Finite-time survival probability and credit default swaps pricing under geometric Lévy markets

Xuemiao Hao∗, Xuan Li† and Yasutaka Shimizu‡§

April 23, 2013

Abstract

We study the first-passage time over a fixed threshold for a pure-jump subordinator with negative drift. We obtain a closed-form formula for its survival function in terms of marginal density functions of the subordinator. We then use this formula to calculate finite-time survival probabilities in a structural model for credit risk, and thus obtain a closed-form pricing formula for a single-name credit default swap (CDS). This pricing formula is well calibrated on market CDS quotes. In particular, it explains why the par CDS credit spread is not negligible when the maturity becomes short.

Keywords: credit default swap; finite-time survival probability; first-passage time; Lévy process; structural model

JEL Classification: C10, G13

1 Introduction

Credit default swaps (CDSs) have become the most popular credit derivatives in the past two decades. Pricing CDSs is based on a reasonable model for credit risk. In the literature, there are basically two main classes of credit risk models: reduced-form models and structural models. In reduced-form models, the precise mechanism leading to default is left unspecified and the default time of a firm is modeled as a non-negative random variable, whose distribution typically depends on economic factors. In this paper we alternatively

∗Corresponding author. Asper School of Business, University of Manitoba, Winnipeg, Manitoba R3T 5V4, Canada; e-mail: xuemiao.hao@ad.umanitoba.ca; phone: +1 204 474 8710; fax: +1 204 474 7545
†Department of Mathematics and Statistics, University of Minnesota Duluth, Duluth, MN 55812, USA; e-mail: xuanli@d.umn.edu
‡Graduate School of Engineering Science, Osaka University; 1-3, Machikaneyama-cho, Toyonaka-shi, Osaka 560-8531, Japan; e-mail: yasutaka@sigmath.es.osaka-u.ac.jp
§Japan Science and Technology Agency, CREST; Sanbancho, Chiyoda-ku, Tokyo 102-0075, Japan
consider a structural model, in which a firm defaults at the first time its asset value breaches a low barrier. This barrier corresponds to the recovery value of the firm’s debt.

Classic structural models use a stochastic process with continuous paths to model a firm’s asset value process. For example, in the pioneering first-passage model developed by Black and Cox (1976), the asset value process was described by a geometric Brownian motion. For an asset value process with continuous paths, default always occurs when the asset value of the firm exactly hits the barrier. It is never the case that the firm’s value would suddenly undershoot the default level. However, it is more reasonable and practical to include jumps in stochastic models of asset value and incorporate skewness in the return distribution, since default events are usually triggered by shocks, i.e. downward jumps in the asset value.

In the last decade more and more attention has been paid to using stochastic processes with jumps in structural models. For example, Zhou (2001) used a geometric jump-diffusion process in modeling the market value of a firm’s asset and gave a pricing formula for the defaultable bond issued by the firm. It was originally discovered by Zhou (2001) that the credit spread of a defaultable bond does not go to zero as its maturity goes to zero. Hilberink and Rogers (2002) assumed that a firm’s market value follows an exponential spectrally negative Lévy process and also studied credit spreads with short maturities. Kou and Wang (2004) used a double exponential jump-diffusion model and obtained analytically tractable pricing formulas for path-dependent options. Still using the double exponential jump-diffusion model, Chen and Kou (2009) studied credit spreads, optimal capital structure, and implied volatility of equity options. Lipton (2002) studied a geometric jump-diffusion model with log-exponential jumps and gave a numerical example in which the par credit default spread is not small with short maturity. More recently, Ruf and Scherer (2011) studied corporate bonds in a geometric jump-diffusion model having two-sided jumps and showed that the limiting credit spread as its maturity goes to zero is given by the product of the local default rate and the expected loss given default. All these papers focused on Poissonian jumps and did not address the important issue of infinite activity. In this paper we consider the asset value process that would possibly be of infinite activity but finite variation in a structural model proposed by Madan and Schoutens (2008).

Why processes with infinite activity are important in modeling asset values? First of all, pure-jump processes with infinite activity are able to capture both frequent small moves and rare large moves, which makes them reasonable alternatives for jump diffusions when describing asset returns. Also, dealing with pure-jump processes with good distributional properties sometimes allows one to calibrate models more quickly and describe dependence
structures between assets in a more straightforward way. More importantly, according to Carr et al. (2002) who empirically investigated the fine structure of asset returns, there is evidence from market prices of equity and option indicating that both physical and risk-neutral processes for equity prices seem to be pure-jump processes of infinite activity and finite variation. Here we highlight some of their findings: (1) index returns tend to be pure-jump processes of infinite activity and finite variation, both physically and risk-neutrally. A diffusion component appears to be statistically insignificant, while it may be present in individual equity returns. (2) Jump components consistently account for significant skewness levels from both equity and option prices. (3) The shape of the mean corrected density for asset returns appears to be a long spike near zero conjoined with two convex curves describing large returns. It apparently departs from that of a normal distribution which is always concave within one standard deviation of the mean. In contrast, the densities of processes with infinite activity and finite variation are consistent with both equity and option prices.

Finite-time survival probability is an essential quantity in the calculation of a single-name CDS. In the literature there are many efficient and fast numerical algorithms to calculate finite-time survival probabilities. For example, Kou and Wang (2004) assumed a double exponential jump diffusion for asset value and found an analytic approximation of the first-passage time. Asmussen et al. (2004) obtained an explicit solution to the first-passage time problem in a jump diffusion with phase-type jumps in both directions. More recently, Asmussen et al. (2008) used a phase-type approximation to the CGMY Lévy process and obtained exact computation of finite-time survival probabilities. Under the same model as we use, Madan and Schoutens (2008) found a fast method to calculate survival probabilities, which exploits the remarkable Wiener-Hopf factorization in combination with results by Rogers (2000) and performs a very fast double Laplace transform inversion. In this paper, we adopt an alternative approach to deal with finite-time survival probabilities that originates from ruin theory. Motivated by Dickson and Waters (1993), who obtained a closed-form formula for finite-time survival probabilities for a gamma process with drift, we will show that a similar type of formula holds true for a general pure-jump subordinator with drift. Using this formula we successfully explain why the par credit spread of a CDS does not go to zero when the maturity goes to zero as discovered by Zhou (2001).

This paper is organized as follows. In Section 2 we describe the structural model and obtain a closed-form formula for finite-time survival probabilities. In Section 3 we investigate two special cases of subordinator for which the closed-form formula from Section 2 is completely known. Finally, calibration results and proofs are given in Section 4 and Section
5, respectively.

2 Structural approach for CDS pricing

2.1 Models and the CDS spread

On a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})\), where \(\mathbb{Q}\) is a risk-neutral measure, suppose that a firm’s asset value follows a stochastic process \(V = \{V_t, t \geq 0\}\) such that

\[ V_t = V_0 e^{X_t}, \]

where \(X = \{X_t, t \geq 0\}\) is a Lévy process of the form

\[ X_t = \mu t - S_t \]

with \(\mu > 0\) and \(S = \{S_t, t \geq 0\}\) a pure-jump subordinator. Recall that a pure-jump subordinator is a non-decreasing Lévy process whose Laplace exponent is given by

\[ \varphi(s) := \log \mathbb{E}(e^{-sS_1}) = \int_0^\infty (e^{-sx} - 1) \Pi(dx), \quad s \geq 0, \]

where \(\Pi(\cdot)\) is the Lévy measure of \(S\). Assume a constant risk-free interest rate \(r\). Since \(\mathbb{E}(V_t) = V_0 e^{rt}\) for \(t \geq 0\) under \(\mathbb{Q}\), we immediately have

\[ \mu = r - \varphi(1). \]

Suppose a person owns a defaultable bond of the firm with face value \(N\) and maturity \(T\) years. For a predetermined barrier level \(L(< V_0)\), the default of the bond is defined as the event that \(V_t \leq L\) for some \(t \in (0, T]\), or, equivalently,

\[ X_t \leq \log(L/V_0) \quad \text{for some } t \in (0, T]. \]

When default occurs, the firm only pays the bondholder \(RN\), where \(R \in (0, 1)\) is called the recovery rate. To protect himself from the default risk, the bondholder enters a CDS contract. The CDS has the same maturity as the bond. Under this CDS, the bondholder makes predetermined payments to the protection seller. The payments continue until the maturity date or until the bond defaults, whichever is earlier. In the case of default, the protection seller is required to pay to the bondholder \((1 - R)N\). The CDS spread (in basis points (bpts)), denoted by \(c\), is the yearly rate paid by the bondholder to enter the CDS contract against the default of the bond.
Note that we assume for simplicity a fixed recovery rate throughout the lifetime of a CDS contract in this paper. However, the recovery rate at default in reality may depend on the firm’s asset value at the time of default. The more the firm value process undershoots below the default threshold, the lower the recovery rate one should expect. Our model can be extended to cover this type of more realistic case since the closed-form formula for finite-time survival probabilities that we will derive allows us to take partial derivatives with respect to both time and threshold (see Theorem 2.1 below). We would like to investigate this extension in future research.

If spread is paid continuously the value of the CDS can be expressed in terms of finite-time survival probabilities. The finite-time survival probability up to time \( t \) \((\leq T)\), denoted by \( \delta(t) \), is defined as the probability under \( Q \) that no default occurs by time \( t \), i.e.

\[
\delta(t) = Q(X_s > \log(L/V_0), \text{ for all } 0 \leq s \leq t) = Q\left(\sup_{0 \leq s \leq t} (S_s - \mu_s) < \log(V_0/L)\right).
\]

(2.1)

According to Schoutens and Cariboni (2009, Section 3.1.1), the value of the CDS is

\[
(1 - R)N\left(- \int_0^T e^{-rt} d\delta(t)\right) - cN \int_0^T e^{-rt} \delta(t) dt,
\]

where the first term and the second term respectively correspond to the present value of the so-called loss leg and premium leg of the CDS contract. Pricing the CDS is to find the par spread \( c = c(T) \) which makes the loss leg equal to the premium leg:

\[
c(T) = \frac{(1 - R)\left(- \int_0^T e^{-rt} d\delta(t)\right)}{\int_0^T e^{-rt} \delta(t) dt} = (1 - R) \left(1 - \frac{e^{-rT} \delta(T)}{\int_0^T e^{-rt} \delta(t) dt} - r\right).
\]

(2.2)

### 2.2 Finite-time survival probability

As shown in formula (2.2), the finite-time survival probability \( \delta(t) \) is the essential quantity for \( c(T) \). Motivated by this, we derive a closed-form formula for the following finite-time survival probability in Theorem 2.1:

\[
\delta(t, u) = Q\left(\sup_{0 \leq s \leq t} (S_s - \mu_s) < u\right), \quad t, u > 0.
\]

(2.3)

Note that \( \delta(t) \), as defined in (2.1), becomes \( \delta(t, \log(V_0/L)) \) in (2.3). Let us also define the right-continuous version of the survival probability as

\[
\delta^*(t, u) = Q\left(\sup_{0 \leq s \leq t} (S_s - \mu_s) \leq u\right), \quad t, u > 0.
\]

(2.4)
From (2.3) and (2.4) it is easy to see that if $\delta^*(t,u)$ is left continuous at $u$, then $\delta(t,u) = \delta^*(t,u)$. The survival probability $\delta^*(t,u)$ has been studied in ruin theory for a long time. If we consider the process $\{u + \mu t - S_t, t \geq 0\}$ as an insurance risk process, then $1 - \delta^*(t,u)$ is the finite-time ruin probability. Usually a positive safety loading condition, i.e. $\mu - \mathbb{E}(S_1) > 0$, is assumed in ruin theory to prevent ruin from happening for sure in the long run. But we do not require it in this paper since we focus within a finite-time horizon.

In the following, for $t \geq 0$ we denote by $F_t$ the cumulative distribution function (cdf) of $S_t$ and by $f_t$ the probability density function (pdf) of $F_t$ if it exists.

**Theorem 2.1** Let $\mu > 0$ and $S = \{S_t, t \geq 0\}$ be a Lévy subordinator without drift. Suppose that $F_t(\cdot) \in C^1(0, \infty)$. For $t, u \geq 0$, we have

$$
\delta(t,u) = \delta^*(t,u) = F_t(u + \mu t) - \int_0^t \frac{1}{s} \left( \int_0^{\mu s} F_s(x) \, dx \right) f_{t-s}(u + \mu(t-s)) \, ds
$$

**Remark 2.1** When $S$ is a compound Poisson process formula (2.6) can be found in, for example, Asmussen and Albrecher (2010, Theorems V.2.1 and V.2.2). See also Dickson and Waters (1993) for $S$ being a gamma process.

**Remark 2.2** By partial integration, formula (2.6) can be rewritten as

$$
\delta(t,u) = F_t(u + \mu t) - \mu \int_0^t F_s(\mu s) f_{t-s}(u + \mu(t-s)) \, ds \\
+ \int_0^t \frac{1}{s} \left( \int_0^{\mu s} x f_s(x) \, dx \right) f_{t-s}(u + \mu(t-s)) \, ds.
$$

We notice that formula (2.7) involves double integrals of $f_t$ with variable upper limits, which inevitably cause various computational difficulties in general. However, $\delta(t,u)$ is explicitly expressed and feasible to calculate provided that $F_t$ and $f_t$ are both explicitly presented. This enables us to perform the integrations and compute $\delta(t,u)$ for some known distributions with valid probability densities (e.g. gamma and inverse Gaussian distributions that are to be discussed in Section 3) using readily available computation packages.

**Remark 2.3** Actually, we can easily extend formula (2.6) to the case where $S$ is a spectrally positive Lévy process with possibly unbounded variation; see e.g. Michna (2011) for $\alpha$-stable cases. Since we focus on processes with infinite activity but finite variation in this paper, we will realize this extension in future research.
Remark 2.4 To apply Theorem 2.1 one needs to check the smoothness of $F_t$. Here we list several criteria:

(i) Let $\widehat{F}_1$ be the characteristic function of $F_1$. If

$$\int_{\mathbb{R}} |\widehat{F}_1(x)||x|^{n-1} \, dx < \infty,$$

for some integer $n$, then $F_t(\cdot) \in C^n(\mathbb{R})$. This result is essentially due to the Fourier inversion theorem; see Sato (1999, Theorem 5.28.2).

(ii) $F_t$ has a pdf in $C^\infty(0, \infty)$ if

$$\liminf_{t \downarrow 0} t^{\alpha - 2} \int_0^t x^2 g(x) \, dx > 0,$$

for some $\alpha \in (0, 2)$, where $g$ is the density of $\Pi$; see, e.g. Orey (1968).

(iii) $F_t(\cdot) \in C^1(0, \infty)$ if $k(x) := xg(x)$ is decreasing, which is equivalent to the self-decomposability of $F_t$, and $k(0+) < \infty$. Moreover, $F_t(\cdot) \in C^\infty(0, \infty)$ if $k(0+) = \infty$. See Sato (1999, Theorem 5.28.4) for these results.

As a direct application of Theorem 2.1, we are able to obtain a closed-form formula for the par credit spread $c(T)$ by plugging formula (2.6) with $u = \log(V_0/L)$ into formula (2.2).

2.3 Term structure of CDS with short maturity

One advantage of a closed-form formula for $\delta(t,u)$ is that it can explain why $c(T)$ does not go to zero as $T$ goes to zero. Actually, if $\lim_{t \downarrow 0} \delta(t) = 1$ and $\delta(\cdot) \in C^1(0, \infty)$, then by L'Hôpital’s rule we obtain from (2.2) that

$$\lim_{T \downarrow 0} c(T) = (1 - R) \left( \lim_{T \downarrow 0} \frac{r e^{-rT} \delta(T) - e^{-rT} \delta'(T)}{e^{-rT} \delta(T)} - r \right)$$

$$= -(1 - R) \lim_{T \downarrow 0} \delta'(T).$$

(2.8)

As Theorem 2.2 will show, downward jumps in $S$ guarantee that $\lim_{T \downarrow 0} \delta'(T) < 0$ and thus a positive limiting credit spread. Note that a positive limiting credit spread can also be achieved in a continuous geometric diffusion model where the information set available is not complete. See Jarrow and Protter (2004) and references therein for details on this direction of research.
Theorem 2.2  Given conditions in Theorem 2.1, we additionally assume that \( F_t(x) \) and \( f_t(x) \) are continuous in a neighborhood of \((t = 0, x)\) for all \( x > 0 \). Furthermore, assume that 

\[
(\partial/\partial t)F_t(x), \quad (\partial/\partial t)f_t(x) \quad \text{and} \quad (\partial/\partial x)f_t(x)
\]
exist and are also continuous at \((t = 0, x)\) for all \( x > 0 \). If the Lévy measure \( \Pi \) admits a density \( g \), then for fixed \( u > 0 \) we have

\[
\lim_{t \downarrow 0} \delta(t, u) = 1 \quad \text{and} \quad \lim_{t \downarrow 0} \frac{\partial}{\partial t} \delta(t, u) = -\Pi([u, \infty)) = - \int_u^\infty g(x)dx. \tag{2.9}
\]

Remark 2.5 Relations (2.8) and (2.9) together give that

\[
\lim_{T \downarrow 0} c(T) = (1 - R) \int_{\log(V_0/L)}^\infty g(x)dx. \tag{2.10}
\]

So the credit spread does not go to zero when maturity becomes short and the limiting spread is determined by the Lévy measure of \( S \) and the recovery rate. Note that interest rate \( r \) does not show up on the right-hand side of relation (2.10), which is reasonable since interest rates should not affect the credit spread with very short maturity.

Remark 2.6 Relation (2.10) has also been obtained by Ruf and Scherer (2011, Theorems 3.1 and 3.2) for the case where \( S \) is a jump-diffusion process having two-sided jumps. Their results show that if \( S \) contains only Poissonian jumps then the limiting credit spread as maturity goes to zero is given by the product of the local default rate and the expected loss given default, which is exactly as our relation (2.10) shows. So our Theorem 2.2 complements their results for the case where \( S \) has infinite downward jumps.

3 Examples

As stated in Remark 2.2, to evaluate \( \delta(t, u) \) by using formula (2.7) one needs to know \( f_s \) for \( 0 < s \leq t \). In this section, we consider two special cases of subordinator: gamma processes and inverse Gaussian processes, for which the marginal pdfs are explicitly known.

3.1 Gamma process

Recall that the gamma\((a, b)\) distribution with parameters \( a, b > 0 \) has the pdf

\[
f(x; a, b) = \frac{b^a}{\Gamma(a)}x^{a-1}e^{-bx}, \quad x > 0.
\]

A Lévy process \( S = \{S_t, t \geq 0\} \) is called a gamma\((a, b)\) process if \( S_t \) follows the gamma\((at, b)\) distribution for every \( t > 0 \). The Lévy density of a gamma\((a, b)\) process is given by \( g(x) = ax^{-1}e^{-bx}, \quad x \in (0, \infty) \).

The following is a corollary of Theorems 2.1 and 2.2:
Corollary 3.1 Let $\mu > 0$ and $S$ be a gamma($a, b$) process with parameters $a, b > 0$.

(i) For $t, u > 0$,
\[
\delta(t, u) = F_t(u + \mu t) - \mu \int_0^t F_s(\mu s) f_{t-s}(u + \mu (t - s)) \, ds \\
+ \frac{a}{b} \int_0^t F_{s+1/a}(\mu s) f_{t-s}(u + \mu (t - s)) \, ds,
\]  
(3.1)

where $f_t$ and $F_t$ are the pdf and cdf of the gamma($at, b$) distribution, respectively.

(ii) For fixed $u > 0$,
\[
\lim_{t \downarrow 0} \delta(t, u) = 1 \quad \text{and} \quad \lim_{t \downarrow 0} \frac{\partial}{\partial t} \delta(t, u) = - \int_u^\infty ax^{-1} e^{-bx} \, dx.
\]

Again, formula (3.1) with $a = b = 1$ was first obtained by Dickson and Waters (1993).

3.2 Inverse Gaussian process

Recall that the inverse Gaussian distribution, IG($a, b$), with parameters $a, b > 0$ has the pdf
\[
f(x; a, b) = \frac{a}{\sqrt{2\pi}} e^{ab} x^{-3/2} \exp\{-a^2x^{-1} + b^2x\}/2\}, \quad x > 0.
\]

A Lévy process $S = \{S_t, t \geq 0\}$ is called an IG($a, b$) process if $S_t$ follows the IG($at, b$) distribution for every $t > 0$. The Lévy density of an IG($a, b$) process is given by $g(x) = ae^{-b^2x^2/2}\sqrt{2\pi x^3}$, $x \in (0, \infty)$.

The following is also a corollary of Theorems 2.1 and 2.2:

Corollary 3.2 Let $\mu > 0$ and $S$ be an IG($a, b$) process with parameters $a, b > 0$.

(i) For $t, u > 0$,
\[
\delta(t, u) = F_t(u + \mu t) - \mu \int_0^t F_s(\mu s) f_{t-s}(u + \mu (t - s)) \, ds \\
+ \int_0^t \frac{1}{s} \left( \int_0^{\mu s} xf_s(x) \, dx \right) f_{t-s}(u + \mu (t - s)) \, ds,
\]

where $f_t$ and $F_t$ are the pdf and cdf of the IG($at, b$) distribution, respectively.

(ii) For fixed $u > 0$,
\[
\lim_{t \downarrow 0} \delta(t, u) = 1 \quad \text{and} \quad \lim_{t \downarrow 0} \frac{\partial}{\partial t} \delta(t, u) = - \int_u^\infty \frac{ae^{-b^2x^2/2}}{\sqrt{2\pi x^3}} \, dx.
\]
4 Calibration

We calibrate the gamma model and the IG model from Section 3 to two families of CDS quotes: CDX.NA.IG index (125 North American companies) and iTraxx.EUR index (125 European companies). The quotes were collected on December 9, 2011. For each company, we have its market CDS par spreads with seven maturities (1 year, 2 years, 3 years, 4 years, 5 years, 7 years, and 10 years). Altogether we have 250 real market CDS term structures.

In the calibration, models will be calibrated in order to match the real market spreads as accurately as possible. Specifically, for each reference entity we minimize the mean absolute error (MAE) given by

\[
MAE = \frac{\sum_{\text{CDS par spreads}} |\text{Market CDS par spread} - \text{Model CDS par spread}|}{\text{Number of CDS par spreads}},
\]

where the sum over CDS par spreads refers to all the CDS quotes from the company’s real market CDS term structure. We carry out the calibration using R packages on a Linux computer with an Intel Core i5 CPU (2.27 GHz). Aided by formula (2.7), the calculations for \( \delta(t,u) \) and \( c(T) \) are very efficient. Under the gamma(IG) model the computer takes about 0.004(0.056) seconds to calculate 10,000 CDS spreads.

Table 1 and Figure 1 are here.

Table 1 shows part of the calibration results for the CDX index. According to Moody’s long-term credit ranking on December 9, 2011, we select two companies from each risk category from Aa2 to Baa3 for demonstration. We apply the gamma model and the IG model on each company and always assume that \( r = 1\% \) and \( R = L/V_0 = 40\% \). Here the 40% recovery rate is used for all companies. In reality, recovery rates vary across industries as shown in Table 3 of Altman and Kishore (1996), which also shows an average recovery rate around 40% of 696 defaulted bonds from different industries during the time period from 1971 to 1995. For simplicity we use this average recovery rate in the calibration. Our target is to find out the optimal pair \((a,b)\) with the minimum MAE for each company. It can be seen from Table 1 that both gamma and IG models well fit CDS term structures in the CDX index. In particular, Figure 1 illustrates the results for McDonald’s CDS term structure.

Table 2 and Figure 2 are here.

Table 2 shows the similar calibration results for the iTraxx index. Note that in iTraxx we are able to find companies such as Nestle and Royal Dutch Shell from a higher category.
Aa1. It can be seen from Table 2 that both gamma and IG models well fit CDS term structures in the iTraxx index. In particular, Figure 2 illustrates the results for Siemens’ CDS term structure.

It is worthwhile to point out that almost all the term structures in the 250 companies are upward sloping, namely, CDS spreads are increasing over years. Both gamma and IG models overall fit this kind of term structure very well. Among the 125 North American companies in the CDX index, 79(80) companies have MAE of less than 5 bpts using the gamma(IG) model. Among the 125 European companies in the iTraxx index, 102(99) companies have MAE of less than 5 bpts using the gamma(IG) model. Moreover, the models can also be used to cope with the situations in which a decreasing or a humped term structure is present. See Figures 3 and 4 for artificial spread term structures of these two kinds from both models. The other side of the coin is that the models do not fit well for the term structures with large CDS spread values. For instance, the MAE is as high as 26 bpts when calibrating the gamma model on Computer Sciences’ CDS term structure (210, 263, 293, 334, 366, 397, 423). This is definitely a big concern for using our model. We are still trying to understand the underlying problem causing this limitation.

Next in Tables 3 and 4 we check that the limiting par credit spread when the maturity goes to zero is indeed determined by formula (2.10). We first calculate the limiting par spread directly by plugging formula (2.7) into (2.2) and letting $T$ go to zero. Then we compare the numerical results with the theoretical results obtained from (2.8). Here we still assume $R = L/V_0 = 0.4$. For a variety of values for $(a, b, r)$, it is clearly seen that the numerical results and the theoretical results are very close and that interest rates do not affect the limiting credit spread.

5 Proofs

5.1 Proof of Theorem 2.1

Although the corresponding result is essentially obtained by Dickson and Waters (1993); see also Dufresne et al. (1991) in the context of a gamma subordinator, we will give an
alternative proof for a general subordinator using a weak convergence result in $\mathbb{D}$-space, which is a space of all the càdlàg functions on $[0, t]$ embedded in the Skorokhod topology.

Since $\{S_t - \mu t, t \geq 0\}$ is a finite variation process with downward drift, the point 0 is irregular for $(0, \infty)$; see, e.g., Kyprianou (2006, Theorem 6.5). Then by the same argument as in the proof of Theorem V.2.2 of Asmussen and Albrecher (2010) we have that

$$\delta^*(t, u) = F_\mu(u + \mu t) - \mu \int_0^t \delta^*(t - s, 0) f_s(u + \mu s) \, ds. \tag{5.1}$$

We can also show (5.1) by the convergence in (5.2) below. Note that $f_t(x)$ is continuous in $x > 0$ under the assumption. Then, relation (5.1) and the dominated convergence theorem imply that the function $\delta^*(t, u)$ for every $t > 0$ is continuous in $u \in (0, \infty)$. Therefore, relation (2.5) holds.

We derive a closed-form formula for $\delta(t, 0)$ first. Consider a sequence of drifted compound Poisson processes

$$S_i(t) - \mu_i t = \int_0^t \int_{z \geq 1/n} z \, N(ds, dz) - \mu_i t, \quad n = 1, 2, \ldots,$$

where $\mu_i \to \mu$ as $n \to \infty$, and $N$ is a jump-counting measure of $S$. Then we see that

$$\{S^{(n)}_t - \mu_n s, 0 \leq s \leq t\}$$

converges in law to $\{S_s - \mu s, 0 \leq s \leq t\}$ in $\mathbb{D}[0, t]$-space; see, e.g., Jacod and Shiryaev (2003, Corollary VII.3.6) or Asmussen et al. (2004, Proposition 1). Moreover, it is easy to see that a supremum $f[\cdot] := \sup_{0 \leq s \leq t}(\cdot)$ is a continuous functional on $\mathbb{D}[0, t]$; see Jacod and Shiryaev (2003, Chapter VI) or Embrechts et al. (1997, Section A.2.5). Hence, it follows from the continuous mapping theorem that

$$\sup_{0 \leq s \leq t} (S^{(n)}_s - \mu_n s) \xrightarrow{D} \sup_{0 \leq s \leq t} (S_s - \mu s), \quad n \to \infty,$$

which yields that

$$\delta^*_n(t, u) := \mathbb{Q}\left(\sup_{0 \leq s \leq t} (S^{(n)}_s - \mu s) \leq u\right) \to \delta^*(t, u), \quad n \to \infty, \tag{5.2}$$

for every $t, u > 0$. Since $\delta^*_n(t, u)$ is non-decreasing, uniformly bounded, and continuous in $u \in (0, \infty)$, we see that the convergence in (5.2) is uniform in $u \in (0, \infty)$. So, for any $\epsilon > 0$, $\sup_{u > 0} |\delta^*_n(t, u) - \delta^*(t, 0)| < \epsilon/3$ for $n$ large enough. Moreover, noticing (5.1) and the similar equality for $\delta^*_n$, we see that $\delta(t, u)$ and $\delta^*_n(t, u)$ are right-continuous at $u = 0$. Hence it follows that for any $\epsilon > 0$ there exists a sequence $\{u_n, n = 1, 2, \ldots\}$ with $|\delta^*(t, u_n) - \delta^*(t, 0)| < \epsilon/3$ and $|\delta^*_n(t, u_n) - \delta^*_n(t, 0)| < \epsilon/3$ such that

$$|\delta^*_n(t, 0) - \delta^*(t, 0)| \leq |\delta^*_n(t, 0) - \delta^*_n(t, u_n)| + |\delta^*_n(t, u_n) - \delta^*(t, u_n)| + |\delta^*(t, u_n) - \delta^*(t, 0)|$$

$$\leq \epsilon/3 + \sup_{u > 0} |\delta^*_n(t, u) - \delta^*(t, 0)| + \epsilon/3 < \epsilon,$$
which yields that
\[ \delta^*_n(t,0) \to \delta^*(t,0), \quad n \to \infty. \tag{5.3} \]
On the other hand, denoted by \( F_t^{(n)} \) the cdf of \( S_t^{(n)} \), we see from Theorem V.2.1 of Asmussen and Albrecher (2010) that
\[ \delta^*_n(t,0) = \frac{1}{\mu t} \int_0^{\mu t} F_t^{(n)}(x) \, dx. \]
Furthermore, it follows that for every \( t, x > 0 \)
\[ F_t^{(n)}(x) \to F_t(x), \quad n \to \infty, \tag{5.4} \]
since any marginal distributions of \( S^{(n)} \) converges to those of \( S \) when \( S^{(n)} \xrightarrow{D} S \). Hence, it follows from (5.3), (5.4) and the dominated convergence theorem that
\[ \delta^*(t,0) = \frac{1}{\mu t} \int_0^{\mu t} F_t(x) \, dx. \]
The above equality and relation (5.1) together yield (2.6).

### 5.2 Proof of Theorem 2.2

First, we prove the first relation of (2.9). Recall formula (2.7):
\[
\delta(t,u) = F_t(u+\mu t) - \mu \int_0^t F_s(\mu s) f_1(t-s)(u+\mu(t-s)) \, ds \\
+ \int_0^t \frac{1}{s} \left( \int_0^s x f_s(x) \, dx \right) f_1(t-s)(u+\mu(t-s)) \, ds \\
= I_1(t) - I_2(t) + I_3(t).
\]
By Corollary 3 of Rüschendorf and Woerner (2002), for each fixed \( x > 0 \), we have
\[ \lim_{t \downarrow 0} \frac{1}{t} (1 - F_t(x)) = \Pi([x, \infty)) \tag{5.5} \]
and
\[ \lim_{t \downarrow 0} \frac{1}{t} f_t(x) = g(x). \tag{5.6} \]
Relation (5.5) and the continuity of \( F_t(x) \) at \( t = 0, x = u \) immediately imply that \( \lim_{t \downarrow 0} I_1(t) = 1 \). For \( I_2(t) \), we derive as follows:
\[
0 \leq I_2(t) \leq \mu \int_0^t f_1(t-s)(u+\mu(t-s)) \, ds \\
= \mu \int_0^t f_s(u+\mu s) \, ds \\
\to 0, \quad \text{as } t \to 0,
\]
where in the last step we use relation (5.6) and the continuity of $f_t(x)$ at $(t = 0, x = u)$. As to $I_3(t)$, since for each $s > 0$, 

$$\frac{1}{s} \int_0^{\mu s} x f_s(x) \, dx \leq \mu F_s(\mu s) \leq \mu,$$

we similarly obtain that $\lim_{t \downarrow 0} I_3(t) = 0$. Hence,

$$\lim_{t \downarrow 0} \delta(t, u) = 1.$$

Next, we prove the second relation of (2.9). Note that

$$\frac{\partial}{\partial t} \delta(t, u) = \mu f_t(u + \mu t) + \frac{\partial}{\partial t} F_t(x) \bigg|_{x = u + \mu t}$$

$$- \mu \int_0^t F_s(\mu s) \frac{\partial}{\partial t} f_{t-s}(u + \mu (t-s)) \, ds$$

$$+ \int_0^t \frac{1}{s} \left( \int_0^{\mu s} x f_s(x) \, dx \right) \frac{\partial}{\partial t} f_{t-s}(u + \mu (t-s)) \, ds$$

$$= J_1(t) + J_2(t) - J_3(t) + J_4(t).$$

Obviously, $\lim_{t \downarrow 0} J_1(t) = 0$ by (5.6). Since $(\partial/\partial t) F_t(x)$ is continuous at $(t = 0, x = u)$, by (5.5) we obtain

$$\lim_{t \downarrow 0} J_2(t) = -\Pi([u, \infty)).$$

Therefore, the proof is complete if we can show that

$$\lim_{t \downarrow 0} J_3(t) = 0 \quad \text{and} \quad \lim_{t \downarrow 0} J_4(t) = 0.$$

Actually,

$$|J_3(t)| \leq \mu \int_0^t \left| \frac{\partial}{\partial t} f_{t-s}(u + \mu (t-s)) \right| \, ds = \mu \int_0^t \left| \frac{\partial}{\partial s} f_s(u + \mu s) \right| \, ds$$

$$= t \left| \frac{\partial}{\partial s} f_s(u + \mu s) \right|_{s = s(t)}$$

$$\leq t \left| \frac{\partial}{\partial s} f_s(x) \right|_{s = s(t), x = u + \mu s(t)} + \mu t \left| \frac{\partial}{\partial x} f_s(x) \right|_{s = s(t), x = u + \mu s(t)}$$

(5.7)

for some $s(t) \in [0, t]$. Since $\lim_{t \downarrow 0} s(t) = 0$, by (5.6) and the continuity of $(\partial/\partial t) f_t(x)$ at $(t = 0, x = u)$ we have

$$\lim_{t \downarrow 0} \frac{\partial}{\partial s} f_s(x) \bigg|_{s = s(t), x = u + \mu s(t)} = g(u).$$

(5.8)
Moreover, by (5.5) and the continuity of \((\partial/\partial x)f_t(x)\) at \((t = 0, x = u)\) we have
\[
\lim_{t \downarrow 0} \frac{\partial}{\partial x} f_s(x) \bigg|_{s = s(t), x = u + \mu s(t)} = 0. \tag{5.9}
\]
Plugging (5.8) and (5.9) into (5.7) we obtain
\[
\lim_{t \downarrow 0} J_3(t) = 0.
\]
Since
\[
|J_4(t)| \leq \int_0^t \frac{1}{s} \left( \int_0^{\mu s} xf_s(x) \, dx \right) \left| \frac{\partial}{\partial t} f_{t-s}(u + \mu(t - s)) \right| \, ds \\
\leq \mu \int_0^t F(\mu s) \left| \frac{\partial}{\partial t} f_{t-s}(u + \mu(t - s)) \right| \, ds \\
\leq \mu \int_0^t \left| \frac{\partial}{\partial t} f_{t-s}(u + \mu(t - s)) \right| \, ds,
\]
we similarly have
\[
\lim_{t \downarrow 0} J_4(t) = 0.
\]
This completes the proof.

### 5.3 Proof of Corollary 3.1

For part (i), since
\[
\int_0^{\mu s} x F_s(dx) = \frac{\Gamma(as + 1)}{\Gamma(as)} \int_0^{\mu s} \frac{b^{as}}{\Gamma(as + 1)} x^{as} e^{-bx} \, dx = \frac{as}{b} F_{s+1/a}(\mu s),
\]
formula (3.1) follows immediately from formula (2.7).

We are going to apply Theorem 2.2 to prove part (ii). For a gamma\((a, b)\) process \(S\), obviously \(F_t(x), f_t(x),\) and \((\partial/\partial x)f_t(x)\) are continuous at \((t = 0, x)\) for all \(x > 0\). Also, by Theorem 1 and Corollary 3 of Rüschendorf and Woerner (2002), relations (5.5) and (5.6) hold. So, to prove the continuity of \((\partial/\partial t)f_t(x)\) and \((\partial/\partial t)F_t(x)\) at \((t = 0, x)\) for \(x > 0\) is equivalent to prove that
\[
\lim_{t \downarrow 0} \frac{\partial}{\partial t} f_t(x) = ax^{-1} e^{-bx} \quad \text{and} \quad \lim_{t \downarrow 0} \frac{\partial}{\partial t} F_t(x) = - \int_x^\infty ay^{-1} e^{-by} \, dy. \tag{5.10}
\]
It is known that
\[
\frac{\partial}{\partial t} f_t(x) = af_t(x) (\log b + \log x - \psi(at)),
\]
where \(\psi(x)\) is the digamma function.
where \( \psi(\cdot) = \Gamma'(\cdot) / \Gamma(\cdot) \) is the so-called \textit{digamma function}. Note that we have the following summation representation for the digamma function:

\[
\psi(t) = -\gamma - \frac{1}{t} + \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k + t} \right), \quad t \neq 0,
\]

\[
= -\gamma - \frac{1}{t} + O(t), \quad \text{(5.11)}
\]

and the following \textit{Laurent expansion} for the gamma function:

\[
\Gamma(t) = \sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)}{k!} t^{k-1}, \quad |t| < 1,
\]

\[
= \frac{1}{t} - \gamma + \frac{1}{6} \left( 3\gamma^2 + \frac{\pi^2}{2} \right) t + O(t^2), \quad \text{(5.12)}
\]

where \( \gamma \) is the \textit{Euler-Mascheroni} constant. Hence, using (5.11) and (5.12),

\[
\frac{\partial}{\partial t} f_t(x) = \frac{ab^a x^{at-1} e^{-bx}}{\Gamma(at)} \left( \log b + \log x + \gamma + \frac{1}{at} + O(t) \right)
\]

\[
= \frac{ab^a x^{at-1} e^{-bx}}{\Gamma(at) at} + o(1)
\]

\[
\rightarrow ax^{-1} e^{-bx} \quad \text{as} \ t \to 0.
\]

Then we prove the second relation in (5.10). Using (5.5) and (5.11) we derive

\[
\frac{\partial}{\partial t} F_t(x) = \int_0^x \frac{\partial}{\partial t} f_t(y) dy
\]

\[
= a(\log b - \psi(at)) \int_0^x f_t(y) dy + a \int_0^x f_t(y) \log y \ dy
\]

\[
= a(\psi(at) - \log b) \int_x^{\infty} f_t(y) dy - a \int_x^{\infty} f_t(y) \log y \ dy
\]

\[
= a\psi(at)(1 - F_t(x)) + o(1)
\]

\[
\rightarrow - \int_x^{\infty} ay^{-1} e^{-by} dy, \quad \text{as} \ t \to 0,
\]

where we use, in the third step, the fact that if a random variable \( X \) has the gamma(\( a, b \)) distribution, then \( \mathbb{E} \log X = \psi(a) - \log b \).

5.4 \textbf{Proof of Corollary 3.2}

Part (i) is a direct application of Theorem 2.1. Again we are going to apply Theorem 2.2 to prove part (ii). For an IG(\( a, b \)) process \( S \), obviously \( F_t(x) \), \( f_t(x) \), and \( (\partial/\partial x)f_t(x) \) are
continuous at \((t = 0, x)\) for all \(x > 0\). Also, by Theorem 1 and Corollary 3 of Rüschendorf and Woerner (2002), relations (5.5) and (5.6) hold. So, to prove the continuity of \(\frac{\partial}{\partial t} f_t(x)\) and \(\frac{\partial}{\partial t} F_t(x)\) at \((t = 0, x)\) for \(x > 0\) is equivalent to prove that

\[
\lim_{t \downarrow 0} \frac{\partial}{\partial t} f_t(x) = \frac{ae^{-b^2tx/2}}{\sqrt{2\pi x^3}} \quad \text{and} \quad \lim_{t \downarrow 0} \frac{\partial}{\partial t} F_t(x) = -\int_x^\infty \frac{ae^{-b^2y/2}}{\sqrt{2\pi y^3}} dy.
\]

It is easy to obtain that

\[
\frac{\partial}{\partial t} f_t(x) = f_t(x) \left(ab + t^{-1} - a^2tx^{-1}\right).
\]

Therefore,

\[
\frac{\partial}{\partial t} f_t(x) = \frac{f_t(x)}{t} + o(1) \rightarrow \frac{ae^{-b^2x/2}}{\sqrt{2\pi x^3}}, \quad \text{as } t \to 0.
\]

And, using (5.5),

\[
\frac{\partial}{\partial t} F_t(x) = \int_0^x \frac{\partial}{\partial t} f_t(y) dy = (ab + t^{-1}) \int_0^x f_t(y) dy - a^2t \int_0^x f_t(y)y^{-1} dy = -(ab + t^{-1}) \int_x^\infty f_t(y) dy + a^2t \int_x^\infty f_t(y)y^{-1} dy = -\frac{1}{t}(1 - F_t(x)) + o(1) \rightarrow -\int_x^\infty \frac{ae^{-b^2y/2}}{\sqrt{2\pi y^3}} dy,
\]

where we use, in the third step, the fact that if a random variable \(X\) has the IG\((a, b)\) distribution, then \(\mathbb{E}X^{-1} = ba^{-1} + a^{-2}\). See, for example, Tweedie (1957, Section 6).

**Acknowledgments.** The authors would like to thank the anonymous reviewer for a number of valuable comments, which helped us improve the paper greatly. XH acknowledges support of the Natural Sciences and Engineering Research Council of Canada (grant no. 386552-2010) and the Start-up Research Fund from the University of Manitoba. XL acknowledges support of the Start-up Funds from the University of Minnesota Duluth. YS acknowledges support of JSPS KAKENHI grant no. 24740061 and JST CREST.

**References**


Table 1: Calibration on CDS term structures in the CDX index

<table>
<thead>
<tr>
<th>Company</th>
<th>Moody's</th>
<th>1y</th>
<th>2y</th>
<th>3y</th>
<th>4y</th>
<th>5y</th>
<th>7y</th>
<th>10y</th>
<th>a</th>
<th>b</th>
<th>MAE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wal-Mart</td>
<td>Aa2</td>
<td>17</td>
<td>26</td>
<td>34</td>
<td>46</td>
<td>54</td>
<td>63</td>
<td>75</td>
<td>1.0723</td>
<td>5.7763</td>
<td>0.9342</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8581</td>
<td>2.9853</td>
<td>0.8779</td>
</tr>
<tr>
<td>GE</td>
<td>Aa2</td>
<td>129</td>
<td>170</td>
<td>193</td>
<td>208</td>
<td>235</td>
<td>230</td>
<td>230</td>
<td>0.9678</td>
<td>3.4109</td>
<td>3.1505</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9388</td>
<td>2.2315</td>
<td>3.1711</td>
</tr>
<tr>
<td>UPS</td>
<td>Aa3</td>
<td>20</td>
<td>28</td>
<td>37</td>
<td>48</td>
<td>54</td>
<td>63</td>
<td>73</td>
<td>0.8402</td>
<td>5.1503</td>
<td>0.8670</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.7165</td>
<td>2.8235</td>
<td>0.8149</td>
</tr>
<tr>
<td>IBM</td>
<td>Aa3</td>
<td>15</td>
<td>20</td>
<td>28</td>
<td>35</td>
<td>43</td>
<td>56</td>
<td>68</td>
<td>0.8581</td>
<td>2.9853</td>
<td>0.8779</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.7165</td>
<td>2.8235</td>
<td>0.8149</td>
</tr>
<tr>
<td>ACE</td>
<td>A1</td>
<td>33</td>
<td>45</td>
<td>57</td>
<td>71</td>
<td>80</td>
<td>88</td>
<td>94</td>
<td>0.7893</td>
<td>4.5058</td>
<td>1.7606</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.7236</td>
<td>2.6438</td>
<td>1.7869</td>
</tr>
<tr>
<td>Cisco</td>
<td>A1</td>
<td>33</td>
<td>53</td>
<td>70</td>
<td>88</td>
<td>106</td>
<td>130</td>
<td>149</td>
<td>2.3463</td>
<td>6.9338</td>
<td>5.3810</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.5774</td>
<td>3.1977</td>
<td>5.0485</td>
</tr>
<tr>
<td>McDonald's</td>
<td>A2</td>
<td>8</td>
<td>12</td>
<td>18</td>
<td>22</td>
<td>25</td>
<td>31</td>
<td>39</td>
<td>0.7844</td>
<td>6.0141</td>
<td>0.3810</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.6503</td>
<td>3.0983</td>
<td>0.4039</td>
</tr>
<tr>
<td>Honeywell</td>
<td>A2</td>
<td>15</td>
<td>22</td>
<td>31</td>
<td>41</td>
<td>47</td>
<td>55</td>
<td>65</td>
<td>0.8812</td>
<td>5.4827</td>
<td>0.8272</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.7569</td>
<td>2.9512</td>
<td>0.7943</td>
</tr>
<tr>
<td>Sherwin-Williams</td>
<td>A3</td>
<td>16</td>
<td>29</td>
<td>42</td>
<td>54</td>
<td>61</td>
<td>74</td>
<td>87</td>
<td>1.3622</td>
<td>6.2352</td>
<td>0.5173</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.1267</td>
<td>3.1649</td>
<td>0.4765</td>
</tr>
<tr>
<td>Baxter Intl</td>
<td>A3</td>
<td>13</td>
<td>19</td>
<td>25</td>
<td>32</td>
<td>36</td>
<td>45</td>
<td>55</td>
<td>0.8030</td>
<td>5.5796</td>
<td>0.7111</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.6590</td>
<td>2.9377</td>
<td>0.5291</td>
</tr>
<tr>
<td>Black &amp; Decker</td>
<td>Baa1</td>
<td>14</td>
<td>21</td>
<td>30</td>
<td>38</td>
<td>46</td>
<td>57</td>
<td>66</td>
<td>1.1855</td>
<td>6.0379</td>
<td>0.4295</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9254</td>
<td>3.1384</td>
<td>0.3700</td>
</tr>
<tr>
<td>Ingersoll-Rand</td>
<td>Baa1</td>
<td>11</td>
<td>22</td>
<td>28</td>
<td>36</td>
<td>40</td>
<td>48</td>
<td>56</td>
<td>0.7513</td>
<td>5.2958</td>
<td>0.6606</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.6422</td>
<td>2.8747</td>
<td>0.7934</td>
</tr>
<tr>
<td>McKesson</td>
<td>Baa2</td>
<td>10</td>
<td>16</td>
<td>22</td>
<td>29</td>
<td>36</td>
<td>45</td>
<td>52</td>
<td>1.0589</td>
<td>6.3852</td>
<td>0.6441</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.7700</td>
<td>3.0986</td>
<td>0.6379</td>
</tr>
<tr>
<td>Cox Comm</td>
<td>Baa2</td>
<td>21</td>
<td>32</td>
<td>44</td>
<td>57</td>
<td>66</td>
<td>79</td>
<td>90</td>
<td>1.3432</td>
<td>6.0853</td>
<td>0.9165</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.0077</td>
<td>3.0250</td>
<td>0.8539</td>
</tr>
<tr>
<td>Cardinal Health</td>
<td>Baa3</td>
<td>19</td>
<td>27</td>
<td>37</td>
<td>49</td>
<td>56</td>
<td>65</td>
<td>76</td>
<td>0.9365</td>
<td>5.3741</td>
<td>1.1603</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8048</td>
<td>2.9138</td>
<td>1.1778</td>
</tr>
<tr>
<td>CSX</td>
<td>Baa3</td>
<td>24</td>
<td>31</td>
<td>41</td>
<td>52</td>
<td>63</td>
<td>76</td>
<td>89</td>
<td>1.3730</td>
<td>6.2221</td>
<td>1.9240</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.0758</td>
<td>3.1140</td>
<td>1.7282</td>
</tr>
</tbody>
</table>
Figure 1: Calibration of gamma and IG models for McDonald’s CDS term structure.
Table 2: Calibration on CDS term structures in the iTraxx index

<table>
<thead>
<tr>
<th>Company</th>
<th>Moody's</th>
<th>1y</th>
<th>2y</th>
<th>3y</th>
<th>4y</th>
<th>5y</th>
<th>7y</th>
<th>10y</th>
<th>a</th>
<th>b</th>
<th>MAE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nestle</td>
<td>Aa1</td>
<td>22</td>
<td>27</td>
<td>36</td>
<td>43</td>
<td>50</td>
<td>57</td>
<td>65</td>
<td>0.7023</td>
<td>4.8862</td>
<td>0.8131</td>
</tr>
<tr>
<td>Royal Dutch Shell</td>
<td>Aa1</td>
<td>46</td>
<td>59</td>
<td>72</td>
<td>83</td>
<td>90</td>
<td>99</td>
<td>105</td>
<td>0.6264</td>
<td>3.8343</td>
<td>0.8977</td>
</tr>
<tr>
<td>Statoil ASA</td>
<td>Aa2</td>
<td>58</td>
<td>72</td>
<td>83</td>
<td>95</td>
<td>102</td>
<td>111</td>
<td>119</td>
<td>0.6008</td>
<td>3.5917</td>
<td>0.7814</td>
</tr>
<tr>
<td>Credit Suisse</td>
<td>Aa2</td>
<td>111</td>
<td>125</td>
<td>138</td>
<td>153</td>
<td>163</td>
<td>164</td>
<td>165</td>
<td>0.4373</td>
<td>2.5501</td>
<td>2.7850</td>
</tr>
<tr>
<td>Munich Reinsurance</td>
<td>Aa3</td>
<td>53</td>
<td>61</td>
<td>70</td>
<td>79</td>
<td>85</td>
<td>92</td>
<td>97</td>
<td>0.4003</td>
<td>3.1307</td>
<td>0.8949</td>
</tr>
<tr>
<td>JTI UK Finance</td>
<td>Aa3</td>
<td>17</td>
<td>23</td>
<td>28</td>
<td>35</td>
<td>40</td>
<td>50</td>
<td>58</td>
<td>0.7229</td>
<td>5.1432</td>
<td>1.1124</td>
</tr>
<tr>
<td>Siemens</td>
<td>A1</td>
<td>34</td>
<td>50</td>
<td>63</td>
<td>73</td>
<td>81</td>
<td>92</td>
<td>100</td>
<td>0.8195</td>
<td>4.5252</td>
<td>0.2489</td>
</tr>
<tr>
<td>ENI SpA</td>
<td>A1</td>
<td>116</td>
<td>141</td>
<td>157</td>
<td>168</td>
<td>175</td>
<td>182</td>
<td>188</td>
<td>0.5637</td>
<td>2.7969</td>
<td>0.6538</td>
</tr>
<tr>
<td>Sanofi-Aventis</td>
<td>A2</td>
<td>44</td>
<td>55</td>
<td>63</td>
<td>72</td>
<td>79</td>
<td>86</td>
<td>92</td>
<td>0.4816</td>
<td>3.5408</td>
<td>0.5692</td>
</tr>
<tr>
<td>Vattenfall AB</td>
<td>A2</td>
<td>40</td>
<td>52</td>
<td>65</td>
<td>75</td>
<td>87</td>
<td>95</td>
<td>105</td>
<td>0.8712</td>
<td>4.5971</td>
<td>1.2847</td>
</tr>
<tr>
<td>Groupe Danone</td>
<td>A3</td>
<td>41</td>
<td>56</td>
<td>68</td>
<td>80</td>
<td>86</td>
<td>100</td>
<td>107</td>
<td>0.7741</td>
<td>4.2896</td>
<td>0.7705</td>
</tr>
<tr>
<td>TeliaSonera AB</td>
<td>A3</td>
<td>31</td>
<td>47</td>
<td>62</td>
<td>76</td>
<td>85</td>
<td>98</td>
<td>107</td>
<td>0.9232</td>
<td>2.6309</td>
<td>0.6592</td>
</tr>
<tr>
<td>Pearson</td>
<td>Baa1</td>
<td>26</td>
<td>35</td>
<td>45</td>
<td>54</td>
<td>64</td>
<td>75</td>
<td>84</td>
<td>0.9055</td>
<td>5.2192</td>
<td>0.9299</td>
</tr>
<tr>
<td>Wolters Kluwer</td>
<td>Baa1</td>
<td>30</td>
<td>46</td>
<td>60</td>
<td>72</td>
<td>79</td>
<td>89</td>
<td>97</td>
<td>0.9148</td>
<td>4.8261</td>
<td>0.9100</td>
</tr>
<tr>
<td>Royal KPN NV</td>
<td>Baa2</td>
<td>55</td>
<td>76</td>
<td>93</td>
<td>108</td>
<td>116</td>
<td>128</td>
<td>136</td>
<td>0.8633</td>
<td>4.1123</td>
<td>0.6157</td>
</tr>
<tr>
<td>British Telecom</td>
<td>Baa2</td>
<td>54</td>
<td>75</td>
<td>91</td>
<td>106</td>
<td>114</td>
<td>124</td>
<td>131</td>
<td>0.8117</td>
<td>4.0355</td>
<td>0.8200</td>
</tr>
<tr>
<td>Royal Ahold</td>
<td>Baa3</td>
<td>54</td>
<td>80</td>
<td>101</td>
<td>121</td>
<td>132</td>
<td>145</td>
<td>154</td>
<td>1.2400</td>
<td>4.7376</td>
<td>1.1205</td>
</tr>
<tr>
<td>Tate &amp; Lyle</td>
<td>Baa3</td>
<td>25</td>
<td>42</td>
<td>57</td>
<td>75</td>
<td>86</td>
<td>98</td>
<td>107</td>
<td>1.4625</td>
<td>5.9228</td>
<td>1.6145</td>
</tr>
</tbody>
</table>

22
Figure 2: Calibration of gamma and IG models for Siemens CDS term structure.
Figure 3: Artificial decreasing term structures for gamma\((a = 0.0122, b = 0.3969)\) and IG\((a = 0.0165, b = 0.3085)\) models.
Figure 4: Artificial humped term structures for gamma($a = 1.7908, b = 3.4437$) and IG($a = 1.6758, b = 2.2185$) models.
Table 3: Limiting credit spread when maturity goes to zero: gamma model

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th>Numerical</th>
<th>Theoretical</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5</td>
<td>1%</td>
<td>5.645955</td>
<td>5.645982</td>
</tr>
<tr>
<td>0.5</td>
<td>5</td>
<td>5%</td>
<td>5.645975</td>
<td>5.645982</td>
</tr>
<tr>
<td>1.0</td>
<td>5</td>
<td>1%</td>
<td>11.29197</td>
<td>11.29196</td>
</tr>
<tr>
<td>1.0</td>
<td>5</td>
<td>5%</td>
<td>11.29199</td>
<td>11.29196</td>
</tr>
<tr>
<td>1.5</td>
<td>5</td>
<td>1%</td>
<td>16.93792</td>
<td>16.93795</td>
</tr>
<tr>
<td>1.5</td>
<td>5</td>
<td>5%</td>
<td>16.93794</td>
<td>16.93795</td>
</tr>
<tr>
<td>2.0</td>
<td>5</td>
<td>1%</td>
<td>22.58394</td>
<td>22.58393</td>
</tr>
<tr>
<td>2.0</td>
<td>5</td>
<td>5%</td>
<td>22.58396</td>
<td>22.58393</td>
</tr>
</tbody>
</table>

Table 4: Limiting credit spread when maturity goes to zero: IG model

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th>Numerical</th>
<th>Theoretical</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>2.5</td>
<td>1%</td>
<td>8.705575</td>
<td>8.705537</td>
</tr>
<tr>
<td>0.25</td>
<td>2.5</td>
<td>5%</td>
<td>8.705595</td>
<td>8.705537</td>
</tr>
<tr>
<td>0.50</td>
<td>2.5</td>
<td>1%</td>
<td>17.41108</td>
<td>17.41107</td>
</tr>
<tr>
<td>0.50</td>
<td>2.5</td>
<td>5%</td>
<td>17.41110</td>
<td>17.41107</td>
</tr>
<tr>
<td>0.75</td>
<td>2.5</td>
<td>1%</td>
<td>26.11665</td>
<td>26.11661</td>
</tr>
<tr>
<td>0.75</td>
<td>2.5</td>
<td>5%</td>
<td>26.11667</td>
<td>26.11661</td>
</tr>
<tr>
<td>1.00</td>
<td>2.5</td>
<td>1%</td>
<td>34.82215</td>
<td>34.82215</td>
</tr>
<tr>
<td>1.00</td>
<td>2.5</td>
<td>5%</td>
<td>34.82224</td>
<td>34.82215</td>
</tr>
</tbody>
</table>