Cyclic polytopes, hyperplanes, and codes

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Abstract

We consider cyclic d-polytopes P that are realizable with vertices on the moment curve $M_d : t \longrightarrow (t, t^2, \ldots, t^d)$ of order $d \geq 3$. A hyperplane H bisects a j-face of P if H meets its relative interior. For $\ell \geq 1$, we investigate the maximum number of vertices that P can have so that for some ℓ hyperplanes, each j-face of P is bisected by one of the hyperplanes. For $\ell > 1$, the problem translates to the existence of certain codes, or equivalently, certain paths on the cube $\{0, 1\}^\ell$.

1 Preliminaries

We assume familiarity with basic properties of convex polytopes and refer the reader to [7] for definitions and terminology.

For a set $X \subset \mathbb{R}^d$, the convex hull of X is denoted by $[X]$. Recall that a (convex) *d-polytope* is the convex hull of some finite subset of $\mathbb{R}^d$ of affine dimension d. Recall also that a hyperplane in R^d is a $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$. For any polytope $P \subset \mathbb{R}^d$ a *face* of P is either the empty set, P itself, or the intersection of P with a supporting hyperplane. A j-face is a face with affine dimension j. The 0-faces are vertices, and $(d-1)$-faces are *facets*. Let $V(P)$ denote the set of vertices of P.

For each $j = -1, 0, 1, \ldots, d-2$, each j-face of a d-polytope is contained in a $(j+1)$-face, and hence the faces of a polytope P are ordered by inclusion, yielding the *face lattice* of P. Two polytopes are called *combinatorially equivalent* if, and only if, they have isomorphic face lattices. A polytope is *simplicial* if each of its faces is a simplex. A d-polytope is *neighbourly* if every set of at most $\lfloor d/2 \rfloor$ vertices determines a face.

For a positive integer d, the set

$$M_d = \{(t, t^2, \ldots, t^d) : t \in \mathbb{R}\} \subset \mathbb{R}^d$$

is called the standard *moment curve* (of order d).

Denote points on M_d by $x(t) = (t, t^2, \ldots, t^d)$. If $n \geq d+1$, and $t_1 < t_2 < \cdots < t_n$ are real numbers, then define the d-polytope

$$C_d(t_1, t_2, \ldots, t_n) = [x(t_1), x(t_2), \ldots, x(t_n)].$$

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If P is such a polytope for some choice of $t_1 < t_2 < \cdots < t_n$, whenever we write the vertex set of P in a list, say $V(P) = \{x_1, x_2, \ldots, x_n\}$, we will always understand that the labelling of the x_i's respects the ordering of the t_i's; that is, if for each i, $x_i = x(t_i)$, then $t_i < t_j$ if, and only if, $i < j$. It is well known that for any choice of $t_1 < t_2 < \cdots < t_n$ and $s_1 < s_2 < \cdots < s_n$, the two moment curve polytopes $C_d(t_1, t_2, \ldots, t_n)$ and $C_d(s_1, s_2, \ldots, s_n)$ are combinatorially equivalent. Hence, the notation $C(n, d)$ will denote any element of the equivalence class of moment curve d-polytopes with n vertices.

Definition 1.1 *A d-polytope P with n vertices is* cyclic *if P is combinatorially equivalent to $C(n, d)$.*

So every cyclic polytope has a natural linear ordering of its vertices given by some $C(n, d)$.

We now review some properties of cyclic polytopes (see [7]). Cyclic polytopes are simplicial and neighbourly. McMullen [9] proved that among all d-polytopes with n vertices, for each $j = 1, \ldots, d-1$, neighbourly d-polytopes have the maximum number of j-faces.

A polytope P is said to satisfy *Gale's Evenness Condition* (GEC) with respect to a linear ordering of vertices $v_1 < v_2 < \cdots < v_n$ if for any $X \subset V(P)$, $[X]$ is a facet of P if, and only if, $|X| = d$ and any two vertices in $V(P) \backslash X$ have an even number of vertices in X between them.

Theorem 1.2 (Gale [4]) *Any cyclic polytope satisfies GEC with respect to the natural linear ordering of vertices.*

Theorem 1.3 (Shemer [10]) *If d is even, any d-subpolytope of a cyclic d-polytope is also cyclic.*

In odd dimensions, however, subpolytopes of a cyclic d-polytope need not be cyclic (for example, see [3]). Two polytopes are *geometrically equivalent* if all respective subpolytopes are combinatorially equivalent.

Theorem 1.4 (Sturmfels [12]) *A d-polytope P and each of its d-subpolytopes is cyclic if, and only if, P is geometrically equivalent to some moment curve polytope.*

So, the moment curve can be used to thoroughly analyze any cyclic polytope whose every subpolytope is cyclic.

Theorem 1.2 characterizes when d vertices of a cyclic polytope form a facet; we now prepare to give a condition for faces in general. Given an ordered set of vertices $V = \{x_1, x_2, \ldots, x_n\}$, we say $Y \subset V$ is *contiguous* if $Y = \{x_i, x_{i+1}, \ldots, x_j\}$ for some i and j satisfying $1 < i \leq j < n$ and $Y \cap \{x_{i-1}, x_{j+1}\} = \emptyset$. A set $Z \subset V$ is called an *end set* if either for some i, $Z = \{x_1, \ldots, x_i\}$ and $x_{i+1} \notin Z$, or for some j, $Z = \{x_j, \ldots, x_n\}$ and $x_{j-1} \notin Z$. Any $V' \subset V$ has a unique decomposition

$$V' = Z_1 \cup Y_1 \cup \cdots \cup Y_a \cup Z_2,$$

where each Z_i is an end set or is empty, and each Y_i is contiguous. We say that a contiguous set $Y_i \subset V$ is even [odd] if $|Y_i|$ is even [resp. odd].

Theorem 1.5 (Shephard [11]) *Let P be a cyclic d-polytope with linearly ordered vertex set $V(P)$ and let $0 \leq j \leq d-1$. Let $V' \subset V(P)$, $|V'| = j+1$, and write V' as the disjoint union of end sets and contiguous sets*

$$V' = Z_1 \cup Y_1 \cup Y_2 \cup \cdots \cup Y_n \cup Z_2.$$

Then $[V']$ is a j-face of P if, and only if, at most $d-j-1$ of the contiguous sets Y_i are odd.

Note that in the case $j = d-1$, Shephard's theorem implies Gale's Evenness Condition.

2 j-bisectors

For a d-polytope P, a hyperplane H is called a *j-bisector* of P if H intersects the relative interior of every j-face of P. For fixed d and j, what is the maximum number of vertices $f(d,j)$ so that a d-polytope with $f(d,j)$ vertices has a j-bisector? We answer this question precisely for those cyclic polytopes with vertices on a moment curve.

Bezdek *et al.* (see, e.g., [1], p.40) noted the following:

Theorem 2.1 *For each j satisfying $0 \leq j \leq \lfloor \frac{d}{2} \rfloor$, a d-polytope does not have a j-bisector.*

Hence, in the search for j-bisectors, we restrict ourselves to the cases $\lfloor \frac{d}{2} \rfloor < j < d$.

Theorem 2.2 *Let $P = [V]$ be a cyclic d-polytope whose every subpolytope is cyclic, and let $d \in \{2m, 2m+1\}$ and $m < j < 2m$. If $|V| \leq 4j - 2m$, then P has a j-bisector.*

Proof: Since all subpolytopes of P are cyclic, it suffices to prove the assertion $P = [V]$ with $|V| = 4j - 2m = 2(2j-m)$, and by Theorem 1.4, we may assume that $V \subset M_d$.

We need to exhibit a hyperplane H which cuts every j-face of P. There are two approaches one could take here; we could first identify vertices of P which are on M_d, and then specify the relative position of points $h_1, h_2, \ldots, h_d$ on $M_d \backslash V$, which determine H, or, we could fix the points $h_1, h_2, \ldots, h_d$ first, and then show the position of the vertices of P relative to these h_i's. We adopt the latter approach here, since it is easier to see the patterns and to perform the counting.

Remark: We show later in Lemma 4.4 that the pattern used in this proof is just one of many that provide the same bound; the pattern here is particularly easy to describe and proving that it works is straightforward.

Fix points $h_1 = x(s_1), h_2 = x(s_2), \ldots, h_d = x(s_d)$, where $s_1 < s_2 < \cdots < s_d$. Let H be the unique hyperplane containing these h_i's. Then $H \cap M_d = \{h_1, h_2, \ldots, h_d\}$ and we note that M_d passes through H at each point of contact. To make this concept precise, let H^+ and H^- be the open half-spaces determined by H. Then $M_d \backslash H$ is the union of $d+1$ disjoint open arcs. If $r < t$ and $(r,t) \cap \{s_1, \ldots, s_d\} = \{s_j\}$, then $x(r)$ and $x(t)$ are in opposite half-spaces. Let $d = 2m$ and refer to Figure 1 for the decomposition of $M_d \backslash H$ into arcs. Without loss, the A_i's are the arcs in $M_d \cap H^+$.

We place the vertices of P as follows: for each $i = 1, 2, \ldots, 2m - j$, $|A_i \cap V(P)| = 1$, for each $i = 2m - j + 1, \ldots, m$, $|A_i \cap V(P)| = 3$, for $i = 1, 2, \ldots, j - m$, $|B_i \cap V(P)| = 3$, and for $i = j - m + 1, \ldots, m$, $|B_i \cap V(P)| = 1$, and finally, leave B_0 empty. (Figure 1 indicates vertices of P in H^+.)

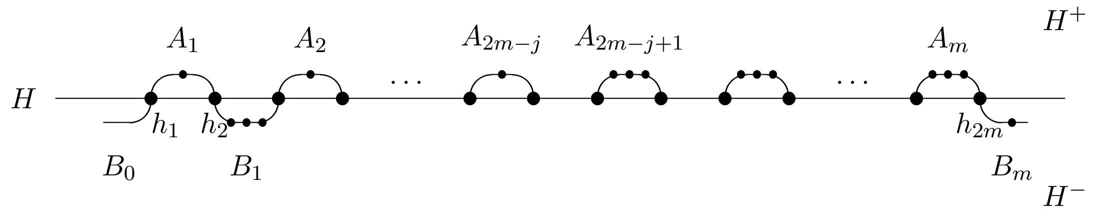

Figure 1: The m arcs of the moment curve in H^+, $d = 2m$.

Now let V' be a set of $j + 1$ points in $V \cap H^+$; we show that V' is not the vertex set of any j-face. Since any contiguous set in V' must be contained in some arc A_i, it has 1,2 or 3 vertices.

Let n_1 be the number of (odd) contiguous sets in V' which are singletons, n_2 the number of even contiguous sets, and n_3 the number of (odd) contiguous sets with 3 points. If there are any end sets, there can only be one, namely the singleton in A_1 (because B_m is non-empty).

Since contiguous sets with pairs or triples come from $A_{2m-j+1}, \ldots, A_m$, we know that $n_2 + n_3 \leq m - (2m - j) = j - m$. Also, $j + 1 = n_1 + 2n_2 + 3n_3 + \epsilon$, where ϵ is 1 if V' has a non-empty end set and 0 otherwise. Then the number of odd contiguous sets is

$$\begin{aligned} n_1 + n_3 &= j + 1 - \epsilon - 2n_2 - 2n_3 \\ &= j + 1 - \epsilon - 2(n_2 + n_3) \\ &\geq j + 1 - \epsilon - 2(j - m) \\ &= 2m - j + 1 - \epsilon \\ &> 2m - j - 1, \end{aligned}$$

and by Shephard's theorem, $[V']$ is not be the vertex set of a j-face of P.

Let V' be a set of $j + 1$ vertices in H^-, and note that the only possible end set is the singleton in B_m. Using the notation analogous for H^+, we again have $n_2 + n_3 \leq j - m$, and $j + 1 = n_1 + 2n_2 + 3n_3 + \epsilon$. The same calculation as above shows that $[V']$ is not a j-face of P.

Since no j-face of P can lie entirely on one side of H, H is a j-bisector. There are $2j - m$ vertices of P in H^+, and $2j - m$ in H^-.

Now let $d = 2m + 1$, and refer to Figure 2 for the decomposition of $M_d \backslash H$ into arcs. Distribute the vertices of P as follows: two vertices in each of A_0 and B_{m+1}, one vertex in each of $A_1, A_2, \ldots, A_{2m-j+1}$, $B_1, \ldots, B_{2m-j+1}$, and three in each of $A_{2m-j+2}, \ldots, A_m$, $B_{2m-j+2}, \ldots, B_m$.

The calculations are just as in the even case. First note that there are $2m - j$ vertices in each open half-space. Let V' be $j + 1$ vertices in $H^+ \cap M_d$. Using the same notation as

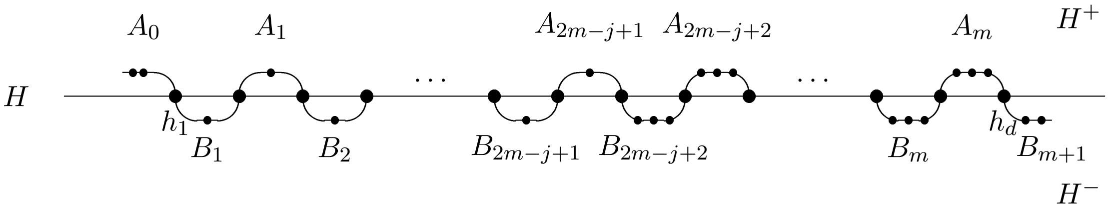

Figure 2: The $m + 1$ arcs of M_d in H^+, $d = 2m + 1$.

in the even case, $n_2 + n_3 \le m - (2m - j + 1) = j - m - 1$ and $j + 1 = n_1 + 2n_2 + 3n_3 + \epsilon$, where $\epsilon \in \{0, 1, 2\}$ is the number of vertices in an end set. These equations imply that the number of odd components is $n_1 + n_3 > d - j - 1$, and so $[V']$ is not a j-face in P.

An argument identical to that for H^+ shows there is no j-face in $H^- \cap M_d$. □

We shall soon see that the value $4j - 2m$ is optimal in every case but one, so let us take care of that rogue case first.

Theorem 2.3 *Let $d = 2m + 1$ and let P be a cyclic d-polytope. Then P has a $2m$-bisector.*

Proof: Let $V(P) = \{x_1, x_2, \ldots, x_n\}$ be given with the natural ordering. First we claim that every facet of P contains at least one of x_1 or x_n. To this end, suppose that $S \subset V(P)\backslash\{x_1, x_n\}$ and $|S| = d = 2m + 1$. Then S can not contain any end sets, and therefore must contain at least one odd contiguous set. But $d - j - 1 = 0$ for $j = d - 1$, and so by Shephard's theorem, S is not the vertex set of a facet.

Since $[x_1, x_n]$ is an edge of P (by Shephard's theorem, for example), there is a hyperplane supporting this edge, that is, there is a hyperplane H which contains both x_1 and x_n, with all other points of P on the same side of H. Translate H slightly toward the rest of these points, thereby cutting every facet. □

In the remaining cases, the value $4j - 2m$ is optimal:

Theorem 2.4 *Let P be a cyclic d-polytope whose every subpolytope is cyclic, $d \in \{2m, 2m + 1\}$, and $m < j < 2m$. If $|V(P)| > 4j - 2m$ vertices, then P has no j-bisector.*

Instead of proving Theorem 2.4, we prove a slightly stronger theorem.

Theorem 2.5 *Let P be a cyclic d-polytope with vertex set $V(P) \subset M_d$, $d \in \{2m, 2m+1\}$ and $m < j < 2m$. Let H be a hyperplane such that $H \cap V(P) = \emptyset$. If one of the open half spaces H^+ or H^- contains more than $2j - m$ vertices of P, then that half space contains a j-face of P.*

Proof: Let $|H^+ \cap V(P)| \ge 2j - m + 1$. Since M_d is of order d, $|H \cap M_d| \le d$. If d is even, either one of $H^+ \cap M_d$ or $H^- \cap M_d$ has at most m arcs and the other has at most $m + 1$ arcs. If d is odd, then both half spaces contain at most $m + 1$ arcs of M_d. So, assume that $H^+ \cap M_d$ consists of at most $m + 1$ disjoint arcs.

Let $W = H^+ \cap V(P)$, and for some $k \leq m+1$, write $W = \cup_{i=1}^{k} W_i$ where the W_i's are ordered by the natural order induced by M_d, and each W_i is either a contiguous set or an end set. Since $2j - m + 1 = j + 1 + (j - m) \geq j + 2$, we have that $|W| \geq j + 2$ and so W is a candidate for containing the vertex set of a j-face. Let γ be the number of odd contiguous sets in W.

Case 1. $\gamma \leq d - j - 1$. We need only to find $W' \subset W$ with $|W'| = j + 1$ which has no more contiguous odd sets than does W. To do this, we delete vertices from odd contiguous sets, or (in pairs) from even contiguous sets.

Case 1a. Suppose that $\gamma \geq |W| - (j+1)$. From each of $|W| - (j+1)$ odd contiguous W_i's, delete an endpoint (so as not to produce two odd contiguous sets from one). The resulting set W' has $j+1$ points and at most $\gamma \leq d - j - 1$ odd contiguous sets, and so by Shephard's theorem, $[W']$ is a j-face of P.

Case 1b. Now suppose that $\gamma < |W| - (j+1)$. Reduce each of the γ odd contiguous sets by one, leaving W', a set with $|W'| = |W| - \gamma > j + 1$ vertices and no odd contiguous sets.

If $|W'| - (j+1)$ is even, then delete as many pairs from the ends of contiguous sets so as to be left with a $(j+1)$-set W'' with no odd contiguous sets. In this case, by Shephard's theorem, $[W'']$ is a j-face of P.

Examine now the case when $|W'| - (j+1)$ is odd. The above deletion technique produces a $(j+2)$-set W'' with no odd contiguous sets. If $j + 2 \leq d$, then by Shephard's theorem, $[W'']$ is a $(j+1)$-face of P, and since every $(j+1)$-face contains a j-face, W'' contains the vertex set of a j-face. If $j + 2 \geq d + 1$, then $j = d - 1$ is even, implying that $d = 2m + 1$ and $j = 2m$, precisely the case not covered in the statement of the theorem (cf. Theorem 2.3).

Case 2. $\gamma > d - j - 1$. Pick any $\gamma - (d - j - 1)$ odd contiguous sets and reduce each by a single vertex, leaving W' with $d - j - 1$ odd contiguous sets and $|W'| = |W| - \gamma + d - j - 1 \geq (2j - m + 1) - \gamma + d - j - 1 = j - m + d - \gamma$ vertices. We now consider a number of cases depending on the size of γ and the parity of d. Note that $\gamma \leq k \leq m + 1$.

Case 2a. If $\gamma = m + 1$, then each W_i is contiguous, that is, $k = m + 1$, which is an impossibility since this requires at least $2(m+1) > d$ points in $H \cap M_d$.

Case 2b. If $\gamma = m$ and $d = 2m + 1$, then $|W'| \geq j - m + 2m + 1 - m = j + 1$, in which case we proceed as in Case 1.

Case 2c. If $\gamma = m$ and $d = 2m$, then $k = m$, (because when $d = 2m$, $k \leq m$ always holds) and so each W_i is an odd contiguous set. Thus, $|W|$ and m have the same parity. Hence $|W| = 2j - m + 1$ is not possible, and so by assumption, $|W| \geq 2j - m + 2$. Thus $|W'| = |W| - \gamma + d - j - 1 \geq (2j - m + 2) - \gamma + d - j - 1 = j + 1$, in which case we proceed as in Case 1.

Case 2d. If $\gamma \leq m - 1$, then $|W'| \geq j - m + d - \gamma \geq j - m + d - (m-1) = j + d - 2m + 1 \geq j + 1$, in which case we proceed as in Case 1. □

Theorem 2.4 now easily follows from Theorem 2.5.

3 Many hyperplanes bisecting every j-face

3.1 The problem and outline for solution

We again assume that $d \in \{2m, 2m+1\}$, $d \geq 3$, and for now, assume that

$$0 \leq j \leq 2m - 1. \tag{1}$$

Definition 3.1 *Let $f(M_d, j, \ell)$ be the maximum number (if it exists) so that for any $n \leq f(M_d, j, \ell)$ if P is a cyclic d-polytope satisfying $V(P) \subset M_d$ and $|V(P)| = n \geq d+1$, then there exist hyperplanes $H_1, \ldots, H_\ell$ so that every j-face of P is bisected by some H_i.*

For $m < j < 2m$, Theorems 2.2 and 2.4 respectively give $f(M_d, j, 1) \geq 4j - 2m$, and $f(M_d, j, 1) \leq 4j - 2m$, and hence $f(M_d, j, 1) = 4j - 2m$. Theorem 2.1 shows that $f(M_d, j, 1)$ does not exist when $j \leq m$ and Theorem 2.3 shows that $f(M_d, j, 1)$ does not exist when $j = 2m$ and $d = 2m+1$ (because $2m$-bisectors exist no matter how many vertices there are).

To find bounds on $f(M_d, j, \ell)$ for general ℓ, we begin (in Section 4) by fixing an arbitrary, family $\mathcal{H}_\ell$ of ℓ hyperplanes. These hyperplanes determine open regions in d-space. We first consider a fixed region R, and the arcs (if any) of M_d which are contained in R. We then calculate precisely (in terms of the numbers of arcs in R) how many vertices of a cyclic d-polytope can appear on these arcs, without having a j-face contained in R. We then sum over all regions, getting an expression $G(M_d, j, \mathcal{H}_\ell)$ for the maximum number of vertices in a d-polytope which has no j-face in any region, that is, which has every j-face bisected by some member of $\mathcal{H}_\ell$. In Section 5 we maximize this expression in various ways over all possible (and perhaps some not possible) hyperplane arrangements giving upper bounds for $f(M_d, j, \ell)$. In Section 7, lower bounds for $f(M_d, j, \ell)$ are given by constructing hyperplane arrangements where each hyperplane intersects M_d in precisely d points.

3.2 Hyperplanes and regions

For a collection $\mathcal{H}_\ell = \{H_1, \ldots, H_\ell\}$ of ℓ hyperplanes in $\mathbb{R}^d$, a maximal connected open set (or component) of $\mathbb{R}^d \backslash (H_1 \cup \cdots \cup H_\ell)$ is called a *region* (or *cell*). For each hyperplane H_i, fix a labelling of each of the open half-spaces determined by H_i; call them H_i^+ and H_i^-. To any point $x \in \mathbb{R}^d \backslash (H_1 \cup \cdots \cup H_\ell)$, assign a *type* $r_x = (r_x(1), r_x(2), \ldots, r_x(\ell)) \in \{0,1\}^\ell$ defined by $r_x(i) = 1$ if $x \in H_i^+$ and $r_x(i) = 0$ if $x \in H_i^-$. Hence any point in $\mathbb{R}^d \backslash (H_1 \cup \cdots \cup H_\ell)$, has one of 2^ℓ types indicating its relative position to each hyperplane and two points from the same region have the same type.

It is well known (for example, see [8]) that for ℓ hyperplanes in general position, there are $\sum_{i=0}^{d} \binom{\ell}{i} \leq 2^\ell$ such regions; when $d \geq \ell$, this number is precisely 2^ℓ. If $\ell > d$, $\sum_{i=0}^{d} \binom{\ell}{i} < 2^\ell$ and so not all 2^ℓ types of regions are possible with one arrangement of ℓ hyperplanes.

For a region R in $\mathbb{R}^d$ determined by some $\mathcal{H}_\ell = \{H_1, \ldots, H_d\}$ and some $V \subset M_d$, $|V| \geq d+1$, we say that $R \cap V$ is *j-empty* if $R \cap V$ does not contain the vertices of a j-face of the polytope $[V]$. For any such region R, define

$$g(M_d, j, R) = \max_{V \subset M_d} \{|R \cap V| : R \cap V \text{ is } j\text{-empty}\}.$$

Since the definition of $g(M_d, j, R)$ makes sense only when $j \geq 1$, this will be henceforth assumed.

4 A fixed family of hyperplanes

4.1 Vertices in a single region

Fix a family of hyperplanes $\mathcal{H}_\ell = \{H_1, \ldots, H_\ell\}$ and let R be a region in $\mathbb{R}^d \backslash (H_1 \cup \cdots \cup H_\ell)$.

Put $A = R \cap M_d$ and note that A is a disjoint union of arcs, say $A = A_1 \cup \cdots \cup A_{\bar{k}}$, where $\bar{k} \leq m + 1$. We now insist that the hyperplanes have no points of tangency with M_d, that is, if M_d touches some H_i, then it passes through H_i. We make this assumption so that consecutive arcs of A have at least one arc between them contained in some other region (and there will be vertices of the polytope in every such region). This assumption comes without loss of generality, because one can perturb any given family of hyperplanes with tangencies to give a family which meets this assumption and bisects precisely the same set of j-faces of the polytope we create on M_d. Also, assume that each hyperplane intersects M_d at least once, since otherwise, a hyperplane will not bisect any j-faces.

We now compute $g(M_d, j, R)$ by putting as many vertices as possible on these A_i's while not creating a j-face, and we do this by progressively identifying properties (Lemmas 4.1–4.4) of a set $V \subset M_d$ so that $[V]$ is a d-polytope and $A \cap V$ is j-empty. Note that if $g(M_d, j, R)$ exists, then $g(M_d, j, R) \geq j$, so we may first assume that $|A \cap V| \geq j$. For each arc A_i which is not an end arc (that is, A_i is bounded), assume that $A_i \cap V \neq \emptyset$. Hence, each $A_i \cap V$ is either an end set (possibly empty) of V, or a (non-empty) contiguous set of V. Let k be the number of $A_i \cap V$'s that are contiguous in V. Then

$$\bar{k} - 2 \leq k \leq \bar{k}.$$

Also note that because any hyperplane intersects M_d in at most d points, $k \leq m$. Further observe that if $d = 2m$, either $k = \bar{k}$ or $k = \bar{k} - 2$, the latter occurring for only the region containing both unbounded arcs; if $d = 2m + 1$, either $k = \bar{k}$ or $k = \bar{k} - 1$, the latter occurring for precisely two regions (since the hyperplanes indeed cut M_d).

Lemma 4.1 *If $k \leq d - j - 1$, then $g(M_d, j, R) = j$.*

Proof: Let $k \leq d - j - 1$. If $|A \cap V| \geq j + 1$, pick $K \subset A \cap V$ with $|K| = j + 1$ vertices so that each $A_i \cap K$ is contiguous. Then K has at most k contiguous sets and so at most $d - j - 1$ odd contiguous sets. By Shephard's theorem, $[K]$ is a j-face. □

Now assume that $k \geq d - j$. Since $k \leq m$, $d - j \leq m$ and so $j \geq d - m$. When $\ell = 1$, Theorem 2.1 says that j-bisectors exist only if $j > \lfloor d/2 \rfloor = m$; here we have only restricted j so that when d is even, $j \geq m$, and when d is odd, $j \geq m + 1$. Thus we now assume

$$\left\lceil \frac{d}{2} \right\rceil \leq j \leq 2m - 1.$$

Lemma 4.2 *$g(M_3, 1, R) = k$, and for any $d \geq 4$, $g(M_d, 1, R) = 1$.*

Proof: In $C(n,3)$, the edges are of the form $[x_1, x_i]$, $[x_i, x_n]$, and $[x_i, x_{i+1}]$, and the best one can do is to pick one point from each contiguous $A_i \cap V$. Since for every $d \geq 4$, every cyclic d-polytope is 2-neighbourly, every pair is an edge, and we can only pick one point from all of A. $\square$

We now consider the remaining possibilities $d \geq 4$ and $j \geq 2$, and derive a key property in our evaluation of $g(M_d, j, R)$.

Lemma 4.3 *Let $A \cap V$ be j-empty with the maximum number of vertices, that is, $|A \cap V| = |R \cap V| = g(M_d, j, R)$, and assume $|A \cap V| > j$. Then for each contiguous $A_i \cap V$, $|A_i \cap V| \leq j$, and $|A_i \cap V|$ is odd.*

Proof: Recall that by Theorem 2.3, when $d = 2m+1$ and $j = 2m$, there is always a j-bisector, that is, $g(M_{2m+1}, 2m, \ell)$ does not exist for any ℓ, so assume that $j < 2m$. (We have already made this assumption in (1) and this is the reason.)

If for some i, $|A_i \cap V| \geq j+1$ and $j+1$ is even, then by Shephard's theorem, any $j+1$ consecutive vertices in $A_i \cap V$ form a j-face, a contradiction. If $|A_i \cap V| \geq j+1$ and $j+1$ is odd, any $j+1$ consecutive vertices forms a j-face unless $d - j - 1 = 0$. However, for $j+1$ odd, $j = d-1$ occurs only when $j = 2m$, which we have precluded. We conclude that for every A_i, $|A_i \cap V| \leq j$.

Now suppose that, for some i, $|A_i \cap V|$ is even; we derive a contradiction by showing $A \cap V$ is not maximal.

Let $x' \in (A_i \cap M_d) \backslash V$, put $V' = V \cup \{x'\}$, and let $X \subset (A \cap V')$ have $j+1$ vertices.

If $x' \notin X$, then $X \subset (A \cap V)$ and $A \cap V$ is j-empty, so $[X]$ is not a j-face.

Let $x' \in X$, and consider the decomposition of X into contiguous sets and end sets. Then x' belongs to a contiguous subset of X. In hopes of a contradiction, suppose that $[X]$ is a j-face. Then at most $d-j-1$ contiguous subsets of X are odd.

Case 1: x' is in an even contiguous subset Y of X.

Then $Y = (Y \cap V) \cup \{x'\}$ and $|Y \cap V|$ is odd. Since $|A_i \cap V|$ is even, there is a $\bar{x} \neq x'$ such that $\bar{x} \in (A_i \cap Y) \backslash Y$. Then $\bar{Y} = (Y \backslash \{x'\}) \cup \{\bar{x}\}$ is an even contiguous set. So replacing Y with $\bar{Y}$ yields a $(j+1)$-element subset $\bar{X}$ of $A \cap V$ such that $[\bar{X}]$ is a j-face, contradicting that $A \cap V$ is j-empty.

Case 2: x' is in an odd contiguous subset U of X.

If $U \neq A_i \cap V'$, then argue as in Case 1, deleting x' and adding an $\bar{x}$. So now suppose that $U = A_i \cap V'$. If one of the other contiguous subsets of X is a proper subset of some $A_p \cap V$, then exchange as in Claim 1 and get a contradiction (since there will still be the same numbers of odd and even contiguous sets). If $X \cap V = (A \cap V) \cup \{x'\}$, then $|A \cap V| = j$, contradicting $|A \cap V| > j$.

We conclude that if, for some i, $|A_i \cap V|$ is even, then for any $x' \in (A_i \cap M_d) \backslash V$ and $V' = V \cup \{x'\}$, $A \cap V'$ is also j-empty, contradicting the maximality of $|A \cap V|$. Therefore, every $A_i \cap V$ is an odd contiguous set. $\square$

Lemma 4.4 *Let $d \geq 4$ and R be a region containing $k \geq d - j$ bounded arcs. When $d = 2m$ and R contains the unbounded arcs then*

$$g(M_d, j, R) = k + 2(j - m) + 1.$$

In all other situations,

$$g(M_d, j, R) = k + 2(j - m)$$

Proof: To prove that $g(M_d, j, R) \geq k + 2(j - m)$ (or in the case when R contains the unbounded arcs, $g(M_d, j, R) \geq k + 2(j - m) + 1$), construct $A \cap V$ as follows: in each of the k bounded A_i's put a vertex $y_i \in A_i$. Add $2(j - m)$ vertices to the A_i's, (bounded or not) so that each A_i receives an additional even number of vertices. Finally, in the case when $d = 2m$ and R is the region containing the unbounded arcs of M_d, add one vertex to an end set. We claim that the resulting $A \cap V$ is j-empty.

Fix $K \subset (A \cap V)$. Let e be the maximum number of mutually disjoint consecutive pairs of vertices $\{x_p, x_{p+1}\}$ in K where each such pair is contained in some A_i (contiguous or end set). By construction, $e \leq j - m$. Suppose that K contains exactly r odd contiguous sets. Then either $|K| = 2e + r$, or in the case $d = 2m$, one could have $|K| = 2e + r + 1$. If $r \geq d - j$, then by Shephard's theorem, $[K]$ is not a j-face. If $r \leq d - j - 1$, the equalities

$$j + 1 = d - j + (2j + 1 - d) = \begin{cases} d - j + 2(j - m) + 1 & \text{if } d = 2m; \\ d - j + 2(j - m) & \text{if } d = 2m + 1, \end{cases} \tag{2}$$

yield $2e + r \leq 2(j - m) + d - j - 1 < j + 1$, and when $d = 2m$, $2e + r + 1 \leq 2(j - m) + d - j - 1 + 1 < j + 1$. In any case $|K| < j + 1$; that is, $[K]$ is not a j-face.

Now we prove that $g(M_d, j, R) \leq k + 2(j - m)$ (or in the case when R contains the unbounded arcs, $g(M_d, j, R) \leq k + 2(j - m) + 1$). Let $A \cap V$ be j-empty, and $|A \cap V| = g(M_d, j, R)$. First observe that $k + 2(j - m) \geq d - j + 2(j - m)$ and so by (2), and the construction above, we may assume that $|A \cap V| > j$. Hence, by Lemma 4.3, all contiguous $A_i \cap V$'s are odd.

Since $A \cap V$ is j-empty, any subset of $A \cap V$ with at most $d - j - 1$ odd contiguous sets has at most j vertices. In each contiguous A_i, fix y_i so that $(A_i \cap V) \backslash \{y_i\}$ is an even contiguous set. Let $b \leq d - j - 1$ and consider a set $X \subset (A \cap V)$ containing elements $y_1, \ldots, y_b$, (which is possible since $k \geq d - j$) and e other disjoint consecutive pairs from $(A \cap V) \backslash \{y_1, \ldots, y_k\}$. Then either $|X| = b + 2e$ or $|X| = b + 2e + 1$, depending whether or not X contains a odd number of vertices from an end set. First suppose that $|X| = b + 2e$. Since $|X| \leq j$, $b + 2e \leq j$ implies that $2e \leq j - b \leq j - (d - j - 1) = 2j - d + 1 \leq 2j - 2m + 1 = 2(j - m) + 1$, and so $X \backslash \{y_1, \ldots, y_b\}$ contains $e \leq j - m$ remaining pairs. Thus $(A \cap V) \backslash \{y_1, \ldots, y_k\}$ contains at most $j - m$ remaining pairs, and $|A \cap V| \leq k + 2(j - m) + 1$. Since each $A_i \cap V$ is odd, the only way to achieve $|A \cap V| = k + 2(j - m) + 1$ is to then have an odd number of vertices in an end set. This can only be done when $d = 2m$, because if $d = 2m + 1$, by (2), one can create a $(j + 1)$-set with $d - j - 1$ odd contiguous sets by using an end set. □

Note that the general pattern described in Lemma 4.4 includes the patterns used in the proof of Theorem 2.2.

Theorem 4.5 *Let $d \geq 4$ and $R \cap M_d$ contain k bounded arcs. When $d = 2m$, if $k \geq d - j$ and $R \cap M_d$ contains the unbounded arcs, then*

$$g(M_d, j, R) = k + 2(j - m) + 1.$$

In all other situations,

$$g(M_d, j, R) = \max\{j, k + 2(j - m)\}.$$

Proof: If $k < d-j$ then $k + 2(j-m) \leq j$ and if $k \geq d-j$ then $k + 2(j-m) \geq j$. Lemmas 4.1 and 4.4 finish the proof. $\square$

4.2 Vertices in all regions

If $j = m$ and $\ell = 1$, there are $t = 2$ regions, and at most j vertices on each side of the hyperplane; $2j = 2m \leq d$ vertices are not enough for a d-polytope, so for $\ell = 1$, we assume $m < j < 2m$. If $\ell \geq 2$, we don't have this argument available to insist that $j > m$ since there are more than 2 regions, so when $\ell > 1$ we could have $j = m$.

For a fixed $\mathcal{H}_\ell$, define

$$G(M_d, j, \mathcal{H}_\ell) = \sum g_j(M_d, j, R),$$

where the sum is taken over all regions determined by $\mathcal{H}_\ell$. By summing the equalities in Lemmas 4.1 and 4.4, we have the following result which gives the maximum number of vertices in V so that no j-face of $[V]$ occurs in any one region.

Theorem 4.6 *Let $d \geq 4$, $\lceil d/2 \rceil \leq j < 2m$ and if $\ell = 1$, let $m < j < 2m$. Let $\mathcal{H}_\ell$ determine t regions, $R_1, R_2, \ldots, R_t$, which contain some arc (unbounded or not) of M_d. For each R_i, let k_i denote the number of bounded arcs in $R_i \cap M_d$ and put $s = |\{i : k_i < d-j\}|$. If $d = 2m$ and the region containing both unbounded arcs also contains at least $d-j$ bounded arcs, then*

$$G(M_d, j, \mathcal{H}_\ell) = sj + \left(\sum_{k_i \geq d-j} k_i \right) + 2(t-s)(j-m) + 1. \tag{3}$$

If $d = 2m$ and no such region exists, or when $d = 2m+1$, then

$$G(M_d, j, \mathcal{H}_\ell) = sj + \left(\sum_{k_i \geq d-j} k_i \right) + 2(t-s)(j-m). \tag{4}$$

There are a number of ways that Theorem 4.6 can be interpreted; we give a few that are useful. In the following corollaries, for brevity we assume: $d \geq 4$; $\lceil d/2 \rceil \leq j < 2m$ and if $\ell = 1$, $m < j < 2m$; $\mathcal{H}_\ell$ determines t regions, $R_1, R_2, \ldots, R_t$, each containing some arc of M_d; the number of bounded arcs in $R_i \cap M_d$ is k_i.

Corollary 4.7 *Let each $k_i \geq d-j$. Then*

$$G(M_d, j, \mathcal{H}_\ell) = \begin{cases} (\sum_{i=1}^t k_i) + 2t(j-m) + 1 & \text{if } d = 2m \\ (\sum_{i=1}^t k_i) + 2t(j-m) & \text{if } d = 2m+1, \end{cases} \tag{5}$$

and if each hyperplane in $\mathcal{H}_\ell$ cuts M_d in d points, then

$$G(M_d, j, \mathcal{H}_\ell) = \begin{cases} \ell d + 2t(j-m) & \text{if } d = 2m \\ \ell d - 1 + 2t(j-m) & \text{if } d = 2m+1. \end{cases} \tag{6}$$

Proof: The proof of (5) is a direct application of Theorem 4.6 with $s = 0$. For (6), we note that ℓd points on M_d decompose M_d into $\ell d + 1$ arcs, $\ell d - 1$ of which are bounded. $\square$

This last corollary yields insight as to how one might maximize the number of vertices in a polytope with vertices on M_d so that every j-face is bisected—find an $\mathcal{H}_\ell$ so that each hyperplane intersects M_d precisely d times while t is as large as possible.

We apply (6) in the case $\ell = 1$ (hence $t = 2$) and $j > m$. If $d = 2m + 1$, then $s = 0$ because for each $i = 1, 2$, $k_i = m = 2m + 1 - (m + 1) \geq d - j$. When $d = 2m$, either $k_i = m$ or $k_i = m - 1$, and so for $i = 1, 2$, $k_i \geq m - 1 = 2m - (m + 1) \geq d - j$ again giving $s = 0$. If $d = 2m$, then $d\ell - 1 + 2t(j - m) + 1 = 2m + 4(j - m) = 4j - 2m$, and if $d = 2m + 1$, then $d\ell - 1 + 2(j - m) = (2m + 1) - 1 + 4(j - m) = 4j - 2m$, (cf. Theorems 2.2 and 2.4).

Corollary 4.8 *$G(M_d, j, \mathcal{H}_\ell) \geq tj$, with equality if each $k_i < d - j$.*

Proof: If some $k_i \geq d - j$, then $k_i + 2(j - m) \geq j$, and the inequality follows from Theorem 4.6. When each $k_i < d - j$, apply Theorem 4.6 with $s = t$, in which case only (4) applies. $\square$

In applying Corollary 4.8, one might note that $t \leq \min\{\ell d + 1, 2^\ell\}$.

We conclude this section by mentioning that finding other bounds for $G(M_d, j, \mathcal{H}_\ell)$ can be obtained by applying the following inequalities in Theorem 4.6: $\sum_{i=1}^{t} k_i \leq \ell d - 1$; $t \leq \min\{\ell d + 1, 2^\ell\}$; if $k \geq d - j$, then $k + 2(j - m) \geq j$; if $k < d - j$ then $k + 2(j - m) \leq j$. and $t \leq \sum_{i=1}^{d} \binom{\ell}{i}$. Some results of such computations are contained in Theorems 5.3 and 5.4 in the next section.

5 Upper bounds for $f(M_d, j, \ell)$

Recall that $f(M_d, j, \ell)$ is the maximum number of vertices in a cyclic d-polytope P with $V(P) \subset M_d$ such that there exists ℓ hyperplanes $H_1, \ldots, H_\ell$ so that every j-face of P is bisected by some H_i, and

$$f(M_d, j, \ell) = \max_{\mathcal{H}_\ell} G(M_d, j, \mathcal{H}_\ell),$$

where the maximum is taken over all hyperplane arrangements $\mathcal{H}_\ell = \{H_1, \ldots, H_\ell\}$, where each H_i has no points of tangency with M_d.

The following global upper bound for $f(M_d, j, \ell)$ follows from Theorem 4.6.

Theorem 5.1 *For $d \geq 4$, $d \in \{2m, 2m + 1\}$, $m \leq j \leq 2m - 1$, and $\ell > 1$,*

$$f(M_d, j, \ell) \leq \ell d + 2^{\ell+1} j.$$

Proof: By (3) and (4), for any hyperplane arrangement $\mathcal{H}_\ell$,

$$G(M_d, j, \mathcal{H}_\ell) \leq sj + \left(\sum_{k_i \geq d-j} k_i \right) + 2(t - s)(j - m) + 1$$

$$
\begin{aligned}
&\le sj + \left(\sum_{i=1}^{t} k_i\right) + 2(t-s)(j-m) + 1 \\
&= sj + \ell d - 1 + 2(t-s)(j-m) + 1 \\
&= \ell d + 2t(j-m) - sj + 2sm \\
&\le \ell d + 2t(j-m) + 2tm \\
&= \ell d + 2tj \\
&\le \ell d + 2 \cdot 2^{\ell} j.
\end{aligned}
$$

□

Sharper upper bounds for $f(M_d, j, \ell)$ are available in special situations by bounding expressions in Theorem 4.6.

Since any consecutive $j+1$ vertices in $C(n,d)$ determine a j-face, if $f(M_d, j, \ell)$ exists, then there is a hyperplane arrangement so that no arc contains more than j vertices, and when d is even, both end arcs are in the same region, and together these arcs contain no more than j vertices. We conclude:

Theorem 5.2 *Let $d \in \{2m, 2m+1\}$, $d \ge 3$, and $1 \le j < 2m$. If $f(M_d, j, \ell)$ exists, then*

$$f(M_d, j, \ell) \le \begin{cases} ldj & \text{for } d = 2m \\ (ld+1)j & \text{for } d = 2m+1. \end{cases} .$$

In Theorem 7.1 we see that the bound in Theorem 5.2 is attained when $\ell d = 2^{\ell}$.

Other upper bounds for $f(M_d, j, \ell)$ which can be obtained with more careful counting are stated in the next two theorems without proof:

Theorem 5.3 *If $d \ge 4$ is even, $\ell \ge 2$ and $\frac{d}{2^\ell} < j < d$ then*

(i) $f(M_d, j, \ell) \le \ell dj$ *if* $d \le \frac{2^\ell}{\ell}$;

(ii) $f(M_d, d-1, \ell) \le \ell d + 2^\ell(d-2)$ *if* $d > \frac{2^\ell}{\ell}$;

(iii) $f(M_d, j, \ell) < \ell d + 2^\ell(j-1)$ *if* $j \le d-2$ *and* $d > \frac{2^\ell}{\ell}$.

Theorem 5.4 *If $d \ge 5$ is odd, $\ell \ge 2$ and $\frac{d}{2^\ell} < j < d-1$ then*

(i) $f(M_d, j, \ell) \le (\ell d + 1)j$ *if* $d < \frac{2^\ell}{\ell}$;

(ii) $f(M_d, d-2, \ell) \le \ell d + 2^\ell(d-3) + 1$ *if* $d > \frac{2^\ell}{\ell}$;

(iii) $f(M_d, j, \ell) < \ell d + 2^\ell(j-1)$ *if* $j \le d-3$ *and* $d > \frac{2^\ell}{\ell}$.

6 Hyperplane arrangements and codes

We now concentrate on finding lower bounds for $f(M_d, j, \ell)$. To accomplish this, we seek examples of families $\mathcal{H}_\ell$ for which $G(M_d, j, \mathcal{H}_\ell)$ is large. We translate the problem of finding maximum values of t into the language of codes, or equivalently, paths on the cube $\{0,1\}^\ell$. In this section, we develop some of this language, and then return in Section 7 to apply the codes. Different techniques are required depending on the relative sizes of d and ℓ.

Let hyperplanes $H_1, H_2, \ldots, H_\ell$ intersect the moment curve M_d each in ℓd distinct points, yielding $\ell d + 1$ arcs in $M_d \backslash (H_1 \cup \cdots \cup H_\ell)$, and refer to the notation in Section 3.2.

Let Q^ℓ denote the graph on vertices $\{0,1\}^\ell$ (the set of ℓ-tuples of 0's and 1's) where two vertices are connected by an edge if, and only if, they differ in precisely one coordinate. A *walk* of length k in Q^ℓ is a sequence $v_0, v_1, v_2, \ldots, v_k$, of vertices v_i, where for each $i = 1, \ldots, k$, $e_i = (v_{i-1}, v_i)$ is an edge. A walk in Q^ℓ is sometimes called a *code with spread 1*. Note: the vertices v_i need not be distinct. A *path* in Q^ℓ is a walk in which no vertex (and no edge) is repeated. A *circuit* is a walk in which no edge is repeated and the first and last vertices are the same. A *cycle* is a circuit which repeats no vertex other than the first and last.

Every walk W in Q^ℓ is fully determined by its first vertex and its transition sequence, $(s_1, s_2, \ldots, s_k)$, where s_i is the coordinate changed between v_{i-1} and v_i. For each $i = 1, \ldots, \ell$, let the transition count $\mathrm{TC}(i)$ be the number of times the i-th coordinate changes in W. We say that a walk W in Q^ℓ of length ℓd is *totally balanced* if each $\mathrm{TC}(i) = d$. Note, if $v_0, v_1, \ldots, v_{\ell d}$ is a totally balanced walk in Q^ℓ and d is even then $v_0 = v_{\ell d}$; if d is odd, then v_0 and $v_{\ell d}$ differ in every coordinate.

When counting types of arcs in M_d cut by ℓ hyperplanes, we can freely interchange the language of geometry with the language of walks in Q^ℓ.

Lemma 6.1 *For every placement of ℓ hyperplanes, each intersecting M_d in d points, there corresponds a totally balanced walk of length ℓd whose vertices corresponds to the types of the $\ell d + 1$ arcs in $M_d \backslash (H_1 \cup \cdots \cup H_\ell)$. Furthermore, for every such walk, there is a placement of hyperplanes yielding that walk.*

The number of types of arcs in $M_d \backslash (H_1 \cup \cdots \cup H_\ell)$ is the number of distinct vertices used in the corresponding walk. Also, since the assignment of "+" or "−" to each side of a hyperplane is arbitrary, we may assume that the walk begins at $(0, 0, \ldots, 0)$. See Figure 3 for an example with $d = 4$, $\ell = 3$ (where the moment is curve is drawn as a straight line, and commas and parentheses are omitted in the types).

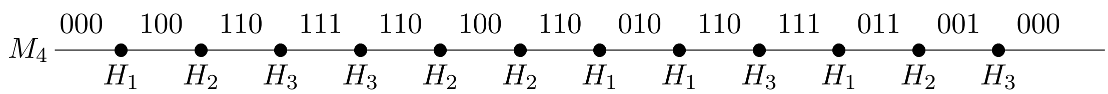

Figure 3: Types of arcs of M_4 with 3 hyperplanes.

Lemma 6.2 *Let W be a totally balanced walk of length ℓd on Q^ℓ. If $d \geq 2$, then the number of distinct vertices in W is at least $\ell + 1$, and this value is attainable for any d.*

Proof: Let W be such a walk. Since each coordinate must be changed at least once, W must have at least $\ell + 1$ vertices.

In fact, such a W using exactly $\ell+1$ vertices exists when d is even. Let $v_0 = (0, 0, \ldots, 0)$, and for each $i = 1, \ldots, \ell$, let $x_i = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$ where the first i coordinates are 1 and the rest 0's. Now let W be the walk on these $\ell + 1$ vertices x_i, going from x_0 to x_ℓ and back, each path repeated d times. $\square$

Trivially, the number of vertices in a walk given in Lemma 6.2 is at most the smaller of $\ell d + 1$ and 2^ℓ.

In the next two sections, we find special kinds of totally balanced walks, those arising from Gray codes (to be used when d is large), and ones constructed by splicing together disjoint chains in a lattice (for use when d is small).

6.1 Gray codes

Recall that in a graph, a *hamiltonian cycle* is a cycle which uses each vertex precisely once; similarly, a *hamiltonian path* is a path which contains every vertex. A graph is called *hamiltonian* if, and only if, it contains a hamiltonian cycle. It is well known that for every ℓ, the graph Q^ℓ is hamiltonian.

When $d = 2^\ell/\ell$, then ℓd is precisely the number of vertices in a hamiltonian cycle, that is, it might be possible to find a totally balanced walk which uses all vertices. In fact, whenever ℓd is a multiple of 2^ℓ, all vertices might be used by just walking a hamiltonian cycle $\ell d/2^\ell$ times. However, it is rare that a hamiltonian cycle in Q^ℓ represents a totally balanced walk. For example, examine the hamiltonian cycle in Q^3: 000, 001, 011, 111, 101, 100, 110, 010, 000. It has $\mathrm{TC}(1) = \mathrm{TC}(3) = 2$ yet $\mathrm{TC}(2) = 4$. The standard inductive construction of a hamiltonian cycle in Q^ℓ gives one coordinate with only 2 changes, and so these cycles are not of much use for hyperplane placements.

An *ℓ-bit Gray code* is an ordered listing of all 2^ℓ ℓ-bit binary strings so that adjacent strings differ in precisely one position. In terms of walks, a Gray code is a walk on Q^ℓ which uses each vertex precisely once. A Gray code is called *cyclic* if the last word differs from the first in precisely one bit. The cyclic Gray code corresponding to the standard hamiltonian cycle (which arises by induction) is called a "binary reflected Gray code" and was patented by Frank Gray in 1953 [6].

If a Gray code is cyclic, for each $i = 1, \ldots, \ell$, define the transition count $\mathrm{TC}(i)$ to be the number of times the i-th bit changes, including the final bit change from the last word to the first. A *balanced* Gray code is a code where the $\mathrm{TC}(i)$'s are as close to being the same value as possible. One of the earliest methods proposed for generating balanced Gray codes is found in [13].

In a cyclic Gray code, since $\sum \mathrm{TC}(i) = 2^\ell$ the most balanced distribution of transition counts is to have each $\mathrm{TC}(i) \sim 2^\ell/\ell$.

Theorem 6.3 (Wagner, West [14]) *An ℓ-bit cyclic Gray code exists with all $TC(i)$'s being equal if, and only if, ℓ is a power of 2.*

In 1996, it was proved that a (nearly) balanced Gray code exists for any ℓ:

Theorem 6.4 (Bhat, Savage [2]) *If a_ℓ is the largest even integer not larger than $2^\ell/\ell$, then there is an ℓ-bit cyclic Gray code with each $TC(i) \in \{a_\ell, a_\ell + 2\}$.*

6.2 Symmetric chain decomposition of $B(\ell)$

Let X be any set with $|X| = \ell$. The Boolean lattice $\mathcal{B}(\ell)$ is the lattice on the power set $\mathcal{P}(X)$, where the partial order is containment. A collection $\mathcal{C} \subseteq \mathcal{P}(X)$ of subsets of X is a *chain* if, and only if, for every $A, B \in \mathcal{C}$, either $A \subset B$ or $B \subset A$. A chain $\mathcal{C}$ in $B(n)$ is *convex* if whenever $A \subset B \subset C$ and $A, C \in \mathcal{C}$, then $B \in \mathcal{C}$. A chain $\mathcal{C} \subseteq \mathcal{P}(X)$ is *symmetric* if for every $C \in \mathcal{C}$ there exists $C' \in \mathcal{C}$ so that $\{|C|, |C'|\} = \{\lceil \ell/2 \rceil + i, \lfloor \ell/2 \rfloor - i\}$ for some $i \geq 0$.

We emphasize that any convex chain from $\emptyset$ to X has precisely one change in each of the coordinates of the characteristic vectors.

There are a number of methods by which $\mathcal{B}(\ell)$ can be partitioned into $\binom{\ell}{\lfloor \ell/2 \rfloor})\}$ disjoint symmetric convex chains (see, for example, [5]). Let $\boldsymbol{\mathcal{C}} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{\binom{\ell}{\lfloor \ell/2 \rfloor}}\}$ be a decomposition of $\mathcal{B}(\ell)$ into disjoint symmetric convex chains. These chains will have lengths $\ell+1, \ell-1, \ell-3, \ldots$, and the number of chains of length $\ell + 1 - 2i$ is $\binom{\ell}{i} - \binom{\ell}{i-1}$. For each k, there are $\binom{\ell}{k}$ chains with length at least $\ell + 1 - 2k$.

If $d \leq \ell$, one can 'splice' together the largest d disjoint convex chains to obtain a totally balanced walk of length ℓd on $2 + d(\ell - 1)$ points in the Boolean lattice. We give an example as to how this is done for $d = \ell = 4$.

Examine the following symmetric chain decomposition of $\mathcal{B}(4)$, the Boolean algebra of subsets of $\{1, 2, 3, 4\}$ into 6 chains (where columns are chains):

$$\begin{array}{cccccc}
\{1,2,3,4\} & & & & & \\
\{1,2,3\} & \{1,3,4\} & \{2,3,4\} & \{1,2,4\} & & \\
\{1,2\} & \{1,3\} & \{2,3\} & \{1,4\} & \{2,4\} & \{3,4\} \\
\{1\} & \{3\} & \{2\} & \{4\} & & \\
\emptyset & & & & &
\end{array}$$

Each subset of $\{1, 2, 3, 4\}$ will have a geometric meaning, and as a first step, we translate the above decomposition replacing sets with characteristic vectors (forgetting commas):

1111					
1110	1011	0111	1101		
1100	1010	0110	1001	0101	0011
1000	0010	0100	0001		
0000					

Consider the walk that goes back and forth between 0000 and 1111 twice: 0000, 1000, 1100, 1110, 1111, 1011, 1010, 0010, 0000, 0100, 0110, 0111, 1111, 1101, 1001, 0001, 0000. Each 4-bit word corresponds to a region, determined by each of four hyperplanes H_1, H_2, H_3, H_4. Notice that in each $0000 \to 1111$ or $1111 \to 0000$ path each bit is changed precisely once, so such a path corresponds to a sequential placement of one point of each $H_i \cap M_d$, giving the points of the hyperplanes placed as follows (where, for example, the first transition $0000 \to 1000$ represents M_d passing through H_1):

$$H_1, H_2, H_3, H_4, H_2, H_4, H_1, H_3, H_2, H_3, H_4, H_1, H_3, H_2, H_1, H_4.$$

This placement ensures that the moment curve passes through fourteen of $2^4 = 16$ possible types of regions; it also gives us that M_d passes through each of twelve regions exactly once, two types of region are missed, region 0000 contains three arcs of M_d, once at each end and once in the middle, and region 1111 contains two arcs.

Generalizing the construction above, the next lemma is easy to verify.

Lemma 6.5 *If $d \leq \ell$, there is a totally balanced walk W of length ℓd in Q^ℓ which uses $\ell + 1 + (d-1)(\ell-1)$ distinct vertices. If $\ell < d \leq \binom{\ell}{2}$, then such a walk exists on $\ell + 1 + (\ell-1)(\ell-1) + (d-\ell)(\ell-2)$ points.*

7 Lower bounds for $f(M_d, j, \ell)$

Theorem 7.1 *Let ℓ be a power of 2 and $d\ell = 2^\ell$ (so $d = 2m$). Then*

$$f(M_{2m}, 2m-1, \ell) \geq 2m\ell + 2^{\ell+1}(m-1),$$

and for each j satisfying, $m < j < 2m-1$,

$$f(M_{2m}, j, \ell) \geq j2^\ell = j \cdot (2m)\ell.$$

Proof: To obtain the bound for $f(M_{2m}, 2m-1, \ell)$, we determine an $\mathcal{H}_\ell$ with $s = 0$ and apply (3). To obtain the bound for $f(M_{2m}, j, \ell)$, we use the same $\mathcal{H}_\ell$, but now with $s = t$, and apply (4). We find such a $\mathcal{H}_\ell$ with $t = 2^\ell$ and each $k_i = 1$ (then $s = 0$ for $j = d-1$ and $s = t$ for $j < d-1$) by using a totally balanced Gray code guaranteed by Theorem 6.3 when ℓ is a power of 2.

Let $W_1, W_2, \ldots, W_{2^\ell}$ be a totally balanced Gray code where for each $i = 1, \ldots, \ell$, $\mathrm{TC}(i) = 2^\ell/\ell = d$. For each $i = 1, \ldots, \ell$ let $H_i \cap M_d = \{h_{i,1}, h_{i,2}, \ldots, h_{i,d}\}$. We now describe how to arrange the points $h_{i,j}$ on M_d. For ease of notation, write $h_{i,j} < h_{i',j'}$ if, and only if, $h_{i,j} = M_d(z)$ and $h_{i',j'} = M_d(z')$ and $z < z'$. For each i, and $j < j'$, we insist that $h_{i,j} < h_{i,j'}$ (points determining one hyperplane are written in order).

Construct the arrangement $\mathcal{H}_\ell$ of hyperplanes as follows. Without loss of generality, put $h_{1,1}$ first. If W_2 differs from W_1 in bit b, then put $h_{b,1}$ next. Now assume that α points have been placed. If W_α and $W_{\alpha+1}$ differ in bit b', then the next $h_{i,j}$ is the smallest available point from $H_{b'}$. $\square$

Theorem 7.2 *Let ℓ be a power of 2, $d = 2m$, and for some integer $c \geq d - j$, let $d\ell = c2^\ell$. Then for every $m < j < 2m$,*

$$f(M_{2m}, j, \ell) \geq 2m\ell + 2^{\ell+1}(j-m).$$

Proof idea: Argue as in the proof of Theorem 7.1 using the same Gray code c times. This yields a hyperplane arrangement with $t = 2^\ell$ and each $k_i = c$. Now apply (3). □

We now want to examine a case when d is very large compared to ℓ, without the assumption that ℓ is a power of 2. Before stating the next theorem, we give an example which makes its proof more transparent.

Consider the case $\ell = 5$; since $\lfloor 2^5/5 \rfloor = 6$, Theorem 6.4 says that there exists a 5-bit Gray code with each $\mathrm{TC}(i) \in \{6, 8\}$. An example of such (from [2]) is:

00000, 10000, 11000, 11100, 11110, 11111, 01111, 01110,
00110, 00010, 00011, 01011, 01001, 00001, 00101, 00111,
10111, 10101, 10001, 11001, 11101, 01101, 01100, 01000,
01010, 11010, 11011, 10011, 10010, 10110, 10100, 00100.

This code has $\mathrm{TC}(3) = 8$, and for $i \neq 3$, $\mathrm{TC}(i) = 6$. The sequence of bit changes (counting the last switch back to the beginning) is:

$$31234515235242341432315341425341 3.$$

Now let $d = 8$, the larger of the two transition counts. Arrange hyperplanes $H_1, \ldots, H_5$ according to the above sequence, that is, the first point on M_8 is a point from H_3; the second point is a point from H_1, and so on, getting a sequence

$$H_3, H_1, H_2, H_3, H_4, H_5, \ldots, H_1, H_3.$$

However, we still have two points from each of H_1, H_2, H_4, H_5 to choose, and we do so (in any order) at the end:

$$H_3, H_1, H_2, H_3, H_4, H_5, \ldots, H_1, H_3, H_1, H_1, H_2, H_2, H_4, H_4, H_5, H_5.$$

The hyperplane arrangement $\mathcal{H}_5$ is now fully determined. Note that the region containing the unbounded arcs also contains a bounded arc between the 32nd and 33rd point of intersection because the code is cyclic.

The bounded arcs with endpoints among the first 32 yield all $t = 32$ types. By Corollary 4.8, for $j \leq 6$, $G(M_8, j, \mathcal{H}_5) \geq 32j$, thus $f(M_8, 4, 5) \geq 128$, $f(M_8, 5, 5) \geq 160$, and $f(M_8, 6, 5) \geq 192$. When $j = 7$, $d - j = 1$, each $k_i \geq 1$, and by Corollary 4.7, equation (6),

$$G(M_8, 7, \mathcal{H}_5) = 5 \cdot 8 + 2 \cdot 32(7 - 4) = 232.$$

Next, we generalize the preceding construction.

Theorem 7.3 *Let d and ℓ be given with $d \geq \lfloor 2^\ell/\ell \rfloor$. Let d^* be largest even integer not larger than $\lfloor 2^\ell/\ell \rfloor + 2$, and suppose that $j \geq d - \lfloor \frac{d}{d^*} \rfloor$. Then*

$$f(M_d, j, \ell) \geq \begin{cases} d\ell + 2^{\ell+1}(j - m) & \text{if } d = 2m \\ d\ell - 1 + 2^{\ell+1}(j - m) & \text{if } d = 2m + 1. \end{cases}$$

Proof: By Theorem 6.4, choose a (nearly) balanced ℓ-bit Gray code with each $TC(i) \in \{d^*-2, d^*\}$. Let $k = \lfloor \frac{d}{d^*} \rfloor$, and apply the code consecutively k times to obtain the placement of $k2^\ell < d\ell$ points of ℓ hyperplanes. Append the remaining $d\ell - k2^\ell$ necessary points from hyperplanes (each hyperplane will intersect the moment curve an even number of times more) to get a legitimate placement of $d\ell$ points determining an arrangement $\mathcal{H}_\ell$ of ℓ hyperplanes. Note that $t = 2^\ell$. One can verify that each region contains at least k bounded arcs.

The fact $j \geq d - \lfloor \frac{d}{d^*} \rfloor$ implies $k \geq d - j$, and by construction, in each region, $k_i \geq k$. To finish the proof, apply equation (6) to find $G(d, j, \mathcal{H}_\ell)$. $\square$

For example, in the case $d = 96$, and $\ell = 5$, then $d^* = 8$ (from our last example), so repeat the Gray code given in the last example $k = \lfloor 96/8 \rfloor = 12$ times and add whatever is necessary to balance the number of times each hyperplane intersects the moment curve. (This can be minimized by applying the Gray code for different permutations of bits.) If $j \geq 84$, then each $k_i \geq 12 \geq d - j$, so there is an arrangement $\mathcal{H}_5$ of five hyperplanes with $G(96, j, \mathcal{H}_5) = 5 \cdot 96 + 64(j - 48) = 64j - 2592$.

Next we examine the case $d = \ell$, where we use the symmetric chain decomposition technique described in Section 6.2. Since $f(M_d, j, \ell) \geq f(M_d, d-1, \ell)$, in the case that d is even, we restrict our attention to facets. We have already shown the example for $d = \ell = 4$.

Theorem 7.4 *For d even, and $d \geq 4$,*

$$d^3 - 2d^2 + 4d - 4 \leq f(M_d, d-1, d) \leq d^3 - d^2.$$

Proof: We first give the construction which realizes the lower bound. Let X be a set of d elements and let $\boldsymbol{\mathcal{C}} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{\binom{d}{d/2}}\}$ be a decomposition of $\mathcal{P}(X)$ into disjoint symmetric convex chains. These chains have lengths $d+1, d-1, d-3, \ldots$, and the number of chains of length $d+1-2i$ is $\binom{d}{i} - \binom{d}{i-1}$ (in particular, there is one chain of length $d+1$ and $d-1$ chains of length $d-1$). Let $\mathcal{C}_1$ have length $d+1$ and let each of $\mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_d$ have length $d-1$. Construct the code of characteristic vectors of the $d+1+(d-1)^2 = d^2-d+2$ sets in these first d chains according the following pattern: start at $\emptyset$, go up $\mathcal{C}_1$ to X, then down through $\mathcal{C}_2$ to $\emptyset$, up through $\mathcal{C}_3$ to X, down through $\mathcal{C}_4$ to $\emptyset$, ..., up through $\mathcal{C}_{d-1}$ to X, and finally down through $\mathcal{C}_d$ to $\emptyset$. Since each path from $\emptyset$ to X corresponds to crossing all $\ell = d$ hyperplanes, the resulting path gives a legitimate placing of the hyperplanes.

Each vector corresponds to a type of region, and so $t = d^2 - d + 2$. The type $11\cdots 1$ occurs $d/2$ times, and so for one value of i, $k_i = d/2$. The type $00\cdots 0$ is realized as a bounded arc $\frac{d}{2} - 1$ times, and as an end arc twice. Thus for one value of i, we have $k_i = \frac{d}{2} - 1$. For the remaining $d^2 - d$ types, $k_i = 1$. By equation (6) with $t = d^2 - d + 2$, one obtains

$$G(M_d, d-1, \mathcal{H}_d) = d^3 - 2d^2 + 4d - 4,$$

proving the lower bound. (If instead, one were to simply put $j = d-1$ points in each region, one would have $(d^2 - d + 2)(d-1) = d^3 - 2d^2 + 3d - 2$ points, a difference of $d - 2$ points, so equation (6) gives a slight improvement.)

If the region in which the curve begins (and ends) contains no inner arc, and if no two bounded arcs occur in the same region, then one could put j points in each region, of which there are $d^2 = d\ell$, yielding a total of $d^2(d-1) = d^3 - d^2$ vertices. (This is the same bound as in (6) with $t = d^2$.) However such a hyperplane arrangement might not be realizable. $\square$

One can generalize the construction above to give bounds when $d \neq \ell$ but d is still 'close to' ℓ.

Theorem 7.5 *Suppose that $4 \leq d \leq \ell$. If d is even, then*

$$f(M_d, d-1, \ell) \geq (d^2 - d)\ell - d^2 + 4d - 4.$$

and if d is odd,

$$f(M_d, d-2, \ell) \geq (d^2 - 2d)\ell + 3d - 7.$$

Proof idea: Perform the aforementioned construction of splicing together d disjoint chains in the Boolean lattice of subsets of an ℓ-element set X. The construction uses only one chain of length $\ell+1$ and $d-1$ chains of length $\ell-1$. This yields $\mathcal{H}_\ell$ with $t = (\ell+1)+(d-1)(\ell-1) = d(\ell-1)+2$, where all but two types are used precisely once. (When d is even, these two types occur on bounded arcs respectively $\frac{d}{2}$ and $\frac{d}{2}-1$ times; when d is odd, they both occur $\frac{d-1}{2}$ times on bounded arcs). So, when d is even, and $j = d-1$, apply equation (6) with $t = d(\ell-1)+2$ (or (3) with $s = 0$), to obtain

$$\begin{aligned} G(M_d, d-1\mathcal{H}_\ell) &= \ell d + 2(d(\ell-1)+2)(2j-d) \\ &= (d^2-d)\ell - d^2 + 4d - 4. \end{aligned}$$

When d is odd and $j = d-2$, we showed above that all regions except two have $k_i = 1 <= d-j$ and $s = d(\ell-1)$. By (3)

$$\begin{aligned} G(M_d, d-2, \mathcal{H}_\ell) &= d(\ell-1)(d-2) + d - 1 + 2 \cdot 2(m-1) + 1 \\ &= d(\ell-1)(d-2) + d - 1 + 2(d-3) + 1 \\ &= (d^2-d)\ell + 3d - 6. \end{aligned}$$

$\square$

Note: the construction of splicing together symmetric convex chains can be used for values of ℓ that are 'near' d. For example, when $\ell < 2d$, one can find symmetric chains of length $d+1$ which intersect in at most two points at each end. When $\ell \leq d$, the number of types realized was $d+1+\ell(d-1)$. When $d < \ell \leq \binom{d}{2}$, the number of types is $d+1+(d-1)^2+(\ell-d)(d-3)$, however there are a few more k_i's with $k_i > 1$. If one further restricts ℓ to be not too much larger than d, the construction still yields nearly optimal values; for example, when $\ell = 2d$ and d is even, we obtain $\mathcal{H}_{2d}$ with $t = d+1+(d-1)^2 d(d-3)$, and each $k_i > 0$, and so by (6),

$$\begin{aligned} G(M_d, d-1, \mathcal{H}_{2d}) &= d\ell + t(d-2) \\ &= 2d^3 - 6d^2 + 10d - 4, \end{aligned}$$

which is fairly close to the optimal of $j(\ell d) = (d-1)(2d^2) = 2d^3 - 2d^2$.

8 Conclusions

The connection between cyclic polytopes and various areas of mathematics is becoming well known. To matroids, Bruhat orders, game theory, K-theory, and others, we can now add coding theory. The question of finding hyperplane arrangements that bisect all j-faces of a cyclic polytope on a moment curve led us to a purely combinatorial problem of how to linearly order ℓd points so as to create as many different types, or ℓ-bit words, as possible. Some success was achieved by finding or constructing certain spread 1 binary codes (binary codes whose adjacent words differ in exactly one bit). It would be very interesting to classify the conditions on codes of spread 1 that give legitimate hyperplane placements and yield the greatest number of types of non-empty regions.

Can one say more about the asymptotic behaviour of $f(M_d, j, \ell)$? It follows from Theorem 7.3 and Corollary 4.8 that for fixed ℓ, as $d \to \infty$, $f(M_d, j, \ell)$ is asymptotically $d\ell + 2^{\ell+1}(j - m)$, and for fixed d, j, as $\ell \to \infty$, $f(M_d, j, \ell) \sim d\ell j$. The probabilistic method might be a worthwhile approach to discovering more properties of f. For example, it seems that for fixed ℓ and large enough d, a random placement of ℓ hyperplanes almost surely yields all 2^ℓ types.

Given that we have some understanding of the behaviour of $f(M_d, j, \ell)$, we can now pose the more practical problem of determining the smallest number $b(M_d, j, n)$ of hyperplanes needed to bisect every j-face of some $C(n, d)$. Assuming that $j > 0$ and noting the trivial case of $b(M_{2m+1}, 2m, n) = 1$, it is clear that

$$n \geq b(M_d, 1, n) \geq b(M_d, 2, n) \geq \cdots \geq b(M_d, d-1, n),$$

and thus the most interesting of these numbers are $b(M_d, 1, n)$, $b(M_{2m+1}, 2m - 1, n)$ and $(M_d, 2m - 1, n)$.

For example, what is $b(M_4, 3, 100)$? By Theorem 7.4, it is greater than four; from Theorem 7.5, we have $f(M_4, 3, \ell) \geq 12\ell - 4$. Thus, $100 \leq 12\ell - 4$ implies that $\ell \geq 9$ and $b(M_4, 3, 100) = 9$. If we knew how to calculate tight bounds for $f(M_d, j, \ell)$, then finding $b(M_d, j, n)$ would be easy. Can one give general theorems about $b(M_d, j, n)$? It would be surprising if not much is known for even $b(M_d, 1, n)$, however we could find no references.

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References

[1] K. Bezdek, T. Bisztriczky, and R. Connelly, On hyperplanes and polytopes, *Mh. Math.* **109** (1990), 39–48.

[2] G. S. Bhat and C. D. Savage, Balanced Gray codes, *Electronic Journal of Combinatorics*, **3** (1996), # R25, 11 pages. (See also comments by K. Kedlaya, following the article.)

[3] T. Bisztriczky and G. Károlyi, Subpolytopes of cyclic polytopes, *Europ. J. Cominatorics* **21** (2000), 13–17.

[4] D. Gale, Neighborly and cyclic polytopes, in *Proc. Symp. Pure Math.* **7** (*Convexity*) (1963), 225–232.

[5] C. Greene and D. J. Kleitman, Strong versions of Sperner's theorem, *J. Combin. Th. Ser. A* **20** (1976), 80–88.

[6] F. Gray, *Pulse code communication*, U.S. Patent No. 2632058, March 15, 1953.

[7] B. Grünbaum, *Convex polytopes*, Pure and Applied Mathematics **16**, Interscience publishers, John Wiley & Sons, New York, 1967.

[8] J. Matoušek, *Lectures on discrete geometry*, Springer (Graduate Texts in Mathematics **212**), 2002.

[9] P. McMullen, The maximum number of faces of a convex polytope, *Mathematika* **17** (1970), 179–184.

[10] I. Shemer, Neighborly Polytopes, *Israel J. Math.* **43**, 291–314.

[11] G. C. Shephard, A theorem on cyclic polytopes, *Israel J. Math.* **6** (1968), 368–372.

[12] B. Sturmfels, Cyclic polytopes and d-order curves, *Geometriae Dedicata* bf 24 (1987), 103–107.

[13] V. E. Vickers and J. Silverman, A technique for generating specialized Gray codes, *IEEE Transactions on Computers*, **C-29** (1980), 329–331.

[14] D. G. Wagner and J. West, Construction of uniform Gray codes, *Congressus Numerantium* **80** (1991) 217–223.