The number e

David S. Gunderson

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In mathematics, there are a few special numbers that occur so often that we give them special names. For example, we use $\pi \sim 3.1416$ for the ratio of a circle's circumference to its diameter. Another famous number is the *golden ratio*, often denoted by τ or ϕ, which has a value $(1+\sqrt{5})/2 \sim 1.61803$. Perhaps the next most famous real number is e, which has a value (to 50 decimal places)

$$e = 2.71828\ 18284\ 59045\ 23536\ 02874\ 71352\ 66249\ 77572\ 47093\ 69995\ ...$$

Where does this number come from? How is it defined? It turns out that e does not have just one simple definition like π, nor does it have a nice algebraic definition like τ.

The Swiss-German mathematician Leonhard Euler first named e back in the 1700's, though its existence was implied by Napier in 1614 while studying logarithms and bases. Did Euler name the constant after himself? Probably not, for then it might be E. My best guess is that since Euler defined it to be the number with "hyperbolic logarithm" equal to 1, and in German, "einheit" means "one-ness", or "unity", that the notation was an abbreviation for unity. (In fact, in general algebraic settings, e is still used to denote an identity element.)

In high school, I was first told that e is defined as an infinite sum. This sum was in fact published by Newton in 1669, but he never called it e. Remember that $n! = 1 \cdot 2 \cdot 3 \cdots n$, so, e.g., $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$. Also, by convention, $0! = 1$. Here is what I was first taught:

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots.$$

What does it mean to add up infinitely many things? You add them up from the beginning, keeping a running total (called a partial sum), and if this running total approaches a finite number, then we say that the infinite sum *converges*. We won't prove that this sum converges, but you can convince yourself by checking the first five terms, getting

$$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.5 + 0.166666... + .04166666... = 2.708333....$$

Adding one more term gives 2.7166666..., and so there is evidence that this infinite sum indeed converges to the value given above. Theoretically, using Newton's formula, you could compute e to as many decimal places as you like. (Over a billion digits have been computed—see

`http:pi.lacim.uqam.ca/eng/records_en.html`

for current record.)

What was the first definition of e? It's really hard to say, however here is Euler's. Pick some number $t \geq 1$ and look at the area under the hyperbola $y = 1/x$, above the x-axis, and between the lines $x = 1$ to $x = t$. Using integral calculus notation, this area is equal to "the definite integral of $f(x) = 1/x$ between $x = 1$ and $x = t$", denoted by

$$\int_1^t \frac{1}{x} dx.$$

This area is a function of t, and so we give this function a name, the "natural logarithm" of t, denoted

$$\ln(t) = \int_1^t \frac{1}{x} dx.$$

When t is close to 1, this area is small, and if you look closely, you can see that for really large t, the area is eventually as large as you like. For t approximately 2.7, this area is 1. The number e is defined so that the area under $y = 1/x$ between $x = 1$ and $x = e$ is precisely 1. Restating this, e is defined to be the number so that $\ln(e) = 1$, or using inverse function notation, $e = \ln^{-1}(1)$.

Another way to define e might be more closely related to the way Napier looked at things, in terms of exponential functions. Some examples of simple exponential functions are given by 2^x or 10^x. In these cases, the number 2 or 10 is called a *base* and, of course, x is the exponent. We will use b for the base, so an exponential function is given by $f(x) = b^x$ for some b. When $b > 1$, these functions grow very fast, and if graphed, they are climbing when they cross the y-axis, and get even steeper to the right.

Many calculus books define e to be the unique base b so that the slope of the graph of $y = b^x$ as it crosses the y-axis is equal to 1. They use this as a definition, since they must first discuss slopes (derivatives) before they discuss areas (integrals). For those having seen some calculus, this definition says that if $f(x) = e^x$, then $f'(0) = 1$. Some books use the notation "$\exp(x)$ before they use e^x, to remind us of exponentiation, and so many think that this is where the "e" comes from. One can then later prove that these two definitions for e (one from ln, and one from slopes) define the same number.

For those who have studied logarithms, we know that logarithms and exponentiation are inverse operations. For example, $\log_{10}(x) = y$ is the same as $x = 10^y$. The natural logarithm of t, which was defined above in terms of area, can be shown to be the same as $\log_e(x)$, logarithm to the base e, hence the name natural "logarithm" (and "ln"). When the base is e, the two statements $\ln(x) = y$ and $x = e^y$ are the same.

How else can e be defined? There are many ways. For example,

$$e = \lim_{n\to\infty} (1 + \frac{1}{n})^n$$

is often given as the definition. This definition can arise by a close look at the slope of the tangent line as discussed above. With $n = 2$, the expression in the limit is $(\frac{3}{2})^2 = \frac{9}{4} = 2.25$, and with $n = 3$, one gets $\frac{64}{27} \sim 2.37$, only a little closer to e. Some give

$$e = \lim_{h\to 0^+} (1 + h)^{1/h},$$

an equivalent definition to the last, using $h = 1/n$.

Continued fractions are another way to define e. Both

$$2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{2 + \cfrac{2}{3 + \cfrac{3}{\ddots}}}}}$$

and $2 + \frac{1}{2}(1 + \frac{1}{3}(1 + \frac{1}{4}(1 + \frac{1}{5}(1 + \ldots))))$ are ways to define e.

None of the ways we have defined e are very simple. It would be nice if we could say that e was a fraction, or maybe a root of some polynomial (like the golden ratio is), but it is not. Hence, we say that e is *irrational* and *transcendental.* These two facts were proved by Euler and Hermite respectively, the latter done in 1873.

We have seen that e is used in terms of exponential functions and calculus, but what else is it good for? This number arises in computation of something called *continuously compounded interest.* Another area in which e often arises is in estimation of probabilities. For example, in the "hat-check problem", if 30 people are randomly returned their hats, the probability that no one receives his/her own hat with likelihood nearly $1/e$. This result follows from

$$\frac{1}{e} = \frac{1}{0!} - 1\frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots,$$

a formula whose proof comes from calculus. Another select result involving e is the following. If you pick real numbers at random from the interval between 0 and 1, how many would you have to pick before the numbers sum to greater than 1? The average of numbers needed is e. (See [3].)

There are many wonderful websites regarding e; here are a few to get you started:

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http://mathforum.org/dr.math/faq/faq.e.html
http://mathforum.org/library/topics/about_e/
http://mathworld.wolfram.com/e.html  (good bibliography)
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http://members.aol.com/jeff570/constants.html
http://www.mu.org/~doug/exp/
http://www.maa.org/mathland/mathtrek_11_9_98.html

(The last one contains "Top $\ln(e^{10})$ reasons why e is better than pi.")

References

[1] M. Gardner, "The transcendental number e", in *The unexpected hanging and other mathematical diversions*, Chicago University Press, Chicago, IL, 1991.

[2] E. Maor, e: *The story of a number*, Princeton University Press, Princeton NJ, 1994.

[3] H. S. Shultz and B. Leonard, Unexpected Occurrences of the Number e, *Mathematics Magazine*, Vol. 62, No. 4, October 1989.