A Uniform Asymptotic Estimate for Discounted Aggregate Claims with Subexponential Tails

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Outline

1. Motivations

The following are some basic questions in insurance mathematics:

- **how to model the claim size distribution?**
  - subexponential distribution class
- **how to model discounted aggregate claims?**
  - constant force of interest
- **how to describe the tail behavior of aggregate claims?**
  - value at risk, expected shortfall, etc.

2. Model Description

Assume that there is a constant force of interest $r > 0$. We model discounted aggregate claims as the stochastic process

$$D_r(t) = \sum_{k=1}^{\infty} X_k e^{-r\tau_k} 1_{(\tau_k \le t)}, \qquad t \ge 0, \tag{1}$$

in which we make the following standard assumptions:

- X_1, X_2, ..., are i.i.d. nonnegative random variables with distribution F representing claim sizes;
- $0 < \tau_1 < \tau_2 < \cdots$ are claim arrival times constituting a renewal counting process
$$N_t = \#\{k = 1, 2, \ldots : \tau_k \le t\}, \qquad t \ge 0,$$
with renewal function $\lambda_t = \mathrm{E}N_t$;
- the sequences $\{X_1, X_2, \ldots\}$ and $\{\tau_1, \tau_2, \cdots\}$ are mutually independent.

3. Subexponentiality

Definition: A distribution F on $[0, \infty)$ is said to be subexponential, denoted by $F \in \mathcal{S}$, if

$$\overline{F^{2*}}(x) \sim 2\overline{F}(x), \qquad x \to \infty.$$

An important property of subexponentiality:

It holds for all $n \geq 2$ that

$$\Pr\left(\sum_{k=1}^{n} X_k > x\right) \sim \Pr\left(\max_{1 \leq k \leq n} X_k > x\right), \qquad x \to \infty.$$

This reveals an interesting phenomenon of subexponentiality that the tail of the maximum dominates that of the sum. It explains the relevance of subexponentiality in modeling heavy-tailed distributions.

Some Examples in the Class $\mathcal{S}$

(F = distribution, f = density)

- **Lognormal:** for $-\infty < \mu < \infty$ and $\sigma > 0$,

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\{-(\ln x - \mu)^2/(2\sigma^2)\};$$

- **Pareto:** for $\alpha > 0,\ \kappa > 0$,

$$\overline{F}(x) = \left(\frac{\kappa}{\kappa + x}\right)^{\alpha};$$

- **Burr:** for $\alpha > 0,\ \kappa > 0,\ \tau > 0$,

$$\overline{F}(x) = \left(\frac{\kappa}{\kappa + x^{\tau}}\right)^{\alpha};$$

- Benktander-type I: for $\alpha > 0,\ \beta > 0,$

$$\overline{F}(x) = (1 + 2(\beta/\alpha)\ln x)\exp\{-\beta(\ln x)^2 - (\alpha+1)\ln x\};$$

- Benktander-type II: for $\alpha > 0,\ 0 < \beta < 1,$

$$\overline{F}(x) = \mathrm{e}^{\alpha/\beta} x^{-(1-\beta)} \exp\{-\alpha x^{\beta}/\beta\};$$

- Weibull: for $c > 0,\ 0 < \tau < 1,$

$$\overline{F}(x) = \exp\{-c x^{\tau}\};$$

- Loggamma: for $\alpha > 0,\ \beta > 0,$

$$f(x) = \frac{\alpha^{\beta}}{\Gamma(\beta)}(\ln x)^{\beta-1} x^{-\alpha-1}.$$

4. Main Results

Notation: Denote $\Lambda = \{t : \lambda_t > 0\} = \{t : \Pr(\tau_1 \le t) > 0\}$.

Theorem 1 If $F \in \mathcal{S}$, then the relation

$$\Pr(D_r(t) > x) \sim \int_{0-}^{t} \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s, \qquad x \to \infty, \tag{2}$$

holds uniformly for all $t \in \Lambda_T = \Lambda \cap [0, T]$ for arbitrarily fixed $T \in \Lambda$. That is to say,

$$\lim_{x\to\infty} \sup_{t\in\Lambda_T} \left| \frac{\Pr(D_r(t) > x)}{\int_{0-}^{t} \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s} - 1 \right| = 0.$$

Theorem 2 If $F \in \mathcal{S}$, $\limsup_{x\to\infty} \overline{F}(vx)/\overline{F}(x) < 1$ for some $v > 1$, and $Pr(\tau_1 > \delta) = 1$ for some $\delta > 0$, then relation (2) holds uniformly for all $t \in \Lambda$.

Notation: If F has a finite expectation μ, then denote by

$$F_e(x) = \frac{1}{\mu} \int_0^x \overline{F}(s)\mathrm{d}s, \quad x \geq 0,$$

the equilibrium distribution function of F.

Theorem 3 Restrict $\{N_t, t \geq 0\}$ to be a Poisson process with intensity $\lambda > 0$. If $F \in \mathcal{S}$, $F_e \in \mathcal{S}$, and $\limsup_{x\to\infty} \overline{F}_e(vx) / \overline{F}_e(x) < 1$ for some $v > 1$, then the relation

$$\Pr(D_r(t) > x) \sim \lambda \int_0^t \overline{F}(x\mathrm{e}^{rs})\mathrm{d}s, \qquad x \to \infty, \tag{3}$$

holds uniformly for all $t \in (0, \infty]$.

5. Two Remarks

Remark 1: Denote by $\tau(x) = \inf\{t : D_r(t) > x\}$ the first time when $D_r(t)$ up-crosses the level $x > 0$. Apply the uniform asymptotic relation (3) to get

$$\mathrm{E}\left(\mathrm{e}^{-u\tau(x)}\right) \sim \lambda \int_0^\infty \mathrm{e}^{-us}\overline{F}(x\mathrm{e}^{rs})\mathrm{d}s, \qquad \forall\, u > 0,$$

which gives an explicit asymptotic expression for the Laplace transform of $\tau(x)$.

Remark 2: Consider the limiting conditional distribution of $\tau(x)$ given $(\tau(x) < \infty)$ as $x \to \infty$. For every fixed $t > 0$, by (3),

$$\Pr(\tau(x) \le t \,|\, \tau(x) < \infty) = \frac{\Pr(D_r(t) > x)}{\Pr(D_r(\infty) > x)} \sim \frac{\int_0^t \overline{F}(x\mathrm{e}^{rs})\mathrm{d}s}{\int_0^\infty \overline{F}(x\mathrm{e}^{rs})\mathrm{d}s}.$$

If $F \in \mathcal{R}_{-\alpha}$ with $\alpha > 0$, then

$$\Pr(\tau(x) \le t \,|\, \tau(x) < \infty) \to 1 - \mathrm{e}^{-\alpha r t},$$

meaning that the limiting conditional distribution of $\tau(x)$ given $(\tau(x) < \infty)$ is exponential.

6. Sketch of the Proof of Theorem 1

Remember we want to prove (2). It is clearly that, for $t \in \Lambda_T$,

$$\begin{aligned}\Pr\left(D_r(t) > x\right) &= \left(\sum_{n=1}^{N} + \sum_{n=N+1}^{\infty}\right) \Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-r\tau_k} > x, N_t = n\right) \\ &= I_1\left(x,t,N\right) + I_2\left(x,t,N\right).\end{aligned}$$

Consider $I_2\left(x,t,N\right)$ first. We have

$$\begin{aligned}I_2\left(x,t,N\right) &\leq \sum_{n=N}^{\infty} \int_{0-}^{t} \Pr\left(\sum_{k=1}^{n+1} X_k > x\mathrm{e}^{rs}\right) \Pr\left(N_{t-s} = n\right) \mathrm{d}\lambda_s \\ &\leq C_\varepsilon \left(1+\varepsilon\right) \mathrm{E}(1+\varepsilon)^{N_T} 1_{(N_T \geq N)} \int_{0-}^{t} \overline{F}(x\mathrm{e}^{rs}) \mathrm{d}\lambda_s.\end{aligned}$$

Given ε small enough, $\mathrm{E}(1+\varepsilon)^{N_T} 1_{(N_T \geq N)} \to 0$ as $N \to \infty$. Therefore, for all $x > 0$,

$$\lim_{N\to\infty} \sup_{t \in \Lambda_T} \frac{I_2\left(x,t,N\right)}{\int_{0-}^{t} \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s} = 0. \qquad (4)$$

Next consider $I_1(x,t,N)$. It holds uniformly for all $t \in \Lambda_T$ that

$$I_1(x,t,N) \sim \left(\sum_{n=1}^{\infty} \sum_{k=1}^{n} - \sum_{n=N+1}^{\infty} \sum_{k=1}^{n} \right) \Pr\left(X_k \mathrm{e}^{-r\tau_k} > x, N_t = n \right)$$
$$= \int_{0-}^{t} \overline{F}(x\mathrm{e}^{rs})\, \mathrm{d}\lambda_s - I_{12}(x,t,N).$$

For $I_{12}(x,t,N)$,

$$I_{12}(x,t,N) \leq \int_{0-}^{t} \overline{F}(x\mathrm{e}^{rs})\, \mathrm{d}\lambda_s \sum_{n=N}^{\infty} (n+1) \Pr(N_T \geq n).$$

It follows that, for all $x > 0$,

$$\lim_{N\to\infty} \sup_{t\in\Lambda_T} \frac{I_{12}(x,t,N)}{\int_{0-}^{t} \overline{F}(x\mathrm{e}^{rs})\, \mathrm{d}\lambda_s} = 0. \tag{5}$$

By (4) and (5), we conclude that the asymptotic relation (2) holds uniformly for all $t \in \Lambda_T$. □

7. Open Problems

- Theorem 2 would look much nicer if we could get rid of the technical assumption on the distribution of the inter-arrival time τ_1.

- Recall (3). It strongly suggests that for $F \in \mathcal{S}$, the relation

$$\Pr(D_r(\infty) > x) \sim \lambda \int_0^\infty \overline{F}(x\mathrm{e}^{rs})\mathrm{d}s$$

  holds as $x \to \infty$. As far as I know, this is still an open problem.

- Whether or not $F \in \mathcal{S}$ implies $F_e \in \mathcal{S}$ is still unknown.

$\sim$ The End $\sim$