114–116.

[18] E. Szemerédi, On sets of integers no $k$ elements in arithmetic progressions, *Acta Arithmetica* **27** (1975), 199–245.

[19] B. L. van der Waerden, Beweis einer Baudetschen Vermutung, *Nieuw. Arch. Wisk.* **15** (1927), 212–216.