ASYMPTOTIC TAIL PROBABILITIES OF RISK PROCESSES IN INSURANCE AND FINANCE

by

Xuemiao Hao

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Statistics
in the Graduate College of
The University of Iowa

July 2009

Thesis Supervisor: Associate Professor Qihe Tang

ABSTRACT

In this thesis we are interested in the impact of economic and financial factors, such as interest rate, tax payment, reinsurance, and investment return, on insurance business. The underlying risk models of insurance business that we consider range from the classical compound Poisson risk model to the newly emerging and more general Lévy risk model. In these risk models, we assume that the claim-size distribution belongs to some distribution classes according to its asymptotic tail behavior. We consider both light-tailed and heavy-tailed cases.

Our study is through asymptotic tail probabilities. Firstly, we study the asymptotic tail probability of discounted aggregate claims in the renewal risk model by introducing a constant force of interest. In this situation we focus on claims with subexponential tails. We derive for the tail probability of discounted aggregate claims an asymptotic formula, which holds uniformly for finite time intervals. For various special cases, we extend this uniformity to be valid for all time horizons.

Then, we investigate the asymptotic tail probability of the maximum exceedance of a sequence of random variables over a renewal threshold. We derive a unified asymptotic formula for this tail probability for both light-tailed and heavy-tailed cases.

By using the previous result, we study how to capture the impact of tax payments on the ruin probability in the Lévy risk model. We introduce periodic taxation under which the company pays tax at a fixed rate on its net income during each period. Assuming the Lévy measure, representing the claim-size distribution in the Lévy risk model, has a subexponential tail, a convolution-equivalent tail, or an exponential-like tail, we derive for the ruin probability several explicit asymptotic relations, in which the prefactor varies with the tax rate, reflecting the impact of tax payments.

Finally, we consider the renewal risk model in which the surplus is invested into a portfolio consisting of both a riskless bond and a risky stock. The price process of the stock is modeled by an exponential Lévy process. We derive an asymptotic formula for the tail probability of the stochastically discounted net loss process.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Statistics at the July 2009 graduation.

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In this thesis we are interested in the impact of economic and financial factors, such as interest rate, tax payment, reinsurance, and investment return, on insurance business. The underlying risk models of insurance business that we consider range from the classical compound Poisson risk model to the newly emerging and more general Lévy risk model. In these risk models, we assume that the claim-size distribution belongs to some distribution classes according to its asymptotic tail behavior. We consider both light-tailed and heavy-tailed cases.

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each period. Assuming the Lévy measure, representing the claim-size distribution in the Lévy risk model, has a subexponential tail, a convolution-equivalent tail, or an exponential-like tail, we derive for the ruin probability several explicit asymptotic relations, in which the prefactor varies with the tax rate, reflecting the impact of tax payments.

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CHAPTER 1
INTRODUCTION

1.1 Notation and Conventions

Throughout this thesis we use the following conventions:

- Without otherwise stated, the limit procedure is according to $x \to \infty$.
- The summation over an empty set of indices produces a value 0.
- The multiplication over an empty set of indices produces a value 1.

We also use these mathematical signs:

a^+	$a \vee 0 = \max\{a, 0\}$
a^-	$-(a \wedge 0) = -\min\{a, 0\}$
$a(x) \lesssim b(x)$	$\limsup_{x\to\infty} a(x)/b(x) \leq 1$ for positive functions $a(\cdot)$ and $b(\cdot)$
$a(x) \gtrsim b(x)$	$\liminf_{x\to\infty} a(x)/b(x) \geq 1$ for positive functions $a(\cdot)$ and $b(\cdot)$
$a(x) \sim b(x)$	both relations $a(x) \lesssim b(x)$ and $a(x) \gtrsim b(x)$ hold
$a(x) \asymp b(x)$	$\limsup_{x\to\infty} a(x)/b(x) < \infty$ and $\limsup_{x\to\infty} b(x)/a(x) < \infty$
$o(1)$, $O(1)$	$\lim_{x\to\infty} o(1) = 0$ and $\limsup_{x\to\infty} \lvert O(1)\rvert < \infty$

Probability notation used is summarized below:

1_E	the indicator function of an event E
a.s.	almost surely
$\overset{d}{=}$	$X \overset{d}{=} Y \iff \Pr(X > x) = \Pr(Y > x)$ for every x
$\overset{d}{\leq}$	$X \overset{d}{\leq} Y \iff \Pr(X > x) \leq \Pr(Y > x)$ for every x

$\overset{d}{\geq}$	$X \overset{d}{\geq} Y \iff \Pr(X > x) \geq \Pr(Y > x)$ for every x
$\mathcal{D}$	the class of distributions with dominatedly-varying tails
$\mathbb{E}$	expectation
ERV	the class of distributions with extended-regularly-varying tails
$\overline{F}$	$1 - F$ for a distribution F
F_e	the equilibrium distribution of a distribution F on $[0, \infty)$ with finite expectation
$f \star g$	the convolution of measurable functions f and g
$f^{n\star}$	the n-fold convolution of a measurable function f
$F * G$	the convolution of distributions F and G
F^{n*}	the n-fold convolution of a distribution F
$F_+(x)$	$F(x)1_{(x\geq 0)}$ for a distribution F on $(-\infty, \infty)$
$\mathbb{J}_F^{\pm}$	Matuszewska indices of a distribution F
$\mathcal{K}$	the class of heavy-tailed distributions
$\mathcal{L}$	the class of long-tailed distributions
$\mathbb{P}$	probability measure
$\Phi(\cdot)$	the standard normal distribution
$\mathcal{R}$	the class of distributions with regularly-varying tails
$\mathcal{S}$	the class of subexponential distributions

Notation for surplus process is summarized below:

B_t	a Brownian motion at time t
càdlàg	right continuous with left limits
$D_r(t)$	the discounted aggregate claims by time t in the presence of the force of interest r
L_t	a Lévy process at time t
λ_t	the renewal function $\mathbb{E}N_t$
Λ	$\{t : \lambda_t > 0\} \cup \{\infty\}$
N_t	a renewal counting process at time t
φ	the Laplace exponent of a Lévy process
ψ	ruin probability of an insurance risk process
Ψ	the characteristic exponent of a Lévy process
τ_k	the kth claim arrival time
θ_k	the inter-arrival time between $(k-1)$th and kth claims
U_t	a general surplus process at time t

1.2 Objectives and Outline of the Thesis

In this thesis, we study the asymptotic tail probabilities of quantities of interest in various risk models, from the compound Poisson risk model to the newly emerging and more general Lévy risk model, to investigate the impact of economic and financial factors, such as interest rate, tax payment, reinsurance, and investment return, on insurance business. We consider both light-tailed and heavy-tailed claim sizes in our models, seeing that the tail behavior of the claim-size distribution may vary in different types of insurance business.

In Chapter 2 we prepare some probability tools that are needed for the following chapters. In Section 2.1 we give the definitions and properties for some stochastic processes that are widely used in insurance mathematics. In Section 2.2 we present some basic theory for stochastic integral with respect to a semimartingale. This part is crucial for us to investigate the impact of risky investment in Chapter 6. Section 2.3 reviews some popular classes of distributions, including both light-tailed and heavy-tailed classes. They are going to be used as assumptions on the claim-size distribution in our risk models and play a very important role in deriving our main results.

The main part of this thesis consists of Chapters 3–6. In Chapter 3 we introduce a constant force of interest in the renewal risk model and study the tail probability of discounted aggregate claims. Since it is usually not possible to get closed-form expressions except for few ideal cases, we instead aim at asymptotic formulas. The question is of much practical interest in insurance risk management. The study can provide an easy and precise approximation when measuring the risk of large losses via Value-at-Risk or Conditional Tail Expectation. Also, such an approximation usually plays a crucial role in pricing some insurance products. We derive for the tail probability of discounted aggregate claims an asymptotic formula, which holds uniformly for all time horizons. A key assumption in our model is that the claim-size distribution is subexponential.

In Chapter 4 we study an interesting problem in the field of probability that will be used as an important tool to give the proof for some main results in Chapter

5. Motivated by the observations that many problems in applied fields, including corporate finance, insurance risk, and production systems, can be reduced to the study of the maximum exceedance of a sequence of random variables over a renewal threshold, we derive a unified asymptotic formula for the tail probability of such a maximum exceedance for both light-tailed and heavy-tailed cases.

In Chapter 5, we use a general Lévy process to model the underlying surplus process of an insurance company in a world without economic factors. This so-called Lévy risk model has recently attracted a lot of attention in the insurance literature. We are particularly interested in how to capture the impact of tax payments on the ruin probability. In a series of papers recently by Albrecher and his coauthors, it is assumed that taxes are paid at a certain fixed rate immediately when the surplus of the company is at a running maximum. In reality, however, taxes are usually paid periodically (e.g. monthly, semi-annually, or annually). Therefore, we introduce periodic taxation under which the company pays tax at a fixed rate on its net income during each period. As main results, we derive for the ruin probability several explicit asymptotic relations, in which the prefactor varies with the tax rate, reflecting the impact of tax payments.

In Chapter 6, we study the tail behavior of the stochastically discounted net loss process in the renewal risk model with risky investment. Consider an insurance company who invests its surplus into a portfolio consisting of both a riskless bond and a risky stock. Suppose the price process of the bond grows with a constant force of interest, while the price process of the stock is modeled by an exponential Lévy process. The study of such a risk model has become a hot topic in the past decade. Assuming a constant mix investment strategy, i.e., the proportions of surplus invested into the riskless and risky assets remain constant, we derive an asymptotic formula for the tail probability of the stochastically discounted net loss process.

CHAPTER 2
PRELIMINARIES

2.1 Brief Review on Stochastic Processes

Stochastic processes as a probabilistic tool have been extensively used for modeling insurance risk processes for a long time. For instance, Lundberg (1903) pointed out that Poisson processes lie at the heart of non-life insurance models.

Definition 2.1. *A counting process* $N = (N_t)_{t\geq 0}$*, defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, is said to be a* ***Poisson process*** *with intensity* $\lambda > 0$ *if*

(i) the paths of N *are* $\mathbb{P}$*-almost surely right continuous with left limits (càdlàg);*

(ii) $\mathbb{P}(N_0 = 0) = 1$*;*

(iii) the process has independent increments;

(iv) the number of events in any interval of length t *is Poisson distributed with mean* λt*, i.e., for every* $s \geq 0$ *and* $t > 0$

$$\mathbb{P}(N_{t+s} - N_s = n) = \mathbb{P}(N_t = n) = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}, \qquad n = 0, 1, \ldots.$$

A useful property of the Poisson process with intensity λ is that the inter-arrival times are independent and identically distributed (i.i.d.) exponential random variables with mean $1/\lambda$. If we allow the inter-arrival times to be i.i.d. copies of an arbitrary nonnegative and not-degenerate-at-zero random variable (a random variable θ is said to be not degenerate at zero if $\mathbb{P}(\theta = 0) < 1$), then Poisson processes are generalized to renewal counting processes.

Definition 2.2. *A counting process* $N = (N_t)_{t \geq 0}$*, defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, is said to be a* ***renewal counting process*** *with parameter* $\lambda > 0$ *if*

(i) the paths of N *are* $\mathbb{P}$*-almost surely càdlàg;*

(ii) $\mathbb{P}(N_0 = 0) = 1$*;*

(iii) the inter-arrival times are i.i.d., nonnegative, and not-degenerate-at-zero random variables with mean $1/\lambda$*.*

Like Poisson processes in actuarial risk theory, Bachelier (1900) recognized that Brownian motions are a key building block for financial models.

Definition 2.3. *A stochastic process* $B = (B_t)_{t \geq 0}$*, defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, is said to be a* ***Brownian motion*** *if*

(i) the paths of B *are* $\mathbb{P}$*-almost surely continuous;*

(ii) $\mathbb{P}(B_0 = 0) = 1$*;*

(iii) B *has stationary and independent increments;*

(iv) for every $t > 0$*,* B_t *is normally distributed with mean* 0 *and variance* $\sigma^2 t$*.*

When $\sigma = 1$ in the above definition, the process B is called a *standard Brownian motion*. Both Poisson processes and Brownian motions are initiated from the origin and have stationary and independent increments. Actually, these properties define a more general class of stochastic processes, which are called Lévy processes.

Definition 2.4. *A stochastic process* $L = (L_t)_{t\geq 0}$*, defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, is said to be a* ***Lévy process*** *if*

(i) the paths of L *are* $\mathbb{P}$*-almost surely càdlàg;*

(ii) $\mathbb{P}(L_0 = 0) = 1$*;*

(iii) L *has stationary and independent increments.*

Lévy processes have an intimate relationship with infinitely divisible distributions, as described below:

Definition 2.5. *A random variable* X *is said to have an* ***infinitely divisible distribution*** *if for each* $n = 1, 2, \ldots,$ *there exists a sequence of i.i.d. random variables* $X_{1,1}, X_{1,2}, \ldots, X_{1,n}$ *such that*

$$X \stackrel{d}{=} X_{1,1} + X_{1,2} + \cdots + X_{1,n}.$$

For a Lévy process $L = (L_t)_{t\geq 0}$, L_t is a random variable with infinitely divisible distribution. This follows from the fact that for every $n = 1, 2, \ldots,$

$$L_t = L_{t/n} + \left(L_{2t/n} - L_{t/n}\right) + \cdots + \left(L_t - L_{(n-1)t/n}\right) \tag{2.1}$$

and the fact that L has stationary and independent increments. For every $t \geq 0$, define

$$\Psi_t(s) = -\log \mathbb{E}e^{isL_t}.$$

From (2.1) it is easy to obtain that for every rational $t > 0$,

$$\Psi_t(s) = t\Psi_1(s). \tag{2.2}$$

If t is an irrational number, then we can choose a decreasing sequence of rationals $\{t_n, n = 1, 2, \ldots\}$ such that $t_n \to t$ as $n \to \infty$. Since L is almost surely right continuous, $\mathbb{E}e^{isL_t} = \exp\{-\Psi_t(s)\}$ is also right continuous in t. Hence, (2.2) still holds.

In conclusion, for every Lévy process L its characteristic function can be written in the form

$$\mathbb{E}e^{isL_t} = e^{-t\Psi(s)},$$

where $\Psi(s) := \Psi_1(s)$ is the *characteristic exponent* of L_1. The famous *Lévy-Khintchine formula* gives the following representation for $\Psi(s)$:

$$\Psi(s) = ias + \frac{1}{2}\sigma^2 s^2 + \int_{-\infty}^{\infty} \left(1 - e^{isx} + isx 1_{\{|x| \leq 1\}}\right) \rho(dx)$$

with $a \in (-\infty, \infty)$, $\sigma \geq 0$, and Lévy measure ρ on $(-\infty, \infty)$ satisfying $\rho(\{0\}) = 0$ and $\int_{-\infty}^{\infty} (x^2 \wedge 1)\, \rho(dx) < \infty$. The triplet (a, σ^2, ρ) (called *Lévy triplet*) uniquely determines the distribution of the Lévy process L. In the following chapters, we also need the *Laplace exponent* of a Lévy process L given by

$$\varphi(s) = -\Psi(is) = \log \mathbb{E}e^{-sL_1}.$$

For more details of Lévy processes, refer to Kyprianou (2006).

Lévy processes form a very rich class of stochastic processes. Besides Poisson processes and Brownian motions, we further give the following typical examples:

(i) Compound Poisson Processes

Consider a *compound Poisson process* $V = (V_t)_{t \geq 0}$ with

$$V_t = \sum_{k=1}^{N_t} \xi_k, \qquad t \geq 0, \tag{2.3}$$

where $N = (N_t)_{t \geq 0}$ is a Poisson process with intensity $\lambda > 0$ and $\xi_1, \xi_2, \ldots$ are i.i.d. random variables independent of N and with common distribution F on $(-\infty, \infty)$. Here, in (2.3) we use the convention that the summation over an empty set of indices produces a value 0. For $s \in (-\infty, \infty)$,

$$\begin{aligned} \mathbb{E}\mathrm{e}^{\mathrm{i}sV_t} &= \sum_{n=0}^{\infty} \mathbb{E}\left(\mathrm{e}^{\mathrm{i}s\sum_{k=1}^{n}\xi_k}\right)\mathrm{e}^{-\lambda}\frac{\lambda^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}sx}F(\mathrm{d}x)\right)^n \mathrm{e}^{-\lambda}\frac{\lambda^n}{n!} \\ &= \mathrm{e}^{-\lambda\int_{-\infty}^{\infty}\left(1-\mathrm{e}^{\mathrm{i}sx}\right)F(\mathrm{d}x)}. \end{aligned} \tag{2.4}$$

From (2.4) we see that the Lévy triplet of the compound Poisson process V is given by $a = -\lambda\int_{0<|x|<1} xF(\mathrm{d}x)$, $\sigma = 0$, and $\rho(\mathrm{d}x) = \lambda F(\mathrm{d}x)$.

(ii) Gamma Processes

A *gamma process* $\Gamma = (\Gamma_t)_{t \geq 0}$ is a stochastic process starting from 0, having stationary and independent increments, and with Γ_1 distributed by the gamma(α,β) distribution with density

$$f(x) = \frac{\alpha^\beta}{\Gamma(\beta)}x^{\beta-1}\mathrm{e}^{-\alpha x}, \qquad \alpha, \beta, x > 0.$$

Its Lévy triplet is given by $a = \beta\left(\mathrm{e}^{-\alpha} - 1\right)/\alpha$, $\sigma = 0$, and $\rho(\mathrm{d}x) = \beta x^{-1}\mathrm{e}^{-\alpha x}\mathrm{d}x$; see Subsection 1.2.4 of Kyprianou (2006) for details.

(iii) Subordinators

A *subordinator* is a Lévy process whose paths are almost surely nondecreasing. The following lemma characterizes subordinators:

Lemma 2.6 (Lemma 2.14 of Kyprianou (2006)). *A Lévy process is a subordinator if and only if* $\rho(-\infty, 0) = 0$, $\int_0^\infty (1 \wedge x)\rho(\mathrm{d}x) < \infty$, $\sigma = 0$, *and* $a + \int_0^{1-} x\rho(\mathrm{d}x) \leq 0$.

(iv) Spectrally One-sided Processes

For a Lévy process L, if $\rho(-\infty, 0) = 0$ and L is not a subordinator, then it is called a *spectrally positive* Lévy process. A spectrally positive Lévy process has no downward jumps. A Lévy process L is called *spectrally negative* if $-L$ is spectrally positive. These two classes of processes are together called *spectrally one-sided.*

2.2 Semimartingales and Stochastic Integrals

In this section, we introduce stochastic integrals and semimartingales, the most general processes of which stochastic integration gives a reasonable meaning. This section is based on Chapter 8 of Klebaner (2005) and Chapter II of Protter (2005).

2.2.1 Semimartingales

We assume as given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In addition, we are given a *filtration* $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq \infty}$. By a filtration we mean a family of σ-fields $(\mathcal{F}_t)_{0 \leq t \leq \infty}$ that is nondecreasing, i.e., $\mathcal{F}_s \subset \mathcal{F}_t$ for all $0 \leq s \leq t \leq \infty$. A nonnegative random variable τ, which is allowed to take the value ∞, is called a *stopping time* (with respect to filtration $\mathbb{F}$) if for each t,

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

In other words, by observing the information contained in $\mathcal{F}_t$ we can decide whether the event $\{\tau \leq t\}$ has or has not occurred.

We then introduce finite variation. If g is a function of real variable, its variation over the interval $[a,b]$ is defined as

$$V_g([a,b]) = \sup \sum_{i=1}^{n} \left| g\left(t_i^n\right) - g\left(t_{i-1}^n\right) \right|, \tag{2.5}$$

where the supremum is taken over the partitions of $[a,b]$:

$$a = t_0^n < t_1^n < \cdots < t_n^n = b.$$

Clearly, (by the triangle inequality) the sum in (2.5) increases as new points are added to the partitions. Therefore, the variation of g is

$$V_g([a,b]) = \lim_{\delta_n \to 0} \sum_{i=1}^{n} \left| g\left(t_i^n\right) - g\left(t_{i-1}^n\right) \right|,$$

where $\delta_n = \max_{1\leq i\leq n}(t_i - t_{i-1})$. If $V_g([a,b])$ is finite then g is said to be a function of finite variation on $[a,b]$. If g is a function of $t \geq 0$, then the variation function of g as a function of t is defined by

$$V_g(t) = V_g([0,t]).$$

Clearly, $V_g(t)$ is a nondecreasing function of t. The function g is of *finite variation* if $V_g(t) < \infty$ for every t. A process $X = (X_t)_{t\geq 0}$ is called a *finite variation process* if the paths of X are almost surely of finite variation.

In the following we define predictable processes and martingales:

Definition 2.7. *A process $X = (X_t)_{t\geq 0}$ is called **adapted** to the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq\infty}$ if for every t, X_t is $\mathcal{F}_t$-measurable.*

Definition 2.8. *A process $H = (H_t)_{t\geq 0}$ is predictable if it is one of the following:*

(i) a left-continuous adapted process, in particular, a continuous adapted process;

(ii) a limit (almost sure, in probability) of left-continuous adapted processes;

(iii) a regular right-continuous process such that, for any stopping time τ, $H\tau$ is $\mathcal{F}_{\tau-}$-measurable, the σ-field generated by the sets $A \cap \{T < t\}$, where $A \in \mathcal{F}_t$;

(iv) a Borel-measurable function of a predictable process.

Definition 2.9. *A stochastic process $M = (M_t)$, where the time t is continuous $0 \leq t \leq T$, or discrete $t = 0, 1, \ldots, T$, adapted to a filtration $\mathbb{F}$ is a* ***martingale*** *if for every t, M_t is integrable (that is, $\mathbb{E}\,|M_t| < \infty$), and for all $0 \leq s \leq t \leq T$,*

$$\mathbb{E}\,(M_t\,|\mathcal{F}_s) = M_s \qquad a.s.$$

To define semimartingales, we still need to introduce local martingales.

Definition 2.10. *A process $(X_t)_{0 \leq t \leq T}$ with $T \in [0, \infty]$ is called* ***uniformly integrable*** *if $\mathbb{E}\left(|X_t|\, 1_{\{|X_t|>n\}}\right)$ converges to 0 as $n \to \infty$ uniformly in t, that is,*

$$\lim_{n\to\infty} \sup_{0 \leq t \leq T} \mathbb{E}\left(|X_t|\, 1_{\{|X_t|>n\}}\right) = 0.$$

where the supremum is over $[0, T]$ in the case of a finite time interval and $[0, \infty)$ if the process is considered on $0 \leq t < \infty$.

Definition 2.11. *An adapted process $M = (M_t)_{0 \leq t \leq T}$ with $T \in [0, \infty]$ is called a* ***local martingale*** *if there exists a sequence of stopping times $\{\tau_n, n = 1, 2, \ldots\}$, such that $\tau_n \to \infty$ as $n \to \infty$ and for each n the stopped process $M_{t \wedge \tau_n}$ is a uniformly integrable martingale in t.*

Now we are ready to give the definition of the core process in this section.

Definition 2.12. *A càdlàg adapted process $S = (S_t)_{t\geq 0}$ is a **semimartingale** if it can be represented as a sum of two processes: a local martingale $M = (M_t)_{t\geq 0}$ and a finite variation process $A = (A_t)_{t\geq 0}$,*

$$S_t = S_0 + M_t + A_t, \qquad \text{with } M_0 = A_0 = 0. \tag{2.6}$$

Here are some examples of semimartingales:

1. A Lévy process is a semimartingale.

2. One way to obtain semimartingale from known semimartingales is by applying a twice continuously differentiable (C^2) transformation. If S is a semimartingale and f is a C^2 function, then $f(S)$ is also a semimartingale.

3. A diffusion, that is, the solution to a stochastic differential equation with respect to a Brownian motion, is a semimartingale.

2.2.2 Quadratic Variation and Covariation

If X, Y are semimartingales on the common space, then their *quadratic covariation process*, also known as the *square bracket process* and denoted by $[X,Y](t)$, is defined, as usual, by

$$[X,Y](t) = \lim \sum_{i=0}^{n-1} \left(X_{t_{i+1}^n} - X_{t_i^n} \right) \left(Y_{t_{i+1}^n} - Y_{t_i^n} \right),$$

where the limit is taken over shrinking partitions $0 = t_0^n < t_1^n < \ldots < t_n^n = t$ when $\delta_n = \max_{1\leq i\leq n} \left(t_i^n - t_{i-1}^n \right) \to 0$ and is in probability. Taking $Y = X$ we obtain the *quadratic variation process* of X.

Here are some properties of quadratic variation and covariation:

1. $[X,Y](t)$ is a regular càdlàg function of finite variation.

2. $\Delta[X,Y] = \Delta X \Delta Y$, that is, the jumps of the quadratic covariation process occur only at points where both processes have jumps.

3. If X or Y is of finite variation, then $[X,Y](t) = \sum_{s<t} \Delta X_s \Delta Y_s$. Notice that although the summation is taken over all s not exceeding t, there are at most countably many terms different from zero.

4. $[X,Y] = 0$ if X or Y is continuous and is of finite variation.

In terms of stochastic integral that to be introduced in the next subsection, we have the following lemma known as integration by parts:

Lemma 2.13. *For semimartingales X and Y, their quadratic covariation process is given by*

$$[X,Y](t) = X_t Y_t - X_0 Y_0 - \int_0^t X_{s-} \mathrm{d}Y_s - \int_0^t Y_{s-} \mathrm{d}X_s.$$

2.2.3 Stochastic Integrals

We are going to define the stochastic integral $\int_0^T H_t \mathrm{d}S_t$ for a semimartingale $S = (S_t)_{t\geq 0}$. Due to (2.6) the integral with respect to S is the sum of two integrals one with respect to a local martingale M and the other with respect to a finite variation process A. The integral with respect to A can be done path by path as the Stieltjes integrals, since A, although random, is of finite variation. So $H = (H_t)_{t\geq 0}$ should be integrable with respect to A. A sufficient condition for that is

$$\int_0^T |H_t| \, V_A(\mathrm{d}t) < \infty, \tag{2.7}$$

where $V_A(t)$ is the variation process of A. Then we need to define the stochastic integral of H with respect to the local martingale M, $\int_0^T H_t \mathrm{d}M_t$.

Stochastic integral with respect to a martingale:

For a simple predictable process H, given by

$$H_t = H_0 1_{\{0\}}(t) + \sum_{i=0}^{n-1} h_i 1_{(T_i, T_{i+1}]}(t),$$

where $0 = T_0 \leq T_1 \leq \cdots \leq T_n \leq T$ are stopping times and h_i is $\mathcal{F}_{T_i}$-measurable, $i = 0, 1, \ldots n-1$, the stochastic integral is defined as the sum

$$\int_0^T H_t \mathrm{d}M_t = \sum_{i=0}^{n-1} h_i \left(M_{T_{i+1}} - M_{T_i} \right).$$

If M is a locally square integrable martingale, then one can extend the stochastic integral from simple predictable processes to the class of predictable processes H such that

$$\sqrt{\int_0^T H_t^2 [M, M](\mathrm{d}t)} \qquad \text{is locally integrable.} \tag{2.8}$$

If M is a continuous local martingale, then the stochastic integral is defined for a wider class of predictable processes H satisfying

$$\int_0^T H_t^2 [M, M](\mathrm{d}t) \overset{a.s.}{<} \infty.$$

Stochastic Integrals with respect to a semimartingale:

Let $S = (S_t)_{t\geq 0}$ be a semimartingale with representation given by (2.6). Let H be a predictable process such that conditions (2.7) and (2.8) hold. Then the stochastic integral is defined as the sum of integrals,

$$\int_0^T H_t \mathrm{d}S_t = \int_0^T H_t \mathrm{d}M_t + \int_0^T H_t \mathrm{d}A_t.$$

Although the decomposition of a semimartingale (2.6) is not unique, the stochastic integral defined above does not depend on the decomposition used. For details, see pages 216–217 of Klebaner (2005).

Since the integral with respect to a local martingale is a local martingale, and the integral with respect to a finite variation process is a finite variation process, it follows that a stochastic integral with respect to a semimartingale is still a semimartingale.

Stochastic exponential:

Definition 2.14. *Let $X = (X_t)_{t\geq 0}$ be a semimartingale. Then the stochastic equation*

$$U_t = 1 + \int_0^t U_{s-} \mathrm{d}X_s \tag{2.9}$$

has a unique solution, denoted by $\mathcal{E}(X)$, called the ***stochastic exponential*** *of X.*

For a Lévy process L, its ordinary exponential and stochastic exponential correspond to different stochastic processes. One may ask, which of the two processes is more suitable for building models for price dynamics. Actually, as pointed out in Subsection 8.4.3 of Cont and Tankov (2004), the two approaches are equivalent: if $Z > 0$ is the stochastic exponential of a Lévy process then it is also the ordinary exponential of another Lévy process, and vice versa. Therefore, the two operations, although they produce different objects when applied to the same Lévy process, end up by giving us the same class of positive processes. The following lemma, due to Goll and Kallsen (2000), gives the relation between ordinary and stochastic exponentials of a Lévy process:

Lemma 2.15 (Lemma A.8 of Goll and Kallsen (2000); Proposition 8.22 of Cont and Tankov (2004)). *(i) Let* $X = (X_t)_{t\geq 0}$ *be a Lévy process with Lévy triplet* (a, σ^2, ρ) *and* $Z = \mathcal{E}(X)$ *its stochastic exponential. If* $Z \overset{a.s.}{>} 0$*, then there exists another Lévy process* $L = (L_t)_{t\geq 0}$ *such that* $Z_t = \mathrm{e}^{L_t}$*, where*

$$L_t = \log Z_t = X_t - \frac{\sigma^2 t}{2} + \sum_{0\leq s\leq t} \left(\log\left(1+\Delta X_s\right) - \Delta X_s\right).$$

Its Lévy triplet $(a_L, \sigma_L^2, \rho_L)$ *is given by*

$$\begin{aligned} a_L &= a + \frac{\sigma^2}{2} - \int_{-\infty}^{\infty} \left(\log\left(1+x\right) 1_{\{-1\leq \log(1+x)\leq 1\}} - x 1_{\{-1\leq x\leq 1\}}\right) \rho\left(\mathrm{d}x\right), \\ \sigma_L &= \sigma, \\ \rho_L(A) &= \rho\left(\{x : \log(1+x) \in A\}\right) = \int_{-\infty}^{\infty} 1_A \log\left(1+x\right) \rho\left(\mathrm{d}x\right). \end{aligned}$$

(ii) Let $L = (L_t)_{t\geq 0}$ *be a Lévy process with Lévy triplet* $(a_L, \sigma_L^2, \rho_L)$ *and* $S_t = \mathrm{e}^{L_t}$ *its ordinary exponential. Then there exists a Lévy process* $X = (X_t)_{t\geq 0}$ *such that* S_t *is the stochastic exponential of* X*, i.e.,* $S = \mathcal{E}(X)$*, where*

$$X_t = L_t + \frac{\sigma^2 t}{2} - \sum_{0\leq s\leq t} \left(1 + \Delta L_s - \mathrm{e}^{\Delta L_s}\right).$$

The Lévy triplet (a, σ^2, ρ) *of* X *is given by*

$$a = a_L - \frac{\sigma_L^2}{2} - \int_{-\infty}^{\infty} \left((\mathrm{e}^x - 1) 1_{\{-1 \le \mathrm{e}^x - 1 \le 1\}} - x 1_{\{-1 \le x \le 1\}} \right) \rho_L(\mathrm{d}x),$$

$$\sigma = \sigma_L,$$

$$\rho(A) = \rho_L(\{x : \mathrm{e}^x - 1 \in A\}) = \int_{-\infty}^{\infty} 1_A(\mathrm{e}^x - 1) \rho_L(\mathrm{d}x).$$

2.3 Heavy-tailed and Light-tailed Distribution Classes

2.3.1 Subexponentiality and Rapid Variation

Let us denote by $\mathcal{K}$ the class of (right) *heavy-tailed* distributions, i.e.,

$$\mathcal{K} = \left\{ F \text{ distribution on } (-\infty, \infty) : \int_{-\infty}^{\infty} \mathrm{e}^{\varepsilon x} F(\mathrm{d}x) = \infty \text{ for all } \varepsilon > 0 \right\}.$$

In insurance mathematics, claim-size distributions are often assumed to belong to some subclass of the class $\mathcal{K}$.

Next we introduce several related subclasses of the class $\mathcal{K}$. A distribution F on $(-\infty, \infty)$ is said to have a *long tail*, denoted by $F \in \mathcal{L}$, if $\overline{F}(x) > 0$ for all x and

$$\lim_{x \to \infty} \frac{\overline{F}(x - y)}{\overline{F}(x)} = 1 \tag{2.10}$$

holds for all (or, equivalently, for some) $y \neq 0$.

Throughout this thesis, for two distributions F and G on $(-\infty, \infty)$ denote by

$$F * G(x) = \int_{-\infty}^{\infty} F(x - y) G(\mathrm{d}y)$$

the convolution of F and G. Write $F^{1*} = F$ and $F^{n*} = F * F^{(n-1)*}$ for every $n = 2, 3, \ldots$. For notational convenience, write F^{0*} as a distribution degenerate at 0. A very important subclass of $\mathcal{L}$ is the subexponential class $\mathcal{S}$. By definition, a distribution F on $[0,\infty)$ is *subexponential*, denoted by $F \in \mathcal{S}$, if $\overline{F}(x) > 0$ for all $x \geq 0$ and the relation

$$\lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n \tag{2.11}$$

holds for all (or, equivalently, for some) $n = 2, 3, \ldots$. More generally, a distribution F on $(-\infty,\infty)$ is also subexponential if $F_+(x) = F(x)1_{(x\geq 0)}$ is. In this case, relation (2.11) still holds. However, for F on $(-\infty,\infty)$, relation (2.11) is not sufficient for $F \in \mathcal{S}$; see the example in Subsection 2.3.2.

Table 2.1, partially copied from Table 1.2.6 of Embrechts *et al.* (1997), gives some examples in the class $\mathcal{S}$.

Name	Tail $\overline{F}$ or density f	Parameters
Lognormal	$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \mathrm{e}^{-(\log x-\mu)^2/\left(2\sigma^2\right)}$	$-\infty < \mu < \infty, \sigma > 0$
Pareto	$\overline{F}(x) = \left(\frac{\kappa}{\kappa+x}\right)^{\alpha}$	$\alpha, \kappa > 0$
Burr	$\overline{F}(x) = \left(\frac{\kappa}{\kappa+x^{\tau}}\right)^{\alpha}$	$\alpha, \kappa, \tau > 0$
Benktander-type-I	$\overline{F}(x)=(1+2(\beta/\alpha)\log x)\, \mathrm{e}^{-\beta(\log x)^2-(\alpha+1)\log x}$	$\alpha, \beta > 0$
Benktander-type-II	$\overline{F}(x) = \mathrm{e}^{\alpha/\beta} x^{-(1-\beta)} \mathrm{e}^{-\alpha x^{\beta}/\beta}$	$\alpha > 0, 0 < \beta < 1$
Weibull	$\overline{F}(x) = \mathrm{e}^{-cx^{\tau}}$	$c > 0, 0 < \tau < 1$
Loggamma	$f(x) = \frac{\alpha^{\beta}}{\Gamma(\beta)} (\log x)^{\beta-1} x^{-\alpha-1}$	$\alpha, \beta > 0$

Table 2.1: Some examples of subexponential distributions

We also want to introduce some subclasses of $\mathcal{S}$ that will play important roles in the following chapters. One useful subclass is $\mathcal{A}$, which was introduced by Konstantinides *et al.* (2002). By definition, a distribution F on $[0,\infty)$ is said to belong to the class $\mathcal{A}$ if $F \in \mathcal{S}$ and, for some $v > 1$,

$$\limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} < 1. \tag{2.12}$$

We remark that the class $\mathcal{A}$ almost coincides with the class $\mathcal{S}$. Indeed, relation (2.12) is satisfied by almost all useful distributions with unbounded supports on the right, including all distributions in Table 2.1.

Another subclass of $\mathcal{S}$ is $\mathcal{S}^*$, which was introduced by Klüppelberg (1988). By definition, a distribution F on $[0,\infty)$ is said to belong to the class $\mathcal{S}^*$ if $\overline{F}(x) > 0$ for all $x \geq 0$, $\nu_F = \int_0^\infty \overline{F}(x)\mathrm{d}x < \infty$, and

$$\lim_{x\to\infty} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)} \overline{F}(y)\mathrm{d}y = 2\nu_F.$$

It is well known that if $F \in \mathcal{S}^*$, then both $F \in \mathcal{S}$ and $F_e \in \mathcal{S}$, where F_e is the equilibrium distribution of F, i.e.,

$$F_e(x) = \frac{1}{\nu_F} \int_0^x \overline{F}(y)\mathrm{d}y, \qquad x \geq 0.$$

For a distribution F on $(-\infty,\infty)$, its equilibrium distribution F_e is defined as the equilibrium distribution of F_+. Again, $\mathcal{S}^*$ contains all distributions in Table 2.1 with finite mean.

A closely related class is the class of dominatedly varying distributions. A distribution F on $(-\infty,\infty)$ is said to have a *dominatedly-varying tail*, denoted by

$F \in \mathcal{D}$, if $\overline{F}(x) > 0$ for all x and

$$\limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} < \infty$$

holds for all (or, equivalently, for some) $0 < v < 1$.

$\mathcal{L} \cap \mathcal{D}$ forms another important subclass of $\mathcal{S}$; see Proposition 1.4.4 of Embrechts *et al.* (1997). In particular, $\mathcal{L} \cap \mathcal{D}$ covers the class ERV of distributions with *extended-regularly-varying tails*. By definition, a distribution F on $(-\infty, \infty)$ is said to belong to the class ERV$(-\alpha, -\beta)$ for some $0 \leq \alpha \leq \beta < \infty$ if $\overline{F}(x) > 0$ holds for all x and the relations

$$v^{-\beta} \leq \liminf_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \leq v^{-\alpha} \tag{2.13}$$

hold for all $v \geq 1$. The class ERV means the union of all ERV$(-\alpha, -\beta)$ over the range $0 \leq \alpha \leq \beta < \infty$. Note that relations (2.13) with $\alpha = \beta$ define the famous class $\mathcal{R}_{-\alpha}$ of *regularly-varying-tailed* distributions with regularity index $-\alpha$. Analogously, the class $\mathcal{R}$ means the union of all $\mathcal{R}_{-\alpha}$ over the range $0 \leq \alpha < \infty$.

An extension of regular variation is rapid variation. By definition, a distribution F on $(-\infty, \infty)$ is said to have a *rapidly-varying tail*, denoted by $\mathcal{R}_{-\infty}$, if $\overline{F}(x) > 0$ for all x and

$$\lim_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} = 0$$

holds for all $v > 1$. Note that lognormal, Benktander-type I & II, and Weibull distributions all belong to the class $\mathcal{R}_{-\infty}$.

2.3.2 Light-tailed Distribution Classes

When the right-hand side of (2.10) is replaced by $\mathrm{e}^{\alpha y}$ for some $\alpha \geq 0$, the class $\mathcal{L}$ is generalized to the class $\mathcal{L}(\alpha)$. By definition, a distribution F on $(-\infty, \infty)$ is said to belong to the class $\mathcal{L}(\alpha)$ for some $\alpha \geq 0$ if $\overline{F}(x) > 0$ for all x and

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = \mathrm{e}^{\alpha y} \tag{2.14}$$

for all $y \in (-\infty, \infty)$. For example, the exponential distribution with density $f(x) = \alpha \mathrm{e}^{-\alpha x}$ for $\alpha > 0$ belongs to the class $\mathcal{L}(\alpha)$. In the literature, a distribution F in $\mathcal{L}(\alpha)$ with $\alpha > 0$ is thus usually said to have an *exponential-like tail.* Furthermore, a distribution F on $[0, \infty)$ is said to belong to the class $\mathcal{S}(\alpha)$ for some $\alpha \geq 0$ if $F \in \mathcal{L}(\alpha)$ and the limit

$$\lim_{x\to\infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2c \tag{2.15}$$

exists and is finite. It is known that the constant c in (2.15) is equal to $\int_{0-}^{\infty} \mathrm{e}^{\alpha x} F(\mathrm{d}x)$; see Rogozin (2000) and references therein. More generally, a distribution F on $(-\infty, \infty)$ is also said to belong to the class $\mathcal{S}(\alpha)$ if F_+ does. In the literature, a distribution F in $\mathcal{S}(\alpha)$ with $\alpha > 0$ is said to have a *convolution-equivalent tail.*

By using a distribution in the class $\mathcal{S}(\alpha)$, we can construct an example showing that for a distribution F on $(-\infty, \infty)$ relation (2.11) with $n = 2$ is not sufficient for $F \in \mathcal{S}$.

Example. Let Y be a nonnegative random variable with a distribution $G \in \mathcal{S}(\alpha)$ for some $\alpha > 0$. Define

$$X = Y - c$$

where c is the positive constant satisfying that $\mathbb{E}e^{\alpha Y} = e^{c\alpha}$. Denote by F the distribution of X. Then relation (2.11) with $n = 2$ holds for F. Actually, as $x \to \infty$,

$$\overline{F^{2*}}(x) = \overline{G^{2*}}(x + 2c) \sim 2\mathbb{E}e^{\alpha Y}\overline{G}(x + 2c) \sim 2e^{-c\alpha}\mathbb{E}e^{\alpha Y}\overline{G}(x + c) = 2\overline{F}(x).$$

However, by Lemma 2.24(ii), $F_+ \in \mathcal{S}(\alpha)$ because, as $x \to \infty$,

$$\overline{F_+}(x) = \overline{F}(x) = \mathbb{P}(Y > x + c) \sim e^{-c\alpha}\overline{G}(x).$$

Hence, in no way can F_+ belong to $\mathcal{L}$, not to say $\mathcal{S}$.

Finally, we need density classes corresponding to the classes $\mathcal{L}(\alpha)$ and $\mathcal{S}(\alpha)$. Throughout this thesis, for two measurable functions f and $g : [0, \infty) \to [0, \infty)$, denote by

$$f \star g(x) = \int_0^x f(x - y)g(y)\mathrm{d}y$$

the convolution of f and g. Write $f^{1\star} = f$ and $f^{n\star} = f \star f^{(n-1)\star}$ for every $n = 2, 3, \ldots$. According to Chover *et al.* (1973) and Klüppelberg (1989), a measurable function $f : [0, \infty) \to [0, \infty)$ is said to belong to the class $\mathcal{S}d(\alpha)$ for some $\alpha \geq 0$ if $f(x) > 0$ for all large x,

$$\lim_{x\to\infty} \frac{f(x - y)}{f(x)} = e^{\alpha y}, \qquad y \in (-\infty, \infty), \tag{2.16}$$

and

$$\lim_{x\to\infty} \frac{f^{2\star}(x)}{f(x)} = 2c \tag{2.17}$$

exists and is finite. Relation (2.16) defines the class $\mathcal{L}d(\alpha)$. It is known that the constant c in (2.17) is equal to $\int_0^\infty e^{\alpha x}f(x)\mathrm{d}x$. For a distribution F with a density $f \in \mathcal{L}d(\alpha)$ for some $\alpha > 0$, it is easy to see that $f(x)/\overline{F}(x) \to \alpha$. Furthermore, for this case $F \in \mathcal{S}(\alpha)$ if and only if $f \in \mathcal{S}d(\alpha)$. The convergence in both (2.14) and (2.16) is automatically uniform on compact y-intervals. See Klüppelberg (1989) for these assertions.

2.3.3 Inverse Gaussian Distributions

The class $\mathcal{S}(\alpha)$ with $\alpha > 0$ contains *inverse Gaussian distributions*, which find important applications in various fields. The inverse Gaussian distributions form a two-parameter family of continuous probability distributions with support on $(0, \infty)$. The general forms of the probability density and distribution of an inverse Gaussian distribution are given by

$$\begin{aligned} f(x) &= \left(\frac{\nu}{2\pi x^3}\right)^{1/2} \exp\left\{\frac{-\nu(x-\mu)^2}{2\mu^2 x}\right\}, \\ F(x) &= \Phi\left(\sqrt{\frac{\nu}{x}}\left(\frac{x}{\mu}-1\right)\right) + \exp\left\{\frac{2\nu}{\mu}\right\}\Phi\left(-\sqrt{\frac{\nu}{x}}\left(\frac{x}{\mu}+1\right)\right), \end{aligned} \tag{2.18}$$

for $x > 0$, where $\Phi(\cdot)$ is the standard normal distribution, $\mu > 0$ is the expectation, and $\nu > 0$ is the shape parameter. The inverse Gaussian distribution with parameters μ and ν is denoted by $IG(\mu, \nu)$. According to the main theorem of Embrechts (1983), $IG(\mu, \nu)$ belongs to the class $\mathcal{S}(\alpha)$ with $\alpha = \nu / (2\mu^2)$.

Here are some basic properties of inverse Gaussian distributions:

1. The moment generating function of $F = IG(\mu, \nu)$ is given by

$$M_F(t) = \exp\left\{\frac{\nu}{\mu}\left(1 - \sqrt{1 - \frac{2\mu^2 t}{\nu}}\right)\right\}.$$

2. If a random variable X is distributed by $IG(\mu, \nu)$, then for each $t > 0$, the random variable tX is distributed by $IG(t\mu, t\nu)$.

3. If $X_1, \ldots, X_n$ are independent random variables with X_i distributed by $IG(\mu_0 w_i, \nu_0 w_i^2)$, $i = 1, \ldots, n$, then

$$\sum_{i=1}^{n} X_i \text{ is distributed by } IG\left(\mu_0 \overline{w}, \nu_0 \overline{w}^2\right),$$

where $\overline{w} = \sum_{i=1}^{n} w_i$.

Inverse Gaussian distributions have an intimate relationship with Brownian motions. Actually, the inverse Gaussian distributions have their origin in Brownian motions as first passage time distributions. Let $B^{(\mu)} = \left(B_t^{(\mu)}\right)_{t \geq 0}$ be a stochastic process such that

$$B_t^{(\mu)} = \mu t + \sigma B_t, \qquad \text{with } B_0^{(\mu)} = 0,$$

where μ is a constant and $(B_t)_{t \geq 0}$ is the standard Brownian motion. Considering

$$T(a) = \inf\left\{t > 0 \,\middle|\, B_t^{(\mu)} = a\right\},$$

the first passage time for a fixed level $a > 0$ by $B^{(\mu)}$, we have the following results:

(i) if $\mu > 0$, then $\mathbb{P}(T(a) < \infty) = 1$ and

$$T(a) \text{ is distributed by } IG\left(\frac{a}{\mu}, \frac{a^2}{\sigma^2}\right);$$

(ii) if $\mu < 0$, then $\mathbb{P}(T(a) < \infty) = \exp\{2a\mu/\sigma^2\}$ and

$$T(a)\,|T(a) < \infty \text{ is distributed by } IG\left(-\frac{a}{\mu}, \frac{a^2}{\sigma^2}\right);$$

(iii) if $\mu = 0$, then $\mathbb{P}(T(a) < \infty) = 1$ and $T(a)$ is distributed by a stable distribution with index 1/2. (See Feller (1966, page 170) for the definition of stable distributions.)

The above results can be proved by identifying the Laplace transform of $T(a)$ given that $T(a) < \infty$; see Prabhu (1965) and Chhikara and Folks (1989, Chapter 3) for the proofs.

The inverse Gaussian distributions have been applied to a wide range of fields. Most of these applications are based on the idea of first passage times of a Brownian motion with drift. These fields include actuarial science, demography, employment management, finance, and even linguistics; see Chhikara and Folks (1989, Chapter 10), Seshadri (1999, Part II), and references therein.

2.3.4 Matuszewska Indices

In this subsection we introduce Matuszewska indices of a distribution, which are connected with many useful properties of heavy-tailed distribution classes. As was done by Tang and Tsitsiashvili (2003b), for each $v > 0$, we set

$$\overline{F}_*(v) = \liminf_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)}, \qquad \overline{F}^*(v) = \limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)},$$

and then define

$$\mathbb{J}_F^+ = \inf_{v>1} \left\{ -\frac{\log \overline{F}_*(v)}{\log v} \right\}, \qquad \mathbb{J}_F^- = \sup_{v>1} \left\{ -\frac{\log \overline{F}^*(v)}{\log v} \right\}.$$

In the terminology of Bingham *et al.* (1987), $\mathbb{J}_F^+$ and $\mathbb{J}_F^-$ are the *upper and lower Matuszewska indices* of the nonnegative and nondecreasing function $f = 1/\overline{F}$. Following Tang and Tsitsiashvili (2003b), we call the quantities $\mathbb{J}_F^+$ and $\mathbb{J}_F^-$ the upper and lower Matuszewska indices of the distribution F, respectively. For more details of the Matuszewska indices, see Bingham *et al.* (1987, Chapter 2.1) and Cline and

Samorodnitsky (1994). Trivially, for a distribution F on $(-\infty, \infty)$ its Matuszewska indices $\mathbb{J}_F^{\pm}$ satisfy $0 \le \mathbb{J}_F^- \le \mathbb{J}_F^+ \le \infty$. The following lemma is very useful:

Lemma 2.16 (Proposition 2.2.1 of Bingham *et al.* (1987))**.** *Let F be a distribution on $(-\infty, \infty)$. Then,*

(i) for every α', $0 < \alpha' < \mathbb{J}_F^- \le \infty$, there are positive constants c_1 and d_1 such that the inequality

$$\frac{\overline{F}(y)}{\overline{F}(x)} \le c_1 \left(\frac{y}{x}\right)^{-\alpha'} \tag{2.19}$$

holds whenever $y \ge x \ge d_1$;

(ii) for every β', $0 \le \mathbb{J}_F^+ < \beta' < \infty$, there are positive constants c_2 and d_2 such that the inequality

$$\frac{\overline{F}(y)}{\overline{F}(x)} \ge c_2 \left(\frac{y}{x}\right)^{-\beta'} \tag{2.20}$$

holds whenever $y \ge x \ge d_2$.

For a distribution $F \in \mathrm{ERV}(-\alpha, -\beta)$, it is easy to see that $\mathbb{J}_F^- \ge \alpha$ and $\mathbb{J}_F^+ \le \beta$. Hence, inequalities (2.19) and (2.20) apply to $F \in \mathrm{ERV}(-\alpha, -\beta)$.

Corollary 2.17. *If $F \in \mathrm{ERV}(-\alpha, -\beta)$ for some $0 < \alpha \le \beta < \infty$, then for arbitrarily chosen $0 < \alpha' < \alpha \le \beta < \beta' < \infty$, there are some $c > 0$ and $x_0 > 0$ such that*

$$\frac{\overline{F}(y)}{\overline{F}(x)} \le c \max\left\{ \left(\frac{y}{x}\right)^{-\alpha'}, \left(\frac{y}{x}\right)^{-\beta'} \right\} \qquad \textit{for all } x, y \ge x_0.$$

Corollary 2.17 is very similar to the well-known Potter's bound for distributions in the class $\mathcal{R}$.

Lemma 2.18 (Theorem 1.5.6 of Bingham *et al.* (1987))**.** *If $F \in \mathcal{R}_{-\alpha}$ for some $\alpha \geq 0$, then for arbitrarily chosen $c > 1$, $\delta > 0$, there exists $x_0 > 0$ such that*

$$\frac{\overline{F}(y)}{\overline{F}(x)} \leq c \max \left\{ \left(\frac{y}{x}\right)^{-\alpha+\delta}, \left(\frac{y}{x}\right)^{-\alpha-\delta} \right\} \qquad \text{for all } x, y \geq x_0.$$

2.3.5 Other Properties

The following interrelations hold for heavy-tailed distribution classes:

(a) $\mathcal{R} \subset \text{ERV} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{K}$;

(b) $\mathcal{D} \not\subset \mathcal{S}$ and $\mathcal{S} \not\subset \mathcal{D}$.

The inclusions $\mathcal{R} \subset \text{ERV} \subset \mathcal{L} \cap \mathcal{D}$ are immediate consequences of their definitions. For the other interrelations above, see Embrechts *et al.* (1997) and Embrechts and Omey (1984) for detailed discussions. Figure 2.1 below clearly shows the interrelations between these heavy-tailed distribution classes.

Subexponential distributions are good candidates for modeling large claim sizes in insurance. Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables with distribution F. Denote the partial sum by $S_n = X_1 + \cdots + X_n$ and the maximum by $M_n = \max(X_1, \ldots, X_n)$. Then for every $n = 2, 3, \ldots$,

$$\mathbb{P}(S_n > x) = \overline{F^{n*}}(x),$$
$$\mathbb{P}(M_n > x) = \overline{F^n}(x) = \overline{F}(x) \sum_{k=0}^{n-1} F^k(x) \sim n\overline{F}(x).$$

If $F \in \mathcal{S}$, then by relation (2.11), we have

$$\mathbb{P}(S_n > x) \sim \mathbb{P}(M_n > x), \tag{2.21}$$

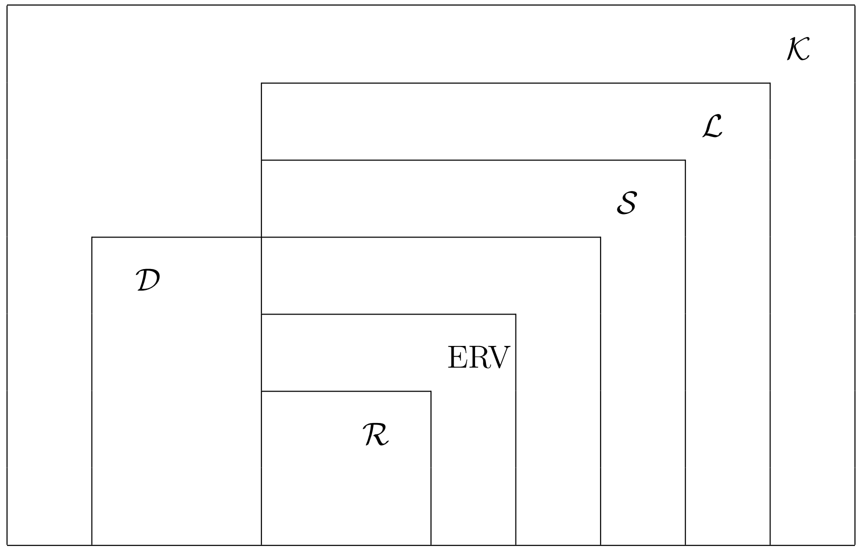

Figure 2.1: Classes of heavy-tailed distributions

which is called *the principle of a single big jump* in the literature. Actually, relation (2.21) is one of the intuitive notions of large claims in insurance business stating that extreme event occurs due to an unusually big jump.

This feature has been discussed in many publications. For instance, when doing a case study on the accumulated loss in the most severe storms encountered by a Swedish insurance group over the period 1982-1993, Rootzén and Tajvidi (1997) wrote that: "It can be seen that the most costly storm contributes about 25% of the total amount for the period, that it is 2.7 times bigger than the second worst storm, and that four storms together make up about half of the claims." Also, the 20-80 rule-of-thumb used by practicing actuaries when large claims are involved states that 20% of the individual claims are responsible for more than 80% of the total claim amount in a well defined portfolio.

Next we will present several properties of the distribution classes introduced in Subsection 2.3.1 that are to be used in the following chapters. The following two lemmas are for heavy-tailed distributions:

Lemma 2.19 (Theorem 1.5.2 of Bingham *et al.* (1987)). *If a distribution* $F \in \mathcal{R}_{-\alpha}$ *for some* $\alpha > 0$*, then for every* $0 < a < \infty$*, the convergence*

$$\lim_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} = v^{-\alpha}$$

holds uniformly in v *on* $[a, \infty)$*, where the uniformity of the convergence means*

$$\lim_{x\to\infty} \sup_{v\geq a} \left| \frac{\overline{F}(vx)}{\overline{F}(x)} - v^{-\alpha} \right| = 0.$$

Lemma 2.20 (Theorem 3.1 of Su and Tang (2003)). *Consider a distribution* F *on* $[0, \infty)$ *with finite mean. If either* $F \in \mathcal{L}$ *or* $F \in \mathcal{D}$*, then the relation*

$$\lim_{x\to\infty} \frac{\overline{F}(x)}{\int_x^\infty \overline{F}(y)\mathrm{d}y} = \alpha \tag{2.22}$$

holds with $\alpha = 0$*.*

Then we give some properties of the classes $\mathcal{L}(\alpha)$ and $\mathcal{S}(\alpha)$:

Lemma 2.21 (Lemma 3.1 of Tang (2007a)). *For each* $\alpha > 0$*, the following three assertions are equivalent:*

$$\text{(i) } F \in \mathcal{L}(\alpha); \quad \text{(ii) } \lim_{x\to\infty} \frac{\overline{F}(x)}{\int_x^\infty \overline{F}(y)\mathrm{d}y} = \alpha; \quad \text{(iii) } F_e \in \mathcal{L}(\alpha).$$

Lemma 2.22. *Assume that* F *on* $(-\infty, \infty)$ *satisfies* $0 < \int_0^\infty \overline{F}(y)\mathrm{d}y < \infty$*. For each* $\alpha \geq 0$*,* $F_e \in \mathcal{L}(\alpha)$ *if and only if relation (2.22) holds.*

Proof. For $\alpha > 0$, see Lemma 2.21. For $\alpha = 0$, observe that

$$0 \leq \frac{\overline{F_e}(x) - \overline{F_e}(x+1)}{\overline{F_e}(x)} \leq \frac{\overline{F}(x)}{\int_x^\infty \overline{F}(y)\mathrm{d}y} \leq \frac{\overline{F_e}(x-1) - \overline{F_e}(x)}{\overline{F_e}(x)},$$

from which the desired equivalence follows. □

Lemma 2.23. *If $F \in \mathcal{L}(\alpha)$ for some $\alpha \geq 0$, then for every $\beta > \alpha$,*

(i) there exist some positive constants c_0 and x_0 such that, for all $x \geq y \geq x_0$,

$$\frac{\overline{F}(y)}{\overline{F}(x)} \leq c_0 \mathrm{e}^{\beta(x-y)};$$

(ii) $\mathrm{e}^{-\beta x} = o\left(\overline{F}(x)\right)$.

Proof. (i) Note that $F \in \mathcal{L}(\alpha)$ if and only if $\overline{F}(\ln x)$ is regularly varying of index $-\alpha$. Thus, the desired conclusion is a straightforward consequence of Lemma 2.18.

(ii) For some β', $\alpha < \beta' < \beta$, by item (i) there exist some positive constants c_0 and x_0 such that, for all $x \geq x_0$,

$$\frac{\overline{F}(x_0)}{\overline{F}(x)} \leq c_0 \mathrm{e}^{\beta'(x-x_0)}.$$

Hence, the relation $\mathrm{e}^{-\beta x} = o\left(\overline{F}(x)\right)$ holds. □

Lemma 2.24. *Let F, G, G_1, G_2 be distributions on $(-\infty, \infty)$.*

(i) If $F \in \mathcal{L}(\alpha)$ for some $\alpha \geq 0$ and $\int_{-\infty}^\infty \mathrm{e}^{\beta y} G(\mathrm{d}y) < \infty$ for some $\beta > \alpha$, then

$$\lim_{x\to\infty} \frac{\overline{F * G}(x)}{\overline{F}(x)} = \int_{-\infty}^\infty \mathrm{e}^{\alpha y} G(\mathrm{d}y).$$

(ii) If $F \in \mathcal{S}(\alpha)$ for some $\alpha \geq 0$ and the limit $c_i = \lim \overline{G_i}(x)/\overline{F}(x)$ exists and belongs to $[0, \infty)$ for $i = 1, 2$, then

$$\lim_{x\to\infty} \frac{\overline{G_1 * G_2}(x)}{\overline{F}(x)} = c_1 \int_{-\infty}^\infty \mathrm{e}^{\alpha y} G_2(\mathrm{d}y) + c_2 \int_{-\infty}^\infty \mathrm{e}^{\alpha y} G_1(\mathrm{d}y).$$

Proof. (i) See Lemma 2.1 of Pakes (2004). Notice that, under the current conditions, the relation $\overline{G}(x) = o\left(\overline{F}(x)\right)$, as required in Lemma 2.1 of Pakes (2004), holds automatically by Lemma 2.23(ii).

(ii) See Proposition 2 of Rogozin and Sgibnev (1999). □

Finally, we give two analogous upper bound properties of the distribution class $\mathcal{S}(\alpha)$ and the density class $\mathcal{S}d(\alpha)$.

Lemma 2.25. *If $F \in \mathcal{S}(\alpha)$ for some $\alpha \geq 0$, then for every $n = 1, 2, \ldots,$*

$$\lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n \left(\int_{-\infty}^{\infty} \mathrm{e}^{\alpha y} F(\mathrm{d}y) \right)^{n-1}. \tag{2.23}$$

Furthermore, for every $\varepsilon > 0$ there exists some $K_\varepsilon > 0$ such that for all $n = 1, 2, \ldots$ and $x > 0$,

$$\overline{F^{n*}}(x) \leq K_\varepsilon \left(\varepsilon + \int_{-\infty}^{\infty} \mathrm{e}^{\alpha y} F(\mathrm{d}y) \right)^{n} \overline{F}(x). \tag{2.24}$$

Proof. For F on $[0, \infty)$, relations (2.23) and (2.24) are known in Cline (1986, Corollary 1) and Embrechts and Goldie (1980). For extension to F on $(-\infty, \infty)$, see Lemma 3.2 of Tang and Tsitsiashvili (2003b) for the proof of (2.23) and Lemma 5.3 of Pakes (2004) for the proof of (2.24). □

Lemma 2.26 (Chover *et al.* (1973); Klüppelberg (1989))**.** *If $f \in \mathcal{S}d(\alpha)$ for some $\alpha \geq 0$, then for every $n = 1, 2, \ldots,$*

$$\lim_{x\to\infty} \frac{f^{n\star}(x)}{f(x)} = n \left(\int_{0}^{\infty} \mathrm{e}^{\alpha y} f(y) \mathrm{d}y \right)^{n-1}.$$

Furthermore, if f is bounded, then for every $\varepsilon > 0$ there exists some $K_\varepsilon > 0$ such that *for all* $n = 1, 2, \ldots$ *and* $x > 0$,

$$f^{n\star}(x) \leq K_\varepsilon \left(\varepsilon + \int_0^\infty \mathrm{e}^{\alpha y} f(y) \mathrm{d}y \right)^n f(x).$$

CHAPTER 3
DISCOUNTED AGGREGATE CLAIMS WITH HEAVY TAILS

In this chapter, we introduce a constant force of interest in the renewal risk model and study the tail probability of discounted aggregate claims. Since it is usually not possible to get closed-form expressions except for few ideal cases, we instead aim at asymptotic formulas. The question is of much practical interest in insurance risk management. The study can provide an easy and precise approximation when measuring the risk of large losses via Value-at-Risk or Conditional Tail Expectation. Also, such an approximation usually plays a crucial role in pricing some insurance products. We derive for the tail probability of discounted aggregate claims an asymptotic formula, which holds uniformly for all time horizons. A key assumption in our model is that the claim-size distribution is subexponential. This chapter is based on the joint research paper Hao and Tang (2008).

3.1 Introduction

Consider the renewal risk model in which claim sizes $X_k, k = 1, 2, \ldots,$ constitute a sequence of i.i.d. nonnegative random variables with common distribution F, while their arrival times $\tau_k, k = 1, 2, \ldots,$ independent of $X_k, k = 1, 2, \ldots,$ constitute a renewal counting process

$$N_t = \#\{k = 1, 2, \ldots : \tau_k \leq t\}, \qquad t \geq 0. \tag{3.1}$$

According to Definition 2.2, the inter-arrival times $\theta_1 = \tau_1, \theta_k = \tau_k - \tau_{k-1}, k = 2, 3, \ldots,$ constitute another sequence of i.i.d., nonnegative, and not-degenerate-at-zero random variables. If $(N_t)_{t\geq 0}$ is a Poisson process, then this model reduces to the commonly used compound Poisson model. Aggregate claims form a random sum $S_t = \sum_{k=1}^{N_t} X_k$, $t \geq 0$. Regarding to the distribution of S_t, we have the following lemma connecting subexponentiality and compound distributions:

Lemma 3.1 (Theorem A3.20 of Embrechts *et al.* (1997))**.** *Suppose p_n, $n = 0, 1, \ldots$, form a probability measure on the set of nonnegative integers such that for some $\varepsilon > 0$,*

$$\sum_{n=0}^{\infty} p_n (1+\varepsilon)^n < \infty,$$

and set

$$G(x) = \sum_{n=0}^{\infty} p_n F^{n*}(x), \qquad x \geq 0.$$

(i) If $F \in \mathcal{S}$, then $G \in \mathcal{S}$, and

$$\lim_{x\to\infty} \frac{\overline{G}(x)}{\overline{F}(x)} = \sum_{n=1}^{\infty} n p_n. \tag{3.2}$$

(ii) Conversely, if (3.2) holds and there exists $n \geq 2$ such that $p_n > 0$, then $F \in \mathcal{S}$.

For example, if $(N_t)_{t\geq 0}$ is a Poisson process with rate λ, then for fixed $t > 0$, the tail probability of S_t is equivalent to $\lambda \overline{F}$ as long as F is subexponential.

We want to investigate the impact of interest rate on the tail behavior of the aggregate claims process. Suppose that there is a constant force of interest $r > 0$. The discounted (actuarial present value of) aggregate claims are expressed as the stochastic process

$$D_r(t) = \int_{0-}^{t} \mathrm{e}^{-rs} \mathrm{d}S_s = \sum_{k=1}^{\infty} X_k \mathrm{e}^{-r\tau_k} 1_{(\tau_k \leq t)}, \qquad t \geq 0. \tag{3.3}$$

From (3.3) we see that $(D_r(t))_{t\geq 0}$ corresponds to a special case of the stochastic integral

$$Z_t = \int_{0-}^{t} \mathrm{e}^{-R_s} \mathrm{d}P_s, \qquad t \geq 0,$$

where $(R_t)_{t\geq 0}$ and $(P_t)_{t\geq 0}$ are two independent stochastic processes fulfilling certain requirements so that Z_∞ is well defined; see Subsection 2.2.3 for stochastic integral. When both of them are Lévy processes, Gjessing and Paulsen (1997) gave a wealth of examples showing the exact distribution or asymptotic tail probability of Z_∞. Related discussions on the distribution of Z_∞ can also be found in Dufresne (1990), Paulsen (1993, 1997), and Nilsen and Paulsen (1996), among others. However, we notice that all these references did not pay particular attention to the important case that $(P_t)_{t\geq 0}$ has heavy-tailed jumps.

In this chapter, we are interested in the asymptotic tail behavior of $D_r(t)$ for all t for which the renewal function

$$\lambda_t = \mathbb{E}N_t = \sum_{k=1}^{\infty} \mathbb{P}\left(\tau_k \leq t\right)$$

is positive. Define $\Lambda = \{t : \lambda_t > 0\} \cup \{\infty\} = \{t : \mathbb{P}\left(\tau_1 \leq t\right) > 0\} \cup \{\infty\}$ for later use.

For two positive bivariate functions $a\left(\cdot,\cdot\right)$ and $b\left(\cdot,\cdot\right)$, we say the asymptotic relation $a\left(x,t\right) \sim b\left(x,t\right)$ holds uniformly over all t in a nonempty set Δ if

$$\lim_{x\to\infty} \sup_{t\in\Delta} \left| \frac{a\left(x,t\right)}{b\left(x,t\right)} - 1 \right| = 0.$$

Tang (2007b) investigated the tail probability of the stochastic process $D_r(t)$ in (3.3) and gave the following result:

Proposition 3.2 (Theorem 1.1 of Tang (2007b)). *Consider the renewal risk model introduced above. If $F \in \mathrm{ERV}$, then the relation*

$$\mathbb{P}\left(D_r(t) > x\right) \sim \int_{0-}^{t} \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s \tag{3.4}$$

holds uniformly for all $t \in \Lambda$.

Formula (3.4) transparently captures all stochastic information of the claim sizes and their arrival times. However, we point out that the assumption $F \in \mathrm{ERV}$ unfortunately excludes some important distributions such as lognormal and Weibull distributions. In the context of ruin theory, Tang (2005) and Wang (2008) obtained some similar asymptotic results as (3.4) for the finite-time ruin probability with a fixed time horizon $t \in \Lambda$.

Our goal in this chapter is to extend Proposition 3.2 from the class ERV to the subexponential class $\mathcal{S}$ so that lognormal and heavy-tailed Weibull distributions are included. The class ERV enjoys some favorable properties like inequalities (2.19) and (2.20), which play a crucial role in establishing Proposition 3.2, but the class $\mathcal{S}$ does not possess such properties. Therefore, to achieve the desired extension we have to employ different approaches.

The rest of this chapter consists of three sections: Section 3.2 presents our main results, Section 3.3 proves them, in turn, after preparing some necessary lemmas, and Section 3.4 gives a simulation result.

3.2 Main Results

The first main result of this chapter is given below:

Theorem 3.3. *Consider the discounted aggregate claims described in relation (3.3). If $F \in \mathcal{S}$, then relation (3.4) holds uniformly for all $t \in \Lambda_T = \Lambda \cap [0, T]$ for arbitrarily fixed finite $T \in \Lambda$.*

In the next two main results below, we extend the set over which relation (3.4) holds uniformly to the maximal set Λ.

Theorem 3.4. *Consider the discounted aggregate claims described in relation (3.3). If $F \in \mathcal{A}$ and $\mathbb{P}(\theta_1 > \delta) = 1$ for some $\delta > 0$, then relation (3.4) holds uniformly for all $t \in \Lambda$.*

Recalling the structure of $D_r(t)$ in (3.3), the purpose of the technical assumption on the distribution of θ_1 in Theorem 3.4 is to let the series $D_r(t)$ converge more easily. Though not nice-looking, it causes no trouble for real applications since δ can be arbitrarily close to 0. Nevertheless, we can get rid of this assumption if we slightly reduce the scope of the claim-size distribution F, as shown below:

Theorem 3.5. *Consider the discounted aggregate claims described in relation (3.3). If $F \in \mathcal{S} \cap \mathcal{R}_{-\alpha}$ for some $0 < \alpha \leq \infty$ and θ_1 is positive, then relation (3.4) holds uniformly for all $t \in \Lambda$.*

Our last main result is given below:

Theorem 3.6. *Consider the discounted aggregate claims described in relation (3.3), in which $(N_t)_{t \geq 0}$ is a Poisson process with intensity $\lambda > 0$. If $F \in \mathcal{S}$ and $F_e \in \mathcal{A}$, then the relation*

$$\mathbb{P}(D_r(t) > x) \sim \lambda \int_0^t \overline{F}(x\mathrm{e}^{rs})\mathrm{d}s \tag{3.5}$$

holds uniformly for all $t \in (0, \infty]$.

We remark that the assumptions $F \in \mathcal{S}$ and $F_e \in \mathcal{A}$ in Theorem 3.6 are satisfied by almost all useful heavy-tailed distributions such as Pareto (with finite expectation), lognormal, and heavy-tailed Weibull distributions.

Let us illustrate the usefulness of the uniformity of (3.5). Denote by

$$\tau(x) = \inf\{t \geq 0 : D_r(t) > x\}, \qquad x > 0,$$

the time when $D_r(t)$ first up-crosses the level x. Clearly, $\tau(x)$ is a defective random variable with total mass $\mathbb{P}(\tau(x) < \infty) = \mathbb{P}(D_r(\infty) > x) < 1$.

Let all conditions of Theorem 3.6 hold. We first consider the asymptotic behavior of the Laplace transform of $\tau(x)$. For every $u > 0$, use integration by parts and the identity $\mathbb{P}(\tau(x) \leq t) = \mathbb{P}(D_r(t) > x)$ for all $t \geq 0$ to get

$$\mathbb{E}\mathrm{e}^{-u\tau(x)} = u \int_0^\infty \mathbb{P}(D_r(t) > x)\, \mathrm{e}^{-ut} \mathrm{d}t.$$

Substituting the uniform asymptotic relation (3.5) into the above then changing the order of integrals, we have

$$\mathbb{E}\mathrm{e}^{-u\tau(x)} \sim \lambda \int_0^\infty \mathrm{e}^{-us} \overline{F}(x\mathrm{e}^{rs}) \mathrm{d}s.$$

This gives an explicit asymptotic expression for the Laplace transform of $\tau(x)$.

We then consider the limiting distribution of $\tau(x)$ conditional on $(\tau(x) < \infty)$. For every fixed $t > 0$, by Theorem 3.6,

$$\mathbb{P}(\tau(x) \leq t |\, \tau(x) < \infty) = \frac{\mathbb{P}(D_r(t) > x)}{\mathbb{P}(D_r(\infty) > x)} \sim \frac{\int_0^t \overline{F}(x\mathrm{e}^{rs}) \mathrm{d}s}{\int_0^\infty \overline{F}(x\mathrm{e}^{rs}) \mathrm{d}s}. \tag{3.6}$$

If $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, then using Lemma 2.19 we see that the convergence

$$\frac{\overline{F}(x\mathrm{e}^{rs})}{\overline{F}(x)} \to \mathrm{e}^{-\alpha rs} \tag{3.7}$$

holds uniformly for all $s \in [0, \infty)$. Therefore, dividing both integrands on the right-hand side of (3.6) by $\overline{F}(x)$ then plugging (3.7), we obtain

$$\mathbb{P}\left(\tau(x) \le t \middle| \tau(x) < \infty\right) \to 1 - \mathrm{e}^{-\alpha rt},$$

meaning that the limiting distribution under discussion is exponential.

3.3 Proofs

3.3.1 Proof of Theorem 3.3

Lemma 3.7. *Let $X_1, \ldots, X_n$ be n independent random variables, each X_k distributed by F_k. If there are n positive constants $l_1, \ldots, l_n$, and a distribution $F \in \mathcal{S}$ such that $\overline{F_k}(x) \sim l_k \overline{F}(x)$ holds for $k = 1, \ldots, n$, then for arbitrarily fixed numbers a and b, $0 < a \le b < \infty$, the relation*

$$\mathbb{P}\left(\sum_{k=1}^{n} c_k X_k > x\right) \sim \sum_{k=1}^{n} \overline{F_k}(x/c_k)$$

holds uniformly for all $(c_1, \ldots, c_n) \in [a, b] \times \cdots \times [a, b]$.

Proof. The proof can be given by going along the same lines of the proof of Proposition 5.1 of Tang and Tsitsiashvili (2003a) with some obvious modifications. □

Lemma 3.8 (Lemma 2.2 of Cai and Kalashnikov (2000))**.** *Consider the renewal counting process $(N_t)_{t \ge 0}$ defined in (3.1). For any $t_1 \ge 0$, $t_2 \ge 0$, there exists a random variable $\widehat{N}_{t_2}$ such that $\widehat{N}_{t_2}$ and N_{t_1} are independent, $\widehat{N}_{t_2} \overset{d}{=} N_{t_2}$, and*

$$N_{t_1+t_2} \le N_{t_1} + \widehat{N}_{t_2} + 1.$$

Lemma 3.9. *Consider the renewal counting process* $(N_t)_{t\geq 0}$ *defined in (3.1). There exists some* $h > 0$ *such that* $\mathbb{E}e^{hN_t} < \infty$ *holds for all* $t \geq 0$.

Proof. It is shown in Stein (1946) that, for arbitrarily fixed $t_0 > 0$, there exists some $h > 0$ such that $\mathbb{E}e^{hN_{t_0}} < \infty$. For every $t \geq 0$, we can find a positive integer k such that $(k-1)\,t_0 \leq t < kt_0$. Inductively applying Lemma 3.8, we can obtain i.i.d. random variables $\widehat{N}_{t_0}(1), \ldots, \widehat{N}_{t_0}(k)$ with common distribution as that of N_{t_0} such that

$$N_t \leq N_{kt_0} \overset{d}{\leq} \sum_{i=1}^{k} \widehat{N}_{t_0}(i) + k - 1,$$

where for two random variables X and Y, the relation $X \overset{d}{\leq} Y$ means that $\mathbb{P}(X > x) \leq \mathbb{P}(Y > x)$ for all x. Therefore, $\mathbb{E}e^{hN_t} < \infty$, as claimed. □

Now we are ready to give the proof for Theorem 3.3.

Proof of Theorem 3.3. Arbitrarily choose some positive integer N. Clearly, for $t \in \Lambda_T$,

$$\begin{aligned}\mathbb{P}\left(D_r(t) > x\right) &= \left(\sum_{n=1}^{N} + \sum_{n=N+1}^{\infty}\right) \mathbb{P}\left(\sum_{k=1}^{n} X_k e^{-r\tau_k} > x, N_t = n\right) \\ &= I_1\left(x, t, N\right) + I_2\left(x, t, N\right).\end{aligned}$$

First consider $I_2(x,t,N)$. We have

$$\begin{aligned}
I_2(x,t,N) &\leq \sum_{n=N+1}^{\infty} \mathbb{P}\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-r\tau_1} > x, \tau_n \leq t < \tau_{n+1}\right) \\
&= \sum_{n=N+1}^{\infty} \int_{0-}^{t} \mathbb{P}\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-rs} > x, \tau_n - \tau_1 \leq t-s < \tau_{n+1}-\tau_1\right) \mathbb{P}(\tau_1 \in \mathrm{d}s) \\
&= \sum_{n=N+1}^{\infty} \int_{0-}^{t} \mathbb{P}\left(\sum_{k=1}^{n} X_k > x\mathrm{e}^{rs}\right) \mathbb{P}(N_{t-s} = n-1)\, \mathbb{P}(\tau_1 \in \mathrm{d}s) \\
&\leq \sum_{n=N}^{\infty} \int_{0-}^{t} \mathbb{P}\left(\sum_{k=1}^{n+1} X_k > x\mathrm{e}^{rs}\right) \mathbb{P}(N_{t-s} = n)\, \mathrm{d}\lambda_s.
\end{aligned}$$

Applying Lemma 2.25 to the above, for every $\varepsilon > 0$ and some $c_\varepsilon > 0$,

$$\begin{aligned}
I_2(x,t,N) &\leq c_\varepsilon (1+\varepsilon) \int_{0-}^{t} \overline{F}(x\mathrm{e}^{rs}) \mathbb{E}(1+\varepsilon)^{N_{t-s}} 1_{\{N_{t-s} \geq N\}} \mathrm{d}\lambda_s \\
&\leq c_\varepsilon (1+\varepsilon)\, \mathbb{E}(1+\varepsilon)^{N_T} 1_{\{N_T \geq N\}} \int_{0-}^{t} \overline{F}(x\mathrm{e}^{rs}) \mathrm{d}\lambda_s.
\end{aligned}$$

By Lemma 3.9, we can choose some ε sufficiently small such that $\mathbb{E}(1+\varepsilon)^{N_T} < \infty$. It follows that $\mathbb{E}(1+\varepsilon)^{N_T} 1_{\{N_T \geq N\}} \to 0$ as $N \to \infty$. Therefore, for all $x > 0$,

$$\lim_{N\to\infty} \sup_{t \in \Lambda_T} \frac{I_2(x,t,N)}{\int_{0-}^{t} \overline{F}(x\mathrm{e}^{rs})\, \mathrm{d}\lambda_s} = 0. \tag{3.8}$$

Next consider $I_1(x,t,N)$. Using Lemma 3.7, it holds uniformly for all $t \in \Lambda_T$ that

$$\begin{aligned}
I_1(x,t,N) &\sim \left(\sum_{n=1}^{\infty}\sum_{k=1}^{n} - \sum_{n=N+1}^{\infty}\sum_{k=1}^{n}\right) \mathbb{P}\left(X_k \mathrm{e}^{-r\tau_k} > x, N_t = n\right) \\
&= I_{11}(x,t) - I_{12}(x,t,N).
\end{aligned}$$

Clearly, for all $t \in \Lambda_T$,

$$I_{11}(x,t) = \sum_{k=1}^{\infty} \mathbb{P}\left(X_k \mathrm{e}^{-r\tau_k} > x, N_t \geq k\right) = \int_{0-}^{t} \overline{F}(x\mathrm{e}^{rs})\, \mathrm{d}\lambda_s. \tag{3.9}$$

For $I_{12}(x,t,N)$, similarly to the derivation for $I_2(x,t,N)$, we have

$$\begin{aligned} I_{12}(x,t,N) &\leq \sum_{n=N+1}^{\infty}\sum_{k=1}^{n}\mathbb{P}\left(X_k \mathrm{e}^{-r\tau_1} > x, N_t = n\right) \\ &\leq \sum_{n=N}^{\infty}\sum_{k=1}^{n+1}\int_{0-}^{t}\overline{F}\left(x\mathrm{e}^{rs}\right)\mathbb{P}\left(N_{t-s}=n\right)\mathrm{d}\lambda_s \\ &\leq \int_{0-}^{t}\overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}\lambda_s \sum_{n=N}^{\infty}(n+1)\,\mathbb{P}\left(N_T \geq n\right). \end{aligned}$$

It follows that, for all $x>0$,

$$\lim_{N\to\infty}\sup_{t\in\Lambda_T}\frac{I_{12}(x,t,N)}{\int_{0-}^{t}\overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}\lambda_s} = 0. \tag{3.10}$$

From (3.8), (3.9), and (3.10) we conclude that the asymptotic relation (3.4) holds uniformly for all $t\in\Lambda_T$. □

3.3.2 Proof of Theorem 3.4

Lemma 3.10. *If a distribution F on $[0,\infty)$ satisfies (2.12) for some $v>1$, then*

$$\lim_{t\to\infty}\limsup_{x\to\infty}\frac{\int_t^{\infty}\overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}\lambda_s}{\int_0^{t}\overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}\lambda_s} = 0, \tag{3.11}$$

where the positive constant r and the renewal function λ_s, $s\geq 0$, are the same as introduced in Section 3.1.

Proof. For every $t\in\Lambda$, apply inequality (2.19) to obtain that, for $x\geq x_0$,

$$\begin{aligned} \frac{\int_t^{\infty}\overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}\lambda_s}{\int_0^{t}\overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}\lambda_s} &= \frac{\int_t^{\infty}\overline{F}\left(x\mathrm{e}^{rs}\right)/\overline{F}\left(x\mathrm{e}^{rt}\right)\mathrm{d}\lambda_s}{\int_0^{t}\overline{F}\left(x\mathrm{e}^{rs}\right)/\overline{F}\left(x\mathrm{e}^{rt}\right)\mathrm{d}\lambda_s} \\ &\leq c^2\frac{\int_t^{\infty}\mathrm{e}^{-pr(s-t)}\mathrm{d}\lambda_s}{\int_0^{t}\mathrm{e}^{pr(t-s)}\mathrm{d}\lambda_s} = c^2\frac{\int_t^{\infty}\mathrm{e}^{-prs}\mathrm{d}\lambda_s}{\int_0^{t}\mathrm{e}^{-prs}\mathrm{d}\lambda_s}. \end{aligned}$$

This implies (3.11). □

Lemma 3.11 (Corollary 3.1 of Chen *et al.* (2005)). *Let X_k, $k = 1, 2, \ldots$, be a sequence of i.i.d. nonnegative random variables with common distribution $F \in \mathcal{A}$. If*

$$\sum_{k=1}^{\infty} \omega_k^{\delta} < \infty, \qquad \textit{for some } 0 < \delta < \frac{\mathbb{J}_F^-}{1 + \mathbb{J}_F^-},$$

where $\mathbb{J}_F^-$ is the lower Matuszewska index of F, then

$$\mathbb{P}\left(\sum_{k=1}^{\infty} \omega_k X_k > x\right) \sim \sum_{k=1}^{\infty} \mathbb{P}\left(\omega_k X_k > x\right).$$

Before presenting the following lemmas, we need to define the association of random variables. We use the definition given by Esary *et al.* (1967) here:

Definition 3.12. *Random variables $X_1, \ldots, X_n$ are **associated** if*

$$\operatorname{Cov}(f(\mathbf{X}), g(\mathbf{X})) \geq 0, \qquad \textit{where } \mathbf{X} = (X_1, \ldots, X_n),$$

for all nondecreasing functions f and g for which $\mathbb{E}f(\mathbf{X})$, $\mathbb{E}g(\mathbf{X})$, and $\mathbb{E}f(\mathbf{X})g(\mathbf{X})$ exist.

Lemma 3.13 (Theorem 2.1 of Esary *et al.* (1967)). *Independent random variables are associated.*

Lemma 3.14. *Under the conditions of Theorem 3.4, we have*

$$\mathbb{P}\left(D_r(\infty) > x\right) \lesssim \int_0^{\infty} \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s. \tag{3.12}$$

Proof. Arbitrarily choose some positive integer N such that $N\delta \in \Lambda$. Since $\mathbb{P}\left(\theta_1 > \delta\right) = 1$, we have

$$\mathbb{P}\left(D_r(\infty) > x\right) \leq \mathbb{P}\left(\sum_{k=1}^{N} X_k \mathrm{e}^{-r\tau_k} + \left(\sum_{k=N+1}^{\infty} X_k \mathrm{e}^{-r(k-N)\delta}\right) \mathrm{e}^{-r\tau_N} > x\right). \tag{3.13}$$

Write $\Sigma_\delta = \sum_{k=N+1}^{\infty} X_k \mathrm{e}^{-r(k-N)\delta}$, whose distribution does not depend on N. Applying Lemma 3.11,

$$\mathbb{P}(\Sigma_\delta > x) = \mathbb{P}\left(\sum_{k=1}^{\infty} X_k \mathrm{e}^{-rk\delta} > x\right) \sim \overline{F}(x) \sum_{k=1}^{\infty} \frac{\overline{F}\left(x\mathrm{e}^{rk\delta}\right)}{\overline{F}(x)}.$$

Hence, by inequality (2.19), there is some constant $c_* > 0$ such that $\mathbb{P}(\Sigma_\delta > x) \leq c_* \overline{F}(x)$ for all $x \in [0, \infty)$. Next we come back to (3.13). Introduce a new random variable $\widetilde{\Sigma_\delta}$ independent of $\{X_k, k = 1, 2, \ldots\}$ and $\{\tau_k, k = 1, 2, \ldots\}$ with a tail satisfying

$$\mathbb{P}\left(\widetilde{\Sigma_\delta} > x\right) = \min\left\{C_* \overline{F}(x), 1\right\}, \qquad x \geq 0.$$

Therefore, $\Sigma_\delta \overset{d}{\leq} \widetilde{\Sigma_\delta}$, and

$$\mathbb{P}(D_r(\infty) > x) \leq \mathbb{P}\left(\sum_{k=1}^{N} X_k \mathrm{e}^{-r\tau_k} + \widetilde{\Sigma_\delta} \mathrm{e}^{-r\tau_N} > x\right). \tag{3.14}$$

To apply Lemma 3.7, we choose some $M_1 > 0$ and derive

$$\begin{aligned}
\mathbb{P}\left(\sum_{k=1}^{N} X_k \mathrm{e}^{-r\tau_k} + \widetilde{\Sigma_\delta} \mathrm{e}^{-r\tau_N} > x\right) &= \mathbb{P}\left(\sum_{k=1}^{N} X_k \mathrm{e}^{-r\tau_k} + \widetilde{\Sigma_\delta} \mathrm{e}^{-r\tau_N} > x, \bigcup_{i=1}^{N} (\theta_i \geq M_1)\right) \\
&\quad + \mathbb{P}\left(\sum_{k=1}^{N} X_k \mathrm{e}^{-r\tau_k} + \widetilde{\Sigma_\delta} \mathrm{e}^{-r\tau_N} > x, \bigcap_{i=1}^{N} (\theta_i < M_1)\right) \\
&= J_1(x, N, M_1) + J_2(x, N, M_1). \qquad (3.15)
\end{aligned}$$

Since $\theta_1, \ldots, \theta_N$ are i.i.d. random variables, by Lemma 3.13 they are associated. By the definition of association we have

$$J_1(x, N, M_1) \leq \mathbb{P}\left(\sum_{k=1}^{N} X_k \mathrm{e}^{-r\tau_k} + \widetilde{\Sigma_\delta} \mathrm{e}^{-r\tau_N} > x\right) \mathbb{P}\left(\bigcup_{i=1}^{N} (\theta_i \geq M_1)\right). \tag{3.16}$$

Substituting (3.16) into (3.15) and rearranging the resulting inequality, we have

$$\mathbb{P}\left(\sum_{k=1}^{N} X_k \mathrm{e}^{-r\tau_k} + \widetilde{\Sigma_\delta} \mathrm{e}^{-r\tau_N} > x\right) \leq \frac{J_2(x, N, M_1)}{1 - \mathbb{P}\left(\bigcup_{i=1}^{N} (\theta_i \geq M_1)\right)}.$$

Further substituting this into (3.14), applying Lemma 3.7 to $J_2(x, N, M_1)$, and letting $M_1 \to \infty$, we obtain that

$$\begin{aligned}\mathbb{P}\left(D_r(\infty) > x\right) &\lesssim \sum_{k=1}^{N} \mathbb{P}\left(X_k \mathrm{e}^{-r\tau_k} > x\right) + \mathbb{P}\left(\widetilde{\Sigma_\delta} \mathrm{e}^{-r\tau_N} > x\right) \\ &\leq \sum_{k=1}^{\infty} \mathbb{P}\left(X_k \mathrm{e}^{-r\tau_k} > x\right) + \int_{N\delta}^{\infty} \mathbb{P}\left(\widetilde{\Sigma_\delta} > x\mathrm{e}^{rs}\right) \mathbb{P}\left(\tau_N \in \mathrm{d}s\right) \\ &\leq \int_0^{\infty} \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s + c_* \int_{N\delta}^{\infty} \overline{F}\left(x\mathrm{e}^{rs}\right) \mathbb{P}\left(\tau_N \in \mathrm{d}s\right). \end{aligned} \tag{3.17}$$

Apply inequality (2.19) again to obtain that, for some $M_2 \in \Lambda \cap (0, N\delta]$ and all large x,

$$\begin{aligned}\frac{\int_{N\delta}^{\infty} \overline{F}\left(x\mathrm{e}^{rs}\right) \mathbb{P}\left(\tau_N \in \mathrm{d}s\right)}{\int_0^{\infty} \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s} &\leq \frac{c\overline{F}\left(x\mathrm{e}^{rM_2}\right) \mathbb{E}\mathrm{e}^{-pr(\tau_N - M_2)}}{\int_0^{M_2} \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s} \\ &\leq \frac{c}{\lambda_{M_2}} \mathbb{E}\mathrm{e}^{-pr(\tau_N - M_2)} \to 0, \end{aligned} \tag{3.18}$$

as $N \to \infty$. From (3.17) and (3.18), the asymptotic relation (3.12) follows immediately. □

Now we are ready to give the proof for Theorem 3.4.

Proof of Theorem 3.4. According to Lemma 3.10, for every $\varepsilon > 0$ there exists some $T_0 > 0$ such that the inequality

$$\int_{T_0}^{\infty} \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s \leq \varepsilon \int_0^{T_0} \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s \tag{3.19}$$

holds for all large x. By Theorem 3.3 and inequality (3.19), it holds uniformly for all

$t \in (T_0, \infty]$ that

$$\begin{aligned}\mathbb{P}\left(D_r\left(t\right) > x\right) &\geq \mathbb{P}\left(D_r\left(T_0\right) > x\right) \\ &\sim \int_0^{T_0} \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s \\ &\geq \left(\int_0^t - \int_{T_0}^{\infty}\right) \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s \\ &\geq (1-\varepsilon) \int_0^t \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s.\end{aligned}$$

Likewise, by Lemma 3.14 and inequality (3.19), it holds uniformly for all $t \in (T_0, \infty]$ that

$$\begin{aligned}\mathbb{P}\left(D_r\left(t\right) > x\right) &\leq \mathbb{P}\left(D_r\left(\infty\right) > x\right) \\ &\lesssim \int_0^{\infty} \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s \\ &\leq \left(\int_0^t + \int_{T_0}^{\infty}\right) \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s \\ &\leq (1+\varepsilon) \int_0^t \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s.\end{aligned}$$

Hence, for all $t \in (T_0, \infty]$ and all large x,

$$(1-2\varepsilon) \int_0^t \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s \leq \mathbb{P}\left(D_r\left(t\right) > x\right) \leq (1+2\varepsilon) \int_0^t \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s. \tag{3.20}$$

By Theorem 3.3 again, the inequalities in (3.20) also hold for all $t \in \Lambda_{T_0}$ (hence for all $t \in \Lambda$) and all large x. As $\varepsilon > 0$ is arbitrary, we complete the proof. □

3.3.3 Proof of Theorem 3.5

For a distribution $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha \leq \infty$, relation (2.12) obviously holds. Hence, Lemma 3.10 still works under the conditions of Theorem 3.5. We need to prepare the following lemma to replace Lemma 3.14:

Lemma 3.15. *Consider the discounted aggregate claims described in relation (3.3). If $F \in \mathcal{S} \cap \mathcal{R}_{-\alpha}$ for some $0 < \alpha \leq \infty$ and θ_1 is positive, then we have*

$$\mathbb{P}\left(D_r(\infty) > x\right) \sim \int_0^\infty \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s. \tag{3.21}$$

Proof. When $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$, relation (3.21) holds by Proposition 3.2. We only need to consider the case $F \in \mathcal{S} \cap \mathcal{R}_{-\infty}$.

The following proof is based on the work of Konstantinides and Tang (2009). Denote $Y_k = \mathrm{e}^{-r\theta_k}$, $k = 1, 2, \ldots$. Then (X_k, Y_k), $k = 1, 2, \ldots$, are i.i.d. random copies, say, of the generic random pair (X, Y). It is clear that

$$D_r(\infty) = \sum_{k=1}^{\infty} X_k \prod_{i=1}^{k} Y_i. \tag{3.22}$$

Furthermore, $D_r(\infty)$ satisfies the random functional equation

$$R \stackrel{d}{=} Y(X + R) \tag{3.23}$$

where R is a random variable independent of (X, Y).

We turn to find out the asymptotic tail probability of the solution R for the random equation (3.23). From (3.22) it is obvious that

$$\mathbb{P}\left(R > x\right) \geq \mathbb{P}\left(X_1 Y_1 > x\right), \qquad -\infty < x < \infty. \tag{3.24}$$

For the upper bound, we consider the random difference equation

$$R_{n+1} = Y_{n+1}(X_{n+1} + R_n), \qquad n = 0, 1, 2, \ldots,$$

where R_0 is independent of $\{(X_k, Y_k), k = 1, 2, \ldots\}$. Since $Y \in (0, 1)$, R_n converges in distribution as $n \to \infty$ to R, the unique solution of the random equation (3.23).

Moreover, the convergence does not depend on R_0. See Theorem 1.5 of Vervaat (1979) for more details.

Let X' be independent of (X, Y) and have the same distribution of X. We have

$$\begin{aligned}\mathbb{P}\left(Y(X+X')>x\right) &= \int_0^1 \mathbb{P}\left(X+X'>\frac{x}{y}\right)\mathbb{P}(Y\in \mathrm{d}y)\\ &\sim 2\int_0^1 \overline{F}\left(\frac{x}{y}\right)\mathbb{P}(Y\in \mathrm{d}y)\\ &= o(1)\overline{F}(x),\end{aligned}$$

where we used $F \in \mathcal{S}$ in the second step and $F \in \mathcal{R}_{-\infty}$ and $\theta_1 > 0$ in the last step. Thus, there exists some $x_0 > 0$ large enough such that for all $x > x_0$,

$$\mathbb{P}\left(Y(X+X')>x\right) \le \overline{F}(x). \tag{3.25}$$

Letting $R_0 = X' |\, X' > x_0$, we claim that

$$Y(X+R_0) \overset{d}{\le} R_0. \tag{3.26}$$

Actually, when $x \le x_0$, it is clear that $\mathbb{P}\left(Y(X+R_0)>x\right) \le 1 = \mathbb{P}\left(R_0>x\right)$. When $x \le x_0$, by (3.25) we have

$$\begin{aligned}\mathbb{P}\left(Y(X+R_0)>x\right) &= \frac{\mathbb{P}\left(Y(X+X')>x, X'>x_0\right)}{\mathbb{P}(X'>x_0)}\\ &\le \frac{\mathbb{P}\left(Y(X+X')>x\right)}{\mathbb{P}(X'>x_0)}\\ &\le \frac{\overline{F}(x)}{\mathbb{P}(X'>x_0)}\\ &= \mathbb{P}(R_0>x).\end{aligned}$$

Hence, relation (3.26) holds for each case. From (3.26) it is clear that

$$R_n \overset{d}{\le} R_{n-1}, \qquad n = 1, 2, \ldots.$$

Therefore,

$$R \overset{d}{\leq} R_n \overset{d}{=} R_0 \prod_{i=1}^{n} Y_i + \sum_{k=1}^{n} X_k \prod_{i=1}^{k} Y_i, \qquad n = 0, 1, 2, \ldots.$$

When $n = 2$, the above random inequality gives that, for all x,

$$\begin{aligned} \mathbb{P}(R > x) &\leq \mathbb{P}(R_0 Y_1 Y_2 + X_1 Y_1 + X_2 Y_1 Y_2 > x) \\ &= \int_0^1 \mathbb{P}\left(R_0 Y_2 + X_1 + X_2 Y_2 > \frac{x}{y}\right) \mathbb{P}(Y_1 \in \mathrm{d}y). \end{aligned} \tag{3.27}$$

By Lemma 2.24(ii),

$$\begin{aligned} \mathbb{P}(R_0 Y_2 + X_2 Y_2 > x) &= \int_0^1 \mathbb{P}\left(R_0 + X_2 > \frac{x}{y}\right) \mathbb{P}(Y_2 \in \mathrm{d}y) \\ &\sim \left(\frac{1}{\overline{F}(x_0)} + 1\right) \int_0^1 \overline{F}\left(\frac{x}{y}\right) \mathbb{P}(Y_2 \in \mathrm{d}y) \\ &= o(1)\overline{F}(x). \end{aligned}$$

Then, applying Lemma 2.24(ii) again to (3.27) yields that

$$\mathbb{P}(R > x) \lesssim \int_0^1 \overline{F}\left(\frac{x}{y}\right) \mathbb{P}(Y_1 \in \mathrm{d}y) = \mathbb{P}(X_1 Y_1 > x). \tag{3.28}$$

A combination of (3.24) and (3.28) gives that

$$\mathbb{P}(R > x) \sim \mathbb{P}(X_1 Y_1 > x). \tag{3.29}$$

It remains to prove that

$$\int_0^\infty \overline{F}(x\mathrm{e}^{rs}) \mathrm{d}\lambda_s \sim \mathbb{P}(X_1 Y_1 > x). \tag{3.30}$$

Since $Y_1 \in (0, 1)$, it is easy to verify that the distribution of $X_1 Y_1$ still belongs to the class $\mathcal{R}_{-\infty}$. By relation (2.19), for arbitrarily chosen $p > 0$ there are $x_0 > 0$ and $c > 0$ such that, uniformly for all $k = 1, 2, \ldots$ and $x \geq x_0$,

$$\frac{1}{\mathbb{P}(X_1 Y_1 > x)} \mathbb{P}\left(X_k \prod_{i=1}^{k} Y_i > x\right) \leq c\mathbb{E}\left(\prod_{i=2}^{k} Y_i\right)^p = c\left(\mathbb{E}Y_1^p\right)^{k-1}.$$

Therefore, by the dominated convergence theorem,

$$\begin{aligned}\lim_{x\to\infty}\frac{\int_0^\infty \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s}{\mathbb{P}\left(X_1Y_1>x\right)} &= \lim_{x\to\infty}\frac{\sum_{k=1}^\infty \mathbb{P}\left(X_k\prod_{i=1}^k Y_i>x\right)}{\mathbb{P}\left(X_1Y_1>x\right)} \\ &= 1+\sum_{k=2}^\infty \lim_{x\to\infty}\frac{\mathbb{P}\left(X_k\prod_{i=1}^k Y_i>x\right)}{\mathbb{P}\left(X_1Y_1>x\right)} \\ &= 1.\end{aligned}$$

From (3.29) and (3.30) we obtain relation (3.21). □

Proof of Theorem 3.5. The proof can be given by copying the proof of Theorem 3.4 with the only modification that we use Lemma 3.15 instead of Lemma 3.14. □

3.3.4 Proof of Theorem 3.6

Konstantinides *et al.* (2002) investigated the asymptotic behavior of the ruin probability of the compound Poisson model. In their model, the surplus process is expressed as

$$S_r\left(t\right)=x\mathrm{e}^{rt}+p\int_0^t \mathrm{e}^{r(t-s)}\mathrm{d}s-\sum_{k=1}^\infty X_k\mathrm{e}^{r(t-\tau_k)}1_{(\tau_k\le t)},\qquad t\ge 0,$$

where $x\ge 0$ is the initial surplus, $p>0$ is the constant premium rate, and $\{X_k,k=1,2,\ldots\}$, $\{\tau_k,k=1,2,\ldots\}$, and r are the same as appearing in relation (3.3). The counting process $(N_t)_{t\ge 0}$ generated by $\{\tau_k,k=1,2,\ldots\}$ is a Poisson process with intensity $\lambda>0$. The ruin probability is defined as

$$\psi_r\left(x\right)=\mathbb{P}\left(\inf_{0<t<\infty}S_r\left(t\right)<0\right).$$

Theorem 2.1 of Konstantinides *et al.* (2002) shows that, if $F_e\in\mathcal{A}$, then

$$\psi_r\left(x\right)\sim\frac{\lambda}{r}\int_x^\infty\frac{\overline{F}(y)}{y}\mathrm{d}y. \tag{3.31}$$

Based on relation (3.31) we produce the following result:

Lemma 3.16. *Consider the discounted aggregate claims described in relation (3.3), in which* $(N_t)_{t\geq 0}$ *is a Poisson process with intensity* $\lambda > 0$. *If* $F_e \in \mathcal{A}$, *then*

$$\mathbb{P}\left(D_r(\infty) > x\right) \sim \frac{\lambda}{r}\int_x^\infty \frac{\overline{F}(y)}{y}\mathrm{d}y. \tag{3.32}$$

Proof. In terms of the model of Konstantinides *et al.* (2002),

$$\psi_r\left(x\right) = \mathbb{P}\left(\sup_{0<t<\infty}\left(D_r\left(t\right) - p\int_0^t \mathrm{e}^{-rs}\mathrm{d}s\right) > x\right).$$

It follows that

$$\psi_r\left(x\right) \leq \mathbb{P}\left(D_r\left(\infty\right) > x\right) \leq \psi_r\left(x - p/r\right). \tag{3.33}$$

By (3.31) and integration by parts,

$$\psi_r\left(x\right) \sim \frac{\mu\lambda}{r}\left(\frac{\overline{F_e}\left(x\right)}{x} - \int_x^\infty \frac{\overline{F_e}\left(y\right)}{y^2}\mathrm{d}y\right) = \frac{\mu\lambda}{r}\left(K_{11}\left(x\right) - K_{12}\left(x\right)\right).$$

Changing x into $x - p/r$ in the above yields that

$$\psi_r\left(x - p/r\right) \sim \frac{\mu\lambda}{r}\left(\frac{\overline{F_e}\left(x - p/r\right)}{x - p/r} - \int_{x-p/r}^\infty \frac{\overline{F_e}\left(y\right)}{y^2}\mathrm{d}y\right) = \frac{\mu\lambda}{r}\left(K_{21}\left(x\right) - K_{22}\left(x\right)\right).$$

Since $F_e \in \mathcal{A} \subset \mathcal{L}$,

$$K_{11}\left(x\right) \sim K_{21}\left(x\right), \qquad K_{12}\left(x\right) \sim K_{22}\left(x\right).$$

In order to infer $\psi_r\left(x\right) \sim \psi_r\left(x - p/r\right)$, it suffices to show that

$$\limsup_{x\to\infty}\frac{K_{12}\left(x\right)}{K_{11}\left(x\right)} < 1. \tag{3.34}$$

Since $F_e \in \mathcal{A}$, there exist some v and ε, $v > 1$ and $0 < \varepsilon < 1$, such that $\overline{F_e}(vx)/\overline{F_e}(x) \leq 1 - \varepsilon$ holds for all large x. Hence, for all large x,

$$\begin{aligned}\frac{K_{12}(x)}{K_{11}(x)} &= \sum_{n=1}^{\infty} \int_{xv^{n-1}}^{xv^n} \frac{\overline{F_e}(y)}{\overline{F_e}(x)} \frac{x}{y^2} \mathrm{d}y \leq \sum_{n=1}^{\infty} \int_{xv^{n-1}}^{xv^n} \frac{\overline{F_e}(xv^{n-1})}{\overline{F_e}(x)} \frac{x}{y^2} \mathrm{d}y \\ &\leq \sum_{n=1}^{\infty} (1-\varepsilon)^{n-1} \int_{xv^{n-1}}^{xv^n} \frac{x}{y^2} \mathrm{d}y = \frac{v-1}{v-1+\varepsilon}.\end{aligned}$$

This proves (3.34). Therefore by (3.31) and (3.33), relation (3.32) follows immediately.

□

Lemma 3.17. *For a distribution F on $[0, \infty)$ with a finite positive expectation, if relation (2.12) with F replaced by F_e holds for some $v > 1$, then*

$$\lim_{t\to\infty} \limsup_{x\to\infty} \frac{\int_t^{\infty} \overline{F}(x\mathrm{e}^{rs})\,\mathrm{d}s}{\int_0^{\infty} \overline{F}(x\mathrm{e}^{rs})\,\mathrm{d}s} = 0. \tag{3.35}$$

Proof. Clearly,

$$\frac{\int_t^{\infty} \overline{F}(x\mathrm{e}^{rs})\,\mathrm{d}s}{\int_0^{\infty} \overline{F}(x\mathrm{e}^{rs})\,\mathrm{d}s} = \frac{-\int_{x\mathrm{e}^{rt}}^{\infty} \frac{1}{y} \mathrm{d}\overline{F_e}(y)}{-\int_x^{\infty} \frac{1}{y} \mathrm{d}\overline{F_e}(y)} = \frac{\frac{\overline{F_e}(x\mathrm{e}^{rt})}{x\mathrm{e}^{rt}} - \int_{x\mathrm{e}^{rt}}^{\infty} \frac{\overline{F_e}(y)}{y^2} \mathrm{d}y}{\frac{\overline{F_e}(x)}{x} - \int_x^{\infty} \frac{\overline{F_e}(y)}{y^2} \mathrm{d}y}.$$

By (3.34), there is some constant $c^* > 0$ such that, uniformly for all $t > 0$,

$$\frac{\int_t^{\infty} \overline{F}(x\mathrm{e}^{rs})\,\mathrm{d}s}{\int_0^{\infty} \overline{F}(x\mathrm{e}^{rs})\,\mathrm{d}s} \leq c^* \frac{\frac{\overline{F_e}(x\mathrm{e}^{rt})}{x\mathrm{e}^{rt}}}{\frac{\overline{F_e}(x)}{x}} \leq c^* \mathrm{e}^{-rt}.$$

Therefore, (3.35) holds. □

Proof of Theorem 3.6. The proof can be given by copying the proof of Theorem 3.4 with the only modification that we use Lemmas 3.16 and 3.17 instead of Lemmas 3.10 and 3.14. □

3.4 Simulation

In this section, we simulate the uniform convergence in (3.5) on a finite time interval $(0, T]$. Set $T = 60$, i.e., the maximum horizon we consider is 60 years. Assume that the constant force of interest is $r = 0.1$, claims arrive according to a Poisson process with intensity $\lambda = 5$, and the claim sizes are i.i.d. with common Pareto distribution F satisfying $\overline{F}(x) = 1/(1+x)^2$ for $x \geq 0$. It is obvious that $F \in \mathcal{S}$ and $F_e \in \mathcal{A}$.

For each x, we find the supremum

$$\sup_{t\in(0,T]} \left| \frac{\mathbb{P}\left(D_r(t) > x\right)}{\lambda \int_0^t \overline{F}(x\mathrm{e}^{rs})\mathrm{d}s} - 1 \right|. \tag{3.36}$$

Then we let x increase to see whether, as predicted by relation (3.5), the above supremum goes to 0 and how fast the convergence is. We execute our simulation in R software. The simulation result is shown in Figure 3.1 below.

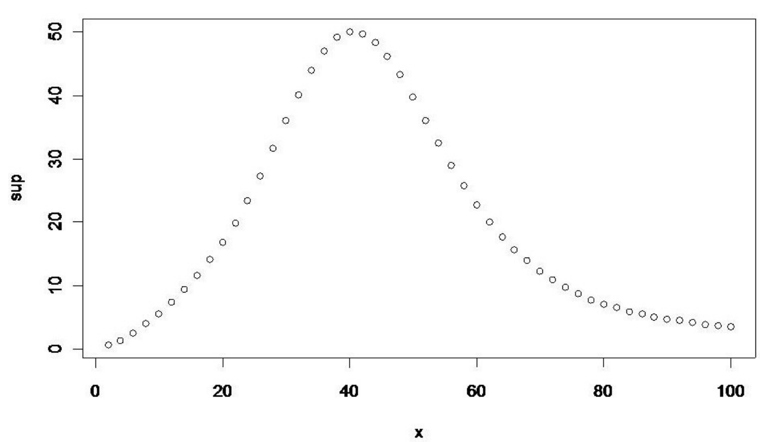

Figure 3.1: Uniform convergence on $(0, T]$ with Pareto F

From Figure 3.1 we see that as x increases the supremum given in (3.36) converges to 0. The convergence speed is reasonable seeing that the expected claim size is 1. Indeed, the supremum is very close to 0 when x comes up to 100.

CHAPTER 4
THE MAXIMUM EXCEEDANCE OVER A RANDOM WALK

In this chapter, we consider a problem in the field of probability. Motivated by the observations that many problems in applied fields, including corporate finance, insurance risk, and production systems, can be reduced to the study of the distribution of the maximum exceedance of a sequence of random variables over a renewal threshold, we derive a unified asymptotic formula for the tail probability of such a maximum exceedance for both light-tailed and heavy-tailed cases. An application of the main result to corporate finance is proposed in Section 4.2. The main result will play an important role in the proofs of two light-tailed cases in Chapter 5. This chapter is based on the joint research paper Hao *et al.* (2009).

4.1 Introduction and Main Result

Let $\{Y_n, n = 1, 2, \ldots\}$ be a sequence of i.i.d. random variables with generic random variable Y, common distribution F on $(-\infty, \infty)$, and $0 < \nu_F = \int_0^\infty \overline{F}(y)\mathrm{d}y < \infty$. For every constant $\mu > 0$, the maximum

$$M_0 = \sup_{n \geq 1} (Y_n - (n-1)\mu)$$

is finite almost surely. If the equilibrium distribution F_e is long tailed, then it is easy to check that

$$\lim_{x \to \infty} \frac{\mathbb{P}(M_0 > x)}{\int_x^\infty \overline{F}(y)\mathrm{d}y} = \frac{1}{\mu}. \tag{4.1}$$

Actually, on the one hand,

$$\mathbb{P}\left(M_0 > x\right) \leq \sum_{n=1}^{\infty} \overline{F}(x + (n-1)\mu) \leq \frac{1}{\mu} \int_{x-\mu}^{\infty} \overline{F}(y)\mathrm{d}y \sim \frac{1}{\mu} \int_{x}^{\infty} \overline{F}(y)\mathrm{d}y.$$

On the other hand,

$$\begin{aligned}
\mathbb{P}\left(M_0 > x\right) &\geq \sum_{n=1}^{\infty} \overline{F}(x + (n-1)\mu) - \sum_{1 \leq n < m < \infty} \overline{F}(x + (n-1)\mu)\overline{F}(x + (m-1)\mu) \\
&\geq \sum_{n=1}^{\infty} \overline{F}(x + (n-1)\mu) - \left(\sum_{n=1}^{\infty} \overline{F}(x + (n-1)\mu) \right)^2 \\
&\geq \frac{1}{\mu} \int_{x}^{\infty} \overline{F}(y)\mathrm{d}y - \left(\frac{1}{\mu} \int_{x-\mu}^{\infty} \overline{F}(y)\mathrm{d}y \right)^2 \\
&\sim \frac{1}{\mu} \int_{x}^{\infty} \overline{F}(y)\mathrm{d}y.
\end{aligned}$$

Hence, relation (4.1) holds.

Motivated by the observation above, in this chapter we study the tail probability of the maximum exceedance of the sequence $\{Y_n, n = 1, 2, \ldots\}$ over a random walk with positive drift. Precisely, let $\{(X_n, Y_n), n = 1, 2, \ldots\}$ be a sequence of i.i.d. random pairs with generic random pair (X, Y). Assume that $\mathbb{E}X = \mu > 0$ and that Y follows a distribution F on $(-\infty, \infty)$. Then, the maximum

$$M = \sup_{n \geq 1} \left(Y_n - S_{n-1}\right), \tag{4.2}$$

with $S_{n-1} = \sum_{i=1}^{n-1} X_i$, is finite almost surely.

For the sake of consistency, for a random variable X with mean $\mu > 0$ we make a convention that

$$\left. \frac{\alpha}{1 - \mathbb{E}\mathrm{e}^{-\alpha X}} \right|_{\alpha=0} = \frac{1}{\mu}.$$

The main result of this chapter is given below:

Theorem 4.1. *Consider the i.i.d. sequence* $\{(X_n, Y_n), n = 1, 2, \ldots\}$ *and the maximum* M *defined in (4.2), where* $\mathbb{E}X = \mu > 0$ *and* Y *is distributed by* F. *Then, the relation*

$$\lim_{x\to\infty} \frac{\mathbb{P}(M > x)}{\int_x^\infty \overline{F}(y)\mathrm{d}y} = \frac{\alpha}{1 - \mathbb{E}\mathrm{e}^{-\alpha X}} \tag{4.3}$$

holds under one of the following groups of conditions:

(i) $F_e \in \mathcal{L}(\alpha)$ *for some* $\alpha \geq 0$, $\mathbb{E}X^2 < \infty$, *and* $\mathbb{E}\mathrm{e}^{-\beta X} < 1$ *for some* $\beta > \alpha$;

(ii) $F_e \in \mathcal{S}(\alpha)$ *for some* $\alpha \geq 0$, $\mathbb{P}(-X > x) = o(\overline{F}(x))$, *and* $\mathbb{E}\mathrm{e}^{-\alpha X} < 1$ *provided* $\alpha > 0$.

Clearly, $\mathbb{E}\mathrm{e}^{-\gamma X}$, as a function of γ, is convex over all γ for which $\mathbb{E}\mathrm{e}^{-\gamma X}$ is finite. Hence for case (i), $\mathbb{E}\mathrm{e}^{-\gamma X} < 1$ for every $\gamma \in (0, \beta]$.

As shown in Lemma 2.22, for every $\alpha \geq 0$, the condition $F_e \in \mathcal{L}(\alpha)$ is equivalent to relation (2.22), i.e.,

$$\lim_{x\to\infty} \frac{\overline{F}(x)}{\int_x^\infty \overline{F}(y)\mathrm{d}y} = \alpha.$$

In particular, the condition $F_e \in \mathcal{L}$ (or, equivalently, relation (2.22) with $\alpha = 0$) is fulfilled by most cited heavy-tailed distributions including all long-tailed or dominatedly-varying-tailed distributions with finite mean; see Lemma 2.20.

It is worth mentioning that Theorem 4.1 allows X and Y to be arbitrarily dependent. However, the asymptotic relation for $\mathbb{P}(M > x)$ completely eliminates impact of the dependence of (X, Y).

Some closely related works are summarized as follows:

(i) Robert (2005) considered a special case of our Theorem 4.1(i) with $\alpha = 0$ and X positive and proposed an application to ruin theory in the presence of dividends paid out at a sequence of random epochs. He considered the renewal risk model for an insurance company. Assume that the insurance company uses a stopping time τ_1 to decide when the surplus $S_{\tau_1} > 0$ is sufficiently large that a part $f(S_{\tau_1})$ is distributed to the shareholders and the other part $S_{\tau_1} - f(S_{\tau_1})$ kept to reinforce the solvency margin. At this time, the surplus reduces to 0. The same rule is then used to define τ_2. In this way, we obtain a sequence of stopping times $\{\tau_n, n = 1, 2, \ldots\}$ and a sequence of dividends $\left\{f\left(S_{\tau_n} - S_{\tau_{n-1}}\right), n = 1, 2, \ldots\right\}$, where $\tau_0 = 0$. If we set $X_n = S_{\tau_n} - S_{\tau_{n-1}} - f\left(S_{\tau_n} - S_{\tau_{n-1}}\right)$ and $Y_n = \sup_{\tau_{n-1} \leq t < \tau_n}\left(S_{\tau_{n-1}} - S_t\right)$, then the ruin probability, i.e., the probability that the surplus process S goes below 0 sometime is $\psi(x) = \mathbb{P}(M > x)$, where M is defined in (4.2). According to Theorem 4.1(i), we can give an explicit asymptotic expression for the ruin probability $\psi(x)$ as

$$\psi(x) \sim \frac{1}{\mathbb{E}X_1}\int_x^\infty \mathbb{P}(Y > y)\,\mathrm{d}y.$$

(ii) Araman and Glynn (2006) systematically studied the same problem in the framework of a perturbed random walk for various cases. Their Theorem 3 corresponds to a special case of our Theorem 4.1(i) with $\alpha > 0$, X, Y independent, and F exponential, but under the condition $\mathbb{E}\mathrm{e}^{-\alpha X} < 1$, which is slightly weaker than our condition $\mathbb{E}\mathrm{e}^{-\beta X} < 1$ for some $\beta > \alpha$. Their Theorem 4, assuming that F has a continuous hazard rate function converging to 0, corresponds to a special case of our Theorem 4.1(i) with $\alpha = 0$ and X, Y independent. Indeed, by L'Hôspital's rule the assumption that F has a continuous hazard rate function converging to 0 implies that $F_e \in \mathcal{L}(0)$.

(iii) Palmowski and Zwart (2007) also studied the same problem but in the framework of a regenerative process. In terms of their model in which the regenerative process $S = (S_t)_{t\geq 0}$ has renewal epochs $0 = T_0 < T_1 < \cdots$, the random variables X_n and Y_n in our theorems correspond to $S_{T_{n-1}} - S_{T_n}$ and $\sup_{T_{n-1}\leq t<T_n} S_t - S_{T_{n-1}}$, respectively. In particular, their Theorem 1 corresponds to our Theorem 4.1(ii) with $\alpha = 0$ under the assumption that the equilibrium distribution of $(-X) \vee Y$ is subexponential and their Theorem 2 corresponds to our Theorem 4.1 with $\alpha > 0$ under the following three assumptions: (1) $F \in \mathcal{L}(\alpha)$, (2) $\mathbb{E}\mathrm{e}^{-\alpha X} < 1$, and (3) $\mathbb{P}(Y - \widetilde{X} > x) \sim \mathbb{P}(Y > x)\mathbb{E}\mathrm{e}^{-\alpha\widetilde{X}}$ with $\widetilde{X}$ identically distributed as X and independent of (X, Y). We need to point out that the assumptions they used in their Theorem 2 are slightly more general than ours.

The rest of this chapter is organized as follows: Section 4.2 proposes an application to corporate finance, Section 4.3 prepares several lemmas, Sections 4.4 and 4.5 respectively prove cases (i) and (ii) of Theorem 4.1, and at last Section 4.6 gives two simulation results.

4.2 Application to Corporate Finance

Consider an incorporated firm whose profit during the nth fiscal year is denoted by Z_n, $n = 1, 2, \ldots$. At the end of each fiscal year, the firm will pay out to shareholders a part of its profit as dividend if it earns money in that year; otherwise, it will issue new equity to raise money. More precisely, introduce two constants Δ and ϵ, $0 < \Delta < 1$ and $\epsilon > 0$, such that the amount ΔZ_n will be paid out if $Z_n > 0$ and the amount $-\epsilon Z_n$ will be raised otherwise. We assume that at the end of each fiscal year the firm liquidates its capital and restarts operation in the very beginning of the next fiscal year. For each real number a, write $a^+ = a \vee 0$ and $a^- = -(a \wedge 0)$. Then, the increment of capital amount during the nth fiscal year after liquidation will be

$$X_n = Z_n - \Delta Z_n^+ + \epsilon Z_n^-, \qquad n = 1, 2, \ldots,$$

and the capital amount up to the end of the nth fiscal year before liquidation will be

$$R_n = x + \sum_{i=1}^{n-1} X_i + Z_n, \qquad n = 1, 2, \ldots,$$

with $R_0 = x > 0$ being the initial capital of the firm.

We are interested in the probability of the so-called bankrupt event. Such an event describes that the financial situation of the firm becomes too bad to survive from budget deficits, or, in other words, the capital surplus of the firm goes below some critical level b. Denote this probability by $\psi(x, b)$, which, when $b = 0$, is called the ruin probability in risk theory. Then,

$$\psi(x, b) = \mathbb{P}\left(\inf_{n \geq 1} R_n < b \,\middle|\, R_0 = x \right).$$

In order to use Theorem 4.1 to derive an asymptotic estimate for $\psi(x, b)$, assume that $\{Z_n, n = 1, 2, \ldots\}$ is a sequence of i.i.d. random variables, implying that the operation of the firm in each year does not depend on its financial situation in the beginning of that year. Let $Y_n = -Z_n$ for $n = 1, 2, \ldots$. Therefore, (X_n, Y_n), $n = 1, 2, \ldots$, constitute a sequence of i.i.d. random pairs, and

$$\psi(x, b) = \mathbb{P}\left\{ \sup_{n \geq 1} \left(Y_n - \sum_{i=1}^{n-1} X_i \right) > x - b \right\}.$$

Denote by Z, X, and Y the generic random variables for Z_n, X_n, and Y_n, respectively, and denote by F the distribution of Y. Then under the conditions of Theorem 4.1, we have

$$\lim_{x-b\to\infty} \frac{\psi(x,b)}{\int_{x-b}^{\infty} \overline{F}(y)\,\mathrm{d}y} = \frac{\alpha}{1-\mathbb{E}\mathrm{e}^{-\alpha X}}. \tag{4.4}$$

To illustrate the conditions of Theorem 4.1(i), assume that Z follows the distribution

$$\mathbb{P}(Z \le z) = \begin{cases} 1 - \frac{\alpha}{\tilde{\alpha}+\alpha}\mathrm{e}^{-\tilde{\alpha} z}, & z \ge 0, \\ \frac{\tilde{\alpha}}{\tilde{\alpha}+\alpha}\mathrm{e}^{\alpha z}, & z < 0, \end{cases} \tag{4.5}$$

for some $\alpha > \tilde{\alpha} > 0$. Recall $X = Z - \Delta Z^+ + \epsilon Z^-$ and $Y = -Z$. Clearly, $F \in \mathcal{L}(\alpha)$ and $0 < \int_0^\infty \overline{F}(y)\mathrm{d}y < \infty$. If $\epsilon \ge 1$, then all conditions of Theorem 4.1 are obviously satisfied. As for $0 < \epsilon < 1$, choose Δ and ϵ satisfying

$$0 < \Delta < 1 \quad \text{and} \quad \frac{\tilde{\alpha}^2 + \tilde{\alpha}\alpha(1-\Delta)}{\tilde{\alpha}^2 + \tilde{\alpha}\alpha(1-\Delta) + \alpha^2(1-\Delta)} < \epsilon < 1. \tag{4.6}$$

Then,

$$\mathbb{E}X = \frac{(1-\Delta)\alpha^2 - (1-\epsilon)\tilde{\alpha}^2}{\tilde{\alpha}\alpha(\tilde{\alpha}+\alpha)} > 0,$$

and for all $\beta \in \left(\alpha, \frac{(1-\Delta)\alpha^2-(1-\epsilon)\tilde{\gamma}^2}{(1-\Delta)(1-\epsilon)(\tilde{\gamma}+\alpha)}\right)$,

$$\mathbb{E}\mathrm{e}^{-\beta X} = \frac{\tilde{\alpha}\alpha}{\tilde{\alpha}+\alpha}\left(\frac{1}{\tilde{\alpha}+\beta(1-\Delta)} + \frac{1}{\alpha-\beta(1-\epsilon)}\right) < 1.$$

Therefor, all conditions of Theorem 4.1(i) are satisfied.

In sum, if Z follows the distribution given in (4.5), then for (Δ, ϵ) belonging to $(0,1) \times [1,\infty)$ or satisfying (4.6), by relation (4.4) we have

$$\lim_{x-b\to\infty} \frac{\psi(x,b)}{\mathrm{e}^{-\alpha(x-b)}} = \frac{\tilde{\alpha}^2\epsilon + \alpha\tilde{\alpha}(1-\Delta)\epsilon}{\alpha^2(1-\Delta)\epsilon - \alpha\tilde{\alpha}(1-\Delta)(1-\epsilon) - \tilde{\alpha}^2(1-\epsilon)}.$$

It is even easier to construct a distribution for the random variable Z such that all conditions of Theorem 4.1(ii) are satisfied.

4.3 Lemmas

Lemma 4.2 (Lemma 3.1 of Robert (2005))**.** *Let* $\{\xi_n, n = 1, 2, \ldots\}$ *be a sequence of i.i.d. random variables with generic random variable* ξ *satisfying* $-\infty < \mathbb{E}\xi < 0$ *and* $\mathbb{P}(\xi > 0) > 0$. *Then,* $\mathbb{E}(\xi^+)^2 < \infty$ *if and only if*

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} \xi_i > 0\right) < \infty.$$

Lemma 4.3. *Let* $\{\xi_n, n = 1, 2, \ldots\}$ *be a sequence of i.i.d. random variables with finite mean* μ. *Then for arbitrarily small* $\varepsilon, \delta > 0$, *there exists some constant* $c > 0$ *such that*

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty}\left(n(\mu-\delta) - c \leq \sum_{i=1}^{n} \xi_i \leq n(\mu+\delta) + c\right)\right) > 1 - \varepsilon. \tag{4.7}$$

Proof. Follow the proof of Lemma 3.1 of Asmussen *et al.* (1999) with some obvious modifications. □

Lemma 4.4 (Theorem 2 of Veraverbeke (1977))**.** *Let* $\{\xi_n, n = 1, 2, \ldots\}$ *be a sequence of i.i.d. random variables with common distribution* F *and finite mean* μ. *Let* $-\alpha$ *be the left abscissa of convergence of* $f(\lambda) = \int_{-\infty}^{\infty} \mathrm{e}^{-\lambda x} F(\mathrm{d}x)$. *Denote by* $W(\cdot)$ *the distribution of* $\sup_{n\geq 0}\sum_{i=1}^{n} \xi_i$.

(i) Suppose $\alpha > 0$. *If* $f(-\alpha) < 1$ *then* $\overline{W}(x) = o(\mathrm{e}^{-\alpha x})$ *and* $F \in \mathcal{S}(\alpha) \iff W \in \mathcal{S}(\alpha)$ *each of which implies that*

$$\lim_{x\to\infty} \frac{\overline{W}(x)}{\overline{F}(x)} = c(\alpha)$$

holds for some constant $c(\alpha) > 0$.

(ii) Suppose $\alpha = 0$. *If* $\mu < 0$, *then* $F_e \in \mathcal{S} \iff W \in \mathcal{S}$ *each of which implies*

$$\lim_{x\to\infty} \frac{\overline{W}(x)}{\int_x^\infty \overline{F}(y)\,\mathrm{d}y} = -\frac{1}{\mu}.$$

Lemma 4.5. *Assume* $F_e \in \mathcal{S}(\alpha)$ *for some* $\alpha \geq 0$. *Let* $\{\xi_n, n = 1, 2, \ldots\}$ *be a sequence of i.i.d. random variables with generic random variable* ξ *satisfying* $-\infty < \mathbb{E}\xi < 0$, $\mathbb{P}(\xi > x) = o(\overline{F}(x))$, *and* $\mathbb{E}\mathrm{e}^{\alpha\xi} < 1$ *provided* $\alpha > 0$. *Then,*

$$\mathbb{P}\left(\sup_{n\geq 0}\sum_{i=1}^{n}\xi_i > x\right) = o(\overline{F_e}(x)).$$

Proof. For arbitrarily fixed N, we have

$$\begin{aligned}\mathbb{P}\left(\sup_{n\geq 0}\sum_{i=1}^{n}\xi_i > x\right) &\leq \mathbb{P}\left(\sup_{0\leq n\leq N}\sum_{i=1}^{n}\xi_i > x\right) + \mathbb{P}\left(\sum_{i=1}^{N}\xi_i + \sup_{n\geq N}\sum_{i=N+1}^{n}\xi_i > x\right)\\ &= I_1(x,N) + I_2(x,N). \end{aligned} \tag{4.8}$$

By Lemma 2.22, $\mathbb{P}(\xi > x) = o(\overline{F}(x)) = o(\overline{F_e}(x))$. Then by Lemma 2.24(ii),

$$I_1(x,N) \leq \mathbb{P}\left(\sum_{i=1}^{N}(\xi_i \vee 0) > x\right) = o(\overline{F_e}(x)). \tag{4.9}$$

To consider $I_2(x,N)$, for arbitrarily small $\varepsilon > 0$, introduce a random variable η satisfying

$$\mathbb{P}(\eta > x) = \mathbb{P}(\xi > x) \vee \varepsilon\overline{F}(x).$$

Clearly, $\mathbb{P}(\eta > x) \sim \varepsilon\overline{F}(x)$. Since $\eta = \eta(\varepsilon)$ converges to ξ in distribution as $\varepsilon \searrow 0$, for all small $\varepsilon > 0$ we have $\mathbb{E}\eta < 0$ and $\mathbb{E}\mathrm{e}^{\alpha\eta} < 1$ provided $\alpha > 0$. Let $\{\eta_n, n = 1, 2, \ldots\}$ be a sequence of i.i.d. copies of η independent of $\{\xi_n, n = 1, 2, \ldots\}$. By Lemma 4.4, it holds for some constant $c(\alpha, \varepsilon) > 0$ that

$$\mathbb{P}\left(\sup_{n\geq N}\sum_{i=N+1}^{n}\eta_i > x\right) = \mathbb{P}\left(\sup_{n\geq 0}\sum_{i=1}^{n}\eta_i > x\right) \sim c(\alpha,\varepsilon)\overline{F_e}(x).$$

When $\alpha > 0$, the expression of $c(\alpha, \varepsilon)$ is rather involved. However, when $\alpha = 0$, we have the transparent expression $c(0, \varepsilon) = -\varepsilon\nu_F / \mathbb{E}\eta$. Then by Lemma 2.24(ii),

$$I_2(x, N) \le \mathbb{P}\left(\sum_{i=1}^{N} \xi_i + \sup_{n \ge N} \sum_{i=N+1}^{n} \eta_i > x\right) \sim \left(\mathbb{E}\mathrm{e}^{\alpha\xi}\right)^N c(\alpha, \varepsilon)\overline{F_e}(x). \tag{4.10}$$

Plugging (4.9) and (4.10) into (4.8) yields that

$$\limsup_{x\to\infty} \frac{1}{\overline{F_e}(x)} \mathbb{P}\left(\sup_{n\ge 0} \sum_{i=1}^{n} \xi_i > x\right) \le \left(\mathbb{E}\mathrm{e}^{\alpha\xi}\right)^N c(\alpha, \varepsilon).$$

If $\alpha > 0$ with ε fixed we let $N \to \infty$, while if $\alpha = 0$ we let $\varepsilon \searrow 0$. Thus, in any case, the right-hand side of the above goes to 0 and the proof is complete. □

4.4 Proof of Theorem 4.1(i)

4.4.1 Preliminary Results

Proposition 4.6. *Under the conditions of Theorem 4.1(i), it holds for arbitrarily small $\varepsilon > 0$, all $0 < \delta < 1$, and all large k that*

$$\sum_{n=k+1}^{\infty} \mathbb{P}\left(Y_n - S_{n-1} > x, S_{n-1} < (n-1)\,\mu\,(1-\delta)\right) \lesssim \varepsilon \int_x^{\infty} \overline{F}(y)\mathrm{d}y. \tag{4.11}$$

Proof. Let $0 < \delta < 1$ and $D > 0$ be arbitrarily fixed. For all $x > D$, according to the range of S_{n-1} we split the left-hand side of (4.11) into three parts as

$$\begin{aligned} &\sum_{n=k+1}^{\infty} \{\mathbb{P}\left(Y_n - S_{n-1} > x, S_{n-1} \in (0, (n-1)\,\mu\,(1-\delta))\right) \\ &\quad + \mathbb{P}\left(Y_n - S_{n-1} > x, S_{n-1} \in (-x + D, 0]\right) \\ &\quad + \mathbb{P}\left(Y_n - S_{n-1} > x, S_{n-1} \in (-\infty, -x + D]\right)\} \\ &= J_1(x, k, \delta) + J_2(x, k, D) + J_3(x, k, D). \end{aligned} \tag{4.12}$$

Using Lemmas 2.22 and 4.2, for arbitrarily small $\varepsilon > 0$ and all large k,

$$J_1(x,k,\delta) \le \overline{F}(x) \sum_{n=k+1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n-1} (\mu(1-\delta) - X_i) > 0 \right) \lesssim \frac{\varepsilon}{2} \int_x^{\infty} \overline{F}(y)\mathrm{d}y. \qquad (4.13)$$

Furthermore, by Lemma 2.23(i), there exist some constants $c_0, D > 0$ such that for all $x \ge x + y - 1 \ge D - 1$ and all large k,

$$\begin{aligned} J_2(x,k,D) &= \sum_{n=k+1}^{\infty} \int_{-x+D}^{0} \overline{F}(x+y)\mathbb{P}(S_{n-1} \in \mathrm{d}y) \\ &\le \sum_{n=k+1}^{\infty} \int_{-x+D}^{0} \left(\int_{x+y-1}^{x+y} \overline{F}(u)\mathrm{d}u \right) \mathbb{P}(S_{n-1} \in \mathrm{d}y) \\ &\le \int_x^{\infty} \overline{F}(y)\mathrm{d}y \sum_{n=k+1}^{\infty} \int_{-x+D}^{0} \frac{\overline{F_e}(x+y-1)}{\overline{F_e}(x)} \mathbb{P}(S_{n-1} \in \mathrm{d}y) \\ &\le c_0 \int_x^{\infty} \overline{F}(y)\mathrm{d}y \sum_{n=k+1}^{\infty} \mathbb{E}\mathrm{e}^{-\beta(S_{n-1}-1)} \\ &\le \frac{\varepsilon}{2} \int_x^{\infty} \overline{F}(y)\mathrm{d}y. \end{aligned} \qquad (4.14)$$

For D specified in (4.14) and for all k, employ Markov's inequality and Lemma 2.23(ii) to obtain that

$$J_3(x,k,D) \le \sum_{n=k+1}^{\infty} \mathbb{P}(S_{n-1} \le -x + D) \le \sum_{n=k+1}^{\infty} \frac{\mathbb{E}\mathrm{e}^{-\beta S_{n-1}}}{\mathrm{e}^{\beta(x-D)}} = o\left(\overline{F_e}(x)\right). \qquad (4.15)$$

Plugging (4.13)–(4.15) into (4.12) yields (4.11). □

Proposition 4.7. *Under the conditions of Theorem 4.1(i), it holds for each* $k = 2, 3, \dots$ *that*

$$\sum_{1 \le n < m \le k} \mathbb{P}(Y_n - S_{n-1} > x, Y_m - S_{m-1} > x) = o\left(\overline{F_e}(x)\right). \qquad (4.16)$$

Proof. Let $\delta > 0$ be a constant satisfying $\beta(1-\delta) > \alpha$. For $1 = n < m \le k$,

$$\begin{aligned}
&\mathbb{P}(Y_1 > x, Y_m - S_{m-1} > x) \\
&\le \mathbb{P}(-S_{m-1} > (1-\delta)x) + \mathbb{P}(Y_1 > x, Y_m - S_{m-1} > x, -S_{m-1} \le (1-\delta)x) \\
&\le \mathrm{e}^{-\beta(1-\delta)x}\mathbb{E}\mathrm{e}^{-\beta S_{m-1}} + \mathbb{P}(Y_1 > x, Y_m > \delta x) \\
&= o\left(\overline{F_e}(x)\right), \qquad (4.17)
\end{aligned}$$

where we used Markov's inequality and Lemmas 2.22 and 2.23(ii). Similarly, for $1 < n < m \le k$,

$$\begin{aligned}
&\mathbb{P}(Y_n - S_{n-1} > x, Y_m - S_{m-1} > x) \\
&\le \mathbb{P}(-S_{n-1} > (1-\delta)x) + \mathbb{P}(Y_n - S_{n-1} > x, Y_m - S_{m-1} > x, -S_{n-1} \le (1-\delta)x) \\
&\le \mathrm{e}^{-\beta(1-\delta)x}\mathbb{E}\mathrm{e}^{-\beta S_{n-1}} \\
&\quad + \int_{-\infty}^{(1-\delta)x} \mathbb{P}(Y_n > x-y, Y_m - S_{n,m-1} > x-y)\,\mathbb{P}(-S_{n-1} \in \mathrm{d}y),
\end{aligned}$$

where $S_{n,m-1} = \sum_{i=n}^{m-1} X_i$. By (4.17), it holds uniformly for all $y \le (1-\delta)x$ that

$$\mathbb{P}(Y_n > x-y, Y_m - S_{n,m-1} > x-y) = o(1)\,\overline{F_e}(x-y).$$

Hence by Lemmas 2.23(ii) and 2.24(i),

$$\begin{aligned}
&\mathbb{P}(Y_n - S_{n-1} > x, Y_m - S_{m-1} > x) \\
&= o\left(\overline{F_e}(x)\right) + o(1)\int_{-\infty}^{(1-\delta)x} \overline{F_e}(x-y)\mathbb{P}(-S_{n-1} \in \mathrm{d}y) \\
&= o\left(\overline{F_e}(x)\right). \qquad (4.18)
\end{aligned}$$

A combination of (4.17) and (4.18) gives (4.16). □

4.4.2 Proof of Theorem 4.1(i) for $\alpha > 0$

We first prove the asymptotic upper bound. For some $0 < \delta < 1$ and each $k = 1, 2, \ldots,$

$$\begin{aligned}
\mathbb{P}(M > x) &\leq \left(\sum_{n=1}^{k} + \sum_{n=k+1}^{\infty} \right) \mathbb{P}(Y_n - S_{n-1} > x) \\
&\leq \sum_{n=1}^{k} \mathbb{P}(Y_n - S_{n-1} > x) + \sum_{n=k+1}^{\infty} \overline{F}(x + (n-1)\mu(1-\delta)) \\
&\quad + \sum_{n=k+1}^{\infty} \mathbb{P}(Y_n - S_{n-1} > x, S_{n-1} < (n-1)\mu(1-\delta)) \\
&= K_1(x,k) + K_2(x,k,\delta) + K_3(x,k,\delta).
\end{aligned} \tag{4.19}$$

By Proposition 4.6, it holds for arbitrarily small $\varepsilon > 0$ and all large k that

$$K_3(x,k,\delta) \lesssim \frac{\varepsilon}{2} \int_x^{\infty} \overline{F}(y)\mathrm{d}y. \tag{4.20}$$

Since $F_e \in \mathcal{L}(\alpha)$, it holds for all large k that

$$K_2(x,k,\delta) \leq \frac{1}{\mu(1-\delta)} \int_{x+(k-1)\mu(1-\delta)}^{\infty} \overline{F}(y)\mathrm{d}y \lesssim \frac{\varepsilon}{2} \int_x^{\infty} \overline{F}(y)\mathrm{d}y. \tag{4.21}$$

With k specified in (4.20) and (4.21), by Lemma 2.24(i) and relation (2.22) we have

$$K_1(x,k) \sim \overline{F}(x) \sum_{n=1}^{k} \mathbb{E}\mathrm{e}^{-\alpha S_{n-1}} \lesssim \frac{\alpha}{1 - \mathbb{E}\mathrm{e}^{-\alpha X}} \int_x^{\infty} \overline{F}(y)\mathrm{d}y. \tag{4.22}$$

Plugging (4.20)–(4.22) into (4.19) and using the arbitrariness of $\varepsilon > 0$, we obtain that

$$\limsup_{x\to\infty} \frac{\mathbb{P}(M > x)}{\int_x^{\infty} \overline{F}(u)\mathrm{d}u} \leq \frac{\alpha}{1 - \mathbb{E}\mathrm{e}^{-\alpha X}}.$$

Next, we turn to prove the asymptotic lower bound. Obviously, for each $k = 1, 2, \ldots,$ using Bonferroni's inequality,

$$\begin{aligned}
\mathbb{P}(M > x) &\geq \mathbb{P}\left(\bigcup_{n=1}^{k} (Y_n - S_{n-1} > x) \right) \\
&\geq K_1(x,k) - \sum_{1 \leq n < m \leq k} \mathbb{P}(Y_n - S_{n-1} > x, Y_m - S_{m-1} > x),
\end{aligned} \tag{4.23}$$

where $K_1(x,k)$ is the same as in (4.19). Similar to (4.22), for arbitrarily small $\varepsilon > 0$ and all large k,

$$K_1(x,k) \gtrsim (1-\varepsilon)\frac{\alpha}{1-\mathbb{E}\mathrm{e}^{-\alpha X}}\int_x^\infty \overline{F}(u)\mathrm{d}u. \tag{4.24}$$

By Proposition 4.7, relation (4.16) holds. Plugging (4.24) and (4.16) into (4.23) and using the arbitrariness of $\varepsilon > 0$, we have

$$\liminf_{x\to\infty}\frac{\mathbb{P}(M>x)}{\int_x^\infty \overline{F}(y)\mathrm{d}y} \geq \frac{\alpha}{1-\mathbb{E}\mathrm{e}^{-\alpha X}}.$$

4.4.3 Proof of Theorem 4.1(i) for $\alpha = 0$

For $\alpha = 0$, relation (4.3) becomes

$$\lim_{x\to\infty}\frac{\mathbb{P}(M>x)}{\int_x^\infty \overline{F}(y)\mathrm{d}y} = \frac{1}{\mu}. \tag{4.25}$$

To derive the asymptotic upper bound, we still use (4.19). By Proposition 4.6, relation (4.20) holds for arbitrarily small $\varepsilon, \delta > 0$ and all large k. With k specified in (4.20), by $F_e \in \mathcal{L}$ we have

$$K_2(x,k,\delta) \leq \frac{1}{\mu(1-\delta)}\int_{x+(k-1)\mu(1-\delta)}^\infty \overline{F}(u)\mathrm{d}u \sim \frac{1}{\mu(1-\delta)}\int_x^\infty \overline{F}(y)\mathrm{d}y, \tag{4.26}$$

while by Lemmas 2.24(i) and 2.22,

$$K_1(x,k) \sim k\overline{F}(x) = o(1)\int_x^\infty \overline{F}(y)\mathrm{d}y. \tag{4.27}$$

Plugging (4.20), (4.26), and (4.27) into (4.19) and using the arbitrariness of $\varepsilon, \delta > 0$, we have

$$\limsup_{x\to\infty}\frac{\mathbb{P}(M>x)}{\int_x^\infty \overline{F}(y)\mathrm{d}y} \leq \frac{1}{\mu}.$$

Next, we consider the asymptotic lower bound. For arbitrarily small $\varepsilon, \delta > 0$, by Lemma 4.3 there exists some constant $C > 0$ such that inequality (4.7) holds. Write $E_n = \{n(\mu - \delta) - C \leq S_n \leq n(\mu + \delta) + C\}$ for $n = 0, 1, \ldots$. Then by Bonferroni's inequality again,

$$\begin{aligned}
\mathbb{P}(M > x) &\geq \mathbb{P}\left(\bigcup_{n=1}^{\infty}\left((Y_n - S_{n-1} > x) \bigcap E_{n-1}\right)\right) \\
&\geq \sum_{n=1}^{\infty} \mathbb{P}\left((Y_n - S_{n-1} > x) \bigcap E_{n-1}\right) \\
&\quad - \sum_{1 \leq n < m < \infty} \mathbb{P}\left((Y_n - S_{n-1} > x) \bigcap (Y_m - S_{m-1} > x) \bigcap E_{n-1} \bigcap E_{m-1}\right) \\
&\geq (1 - \varepsilon) \sum_{n=1}^{\infty} \overline{F}(x + (n-1)(\mu + \delta) + C) \\
&\quad - \sum_{1 \leq n < m < \infty} \overline{F}(x + (n-1)(\mu - \delta) - C)\, \overline{F}(x + (m-1)(\mu - \delta) - C) \\
&\geq \frac{1 - \varepsilon}{\mu + \delta} \int_{x+C}^{\infty} \overline{F}(y)\mathrm{d}y - \left(\frac{1}{\mu - \delta} \int_{x-(\mu-\delta)-C}^{\infty} \overline{F}(y)\mathrm{d}y\right)^2.
\end{aligned}$$

Since $F_e \in \mathcal{L}$, by the arbitrariness of $\varepsilon, \delta > 0$ it follows that

$$\liminf_{x \to \infty} \frac{\mathbb{P}(M > x)}{\int_x^{\infty} \overline{F}(y)\mathrm{d}y} \geq \frac{1}{\mu}.$$

4.5 Proof of Theorem 4.1(ii)

4.5.1 Preliminary Results

We establish the counterparts of Propositions 4.6 and 4.7 for the case $\alpha > 0$, respectively.

Proposition 4.8. *Under the conditions of Theorem 4.1(ii) for the case $\alpha > 0$, relation (4.11) holds for arbitrarily small $\varepsilon > 0$, $0 < \delta < 1$ arbitrarily close to 1, and all large k.*

Proof. For $0 < \delta, d < 1$, introduce the maximum $M_\delta = \sup_{n\geq 1}\sum_{i=1}^{n-1}(\mu(1-d\delta)-X_i)$, which is finite almost surely. For every $n \geq k+1$, we derive

$$S_{n-1} = (n-1)\mu(1-d\delta) - \sum_{i=1}^{n-1}(\mu(1-d\delta)-X_i) \geq (n-1)\mu(1-d\delta) - M_\delta.$$

Therefore, for every $k \geq 1$,

$$\begin{aligned}
&\sum_{n=k+1}^{\infty}\mathbb{P}(Y_n - S_{n-1} > x, S_{n-1} < (n-1)\mu(1-\delta)) \\
&\leq \sum_{n=k+1}^{\infty}\mathbb{P}(Y_n - (n-1)\mu(1-d\delta) + M_\delta > x, M_\delta > k\mu(1-d)\delta) \\
&= \int_{k\mu(1-d)\delta}^{\infty}\sum_{n=k+1}^{\infty}\overline{F}(x-y+(n-1)\mu(1-d\delta))\,\mathbb{P}(M_\delta \in \mathrm{d}y) \\
&\leq \frac{1}{\mu(1-d\delta)}\left(\int_{k\mu(1-d)\delta}^{x} + \int_{x}^{\infty}\right)\left(\int_{x-y}^{\infty}\overline{F}(u)\mathrm{d}u\right)\mathbb{P}(M_\delta \in \mathrm{d}y) \\
&\leq \frac{1}{\mu(1-d\delta)}\left(\nu_F\int_{k\mu(1-d)\delta}^{x}\overline{F_e}(x-y) + \int_{x}^{\infty}(y-x+\nu_F)\right)\mathbb{P}(M_\delta \in \mathrm{d}y). \quad (4.28)
\end{aligned}$$

To apply Lemma 4.5, we need to choose δ and d close to 1 such that $\mathbb{E}\mathrm{e}^{\alpha(\mu(1-d\delta)-X)} < 1$. Let F^* be a distribution defined as $F^*(x) = F(x-\mu(1-d\delta))$. Then, $\mathbb{P}(\mu(1-d\delta) - X > x) = o(\overline{F^*}(x))$ and $F_e^* \in \mathcal{S}(\alpha)$. By Lemma 4.5, we have

$$\mathbb{P}(M_\delta > x) = o(\overline{F_e^*}(x)) = o(\overline{F_e}(x)). \quad (4.29)$$

By Lemma 2.24(ii) and the local uniformity of the convergence in relation (2.14), it holds for arbitrarily fixed $k \geq 1$ that

$$\begin{aligned}
\int_{k\mu(1-d)\delta}^{x}\overline{F_e}(x-y)\mathbb{P}(M_\delta \in \mathrm{d}y) &\leq \left(\int_{0-}^{\infty} - \int_{0-}^{k\mu(1-d)\delta}\right)\overline{F_e}(x-y)\mathbb{P}(M_\delta \in \mathrm{d}y) \\
&\sim \overline{F_e}(x)\mathbb{E}\mathrm{e}^{\alpha M_\delta}1_{\{M_\delta > k\mu(1-d)\delta\}}. \quad (4.30)
\end{aligned}$$

Moreover, by Lemma 2.22,

$$\int_x^\infty (y-x)\,\mathbb{P}\left(M_\delta \in \mathrm{d}y\right) = \int_x^\infty \mathbb{P}\left(M_\delta > y\right)\mathrm{d}y = o(1)\int_x^\infty \overline{F_e}(y)\mathrm{d}y = o(\overline{F_e}(x)). \tag{4.31}$$

Plugging (4.29)-(4.31) into (4.28) yields the desired assertion. □

Proposition 4.9. *Under the conditions of Theorem 4.1(ii) for the case $\alpha > 0$, relation (4.16) holds for each $k = 2, 3, \ldots$.*

Proof. By Lemma 2.22, $F_e \in \mathcal{S}(\alpha)$ for some $\alpha > 0$ implies $F \in \mathcal{S}(\alpha)$. When $1 = n < m \leq k$, for arbitrarily fixed $D > 0$, we have

$$\begin{aligned} &\mathbb{P}\left(Y_1 > x, Y_m - S_{m-1} > x\right) \\ &\leq \mathbb{P}\left(-S_{m-1} > x - D\right) + \mathbb{P}\left(Y_1 > x, Y_m - S_{m-1} > x, -S_{m-1} \leq x - D\right) \\ &\leq \mathbb{P}\left(-S_{m-1} > x - D\right) + \mathbb{P}\left(Y_1 > x, Y_m > D\right). \end{aligned} \tag{4.32}$$

By Lemma 2.24(ii),

$$\mathbb{P}\left(-S_{m-1} > x - D\right) = o\left(\overline{F}(x-D)\right) = o\left(\overline{F}(x)\right).$$

Substitute this into (4.32) then notice that D can be arbitrarily large. It follows that

$$\mathbb{P}\left(Y_1 > x, Y_m - S_{m-1} > x\right) = o(\overline{F}(x)) = o(\overline{F_e}(x)). \tag{4.33}$$

Similarly, when $1 < n < m \leq k$, for arbitrarily fixed $D > 0$,

$$\begin{aligned} &\mathbb{P}\left(Y_n - S_{n-1} > x, Y_m - S_{m-1} > x\right) \\ &\leq \mathbb{P}\left(-S_{n-1} > x - D\right) + \int_{-\infty}^{x-D} \mathbb{P}\left(Y_n > x - y, Y_m - S_{n,m-1} > x - y\right)\mathbb{P}(-S_{n-1} \in \mathrm{d}y), \end{aligned}$$

where $S_{n,m-1} = \sum_{i=n}^{m-1} X_i$ as before. By (4.33), for arbitrarily small $\varepsilon > 0$, choose $D > 0$ such that

$$\mathbb{P}\left(Y_n > x, Y_m - S_{n,m-1} > x\right) \leq \varepsilon \overline{F_e}(x)$$

for $1 < n < m \leq k$ and all $x \geq D$. Using this inequality and Lemma 2.24(ii), we obtain that

$$\begin{aligned}\mathbb{P}\left(Y_n - S_{n-1} > x, Y_m - S_{m-1} > x\right) &\leq o(\overline{F_e}(x)) + \varepsilon \int_{-\infty}^{x-D} \overline{F_e}(x-y) \mathbb{P}(-S_{n-1} \in \mathrm{d}y) \\ &\lesssim \varepsilon \mathbb{E} \mathrm{e}^{-\alpha S_{n-1}} \overline{F_e}(x).\end{aligned}$$

This proves that

$$\mathbb{P}\left(Y_n - S_{n-1} > x, Y_m - S_{m-1} > x\right) = o(\overline{F_e}(x)). \tag{4.34}$$

A combination of (4.33) and (4.34) gives (4.16). □

4.5.2 Proof of Theorem 4.1(ii)

The proof for the case $\alpha > 0$ can be given by copying the proof of Theorem 4.1(i) for the case $\alpha > 0$ with only modifications that we use Lemma 2.24(ii) and Propositions 4.8 and 4.9 instead of Lemma 2.24(i) and Propositions 4.6 and 4.7.

Consider the case $\alpha = 0$ and we aim at relation (4.25). The proof of the asymptotic lower bound is the same as that in Theorem 4.1(i). The proof of the asymptotic upper bound can be found in Palmowski and Zwart (2007). Nevertheless, for the sake of self-containedness, we copy their proof here.

For some arbitrarily large but fixed $\zeta > 0$, define

$$Z = (-X) 1_{\{(-X)\vee Y \leq \zeta\}} + ((-X) \vee Y)\, 1_{\{(-X)\vee Y > \zeta\}}.$$

Clearly, $Z = Z(\zeta)$ converges to $-X$ in distribution as $\zeta \to \infty$ and $\mathbb{E}Z < 0$ for all large ζ. Moreover, it is easy to see that the relation $\mathbb{P}(Z > x) \sim \overline{F}(x)$ holds for arbitrarily fixed ζ. Define Z_n in a similar way in terms of X_n and Y_n, $n = 1, 2, \ldots$, so that $\{Z_n, n = 1, 2, \ldots\}$ forms a sequence of i.i.d. copies of Z. Then, we arrive at a key inequality of Palmowski and Zwart (2007) that

$$M = \sup_{n \geq 1} (Y_n - S_{n-1}) \leq \sup_{n \geq 1} \sum_{i=1}^{n-1} Z_i + \zeta.$$

Therefore, by Lemma 4.4(ii),

$$\mathbb{P}(M > x) \leq \mathbb{P}\left(\sup_{n \geq 1} \sum_{i=1}^{n-1} Z_i > x - \zeta \right) \sim -\frac{1}{\mathbb{E}Z} \int_{x-\zeta}^{\infty} \overline{F}(y) \mathrm{d}y.$$

Since $F_e \in \mathcal{S}$ and ζ can be arbitrarily large, it follows that

$$\mathbb{P}(M > x) \lesssim \frac{1}{\mu} \int_{x}^{\infty} \overline{F}(y) \mathrm{d}y.$$

4.6 Simulations

In this section, we present two simulation results for Theorem 4.1, one for light-tailed F and the other for heavy-tailed F. We want to see from the two special cases how fast the convergence in relation (4.3) is.

(i) Light-tailed Case:

Let (X, Y) have the following marginal density functions

$$f_X(x) = \mathrm{e}^{4x} 1_{\{x<0\}} + \frac{3}{2} \mathrm{e}^{-2x} 1_{\{x \geq 0\}}, \qquad f_Y(y) = \mathrm{e}^{-x} 1_{\{x \geq 0\}}.$$

Then, $F_Y \in \mathcal{L}(1)$, $\mathbb{E}X = 5/16 > 0$, $\mathbb{E}X^2 < \infty$, and $\mathbb{E}\mathrm{e}^{-\beta X} < 1$ for all $\beta \in (1, 5/2)$. Hence, all conditions of Theorem 4.1(i) are satisfied. To model the dependence structure between X and Y, we apply the so-called Farlie-Gumbel-Morgenstern copula, i.e., $C(u_1, u_2) = u_1 u_2(1 + c(1 - u_1)(1 - u_2))$ with $c \in [-1, 1]$ and $u_1, u_2 \in [0, 1]$. In other words,

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)\left(1 + c\overline{F_X}(x)\overline{F_Y}(y)\right),$$

where $F_{X,Y}(\cdot)$, $F_X(\cdot)$, and $F_Y(\cdot)$ are the corresponding joint and marginal distributions. We use the following algorithm (see Johnson (1986)) to generate random values for $F_X(\cdot)$ and $F_Y(\cdot)$:

1. Generate two independent uniform $(0, 1)$ variates v_1, v_2;
2. Set $a = 1 + c(1 - 2v_1)$, $b = \sqrt{a^2 - 4(a - 1)v_2}$;
3. Set $u_1 = v_1$, $u_2 = 2v_2/(a + b)$;
4. Then (u_1, u_2) is one outcome of $(F_X(\cdot), F_Y(\cdot))$.

We set $c = 1/2$ and execute our simulation in R software. The simulation result is shown in Figure 4.1 below.

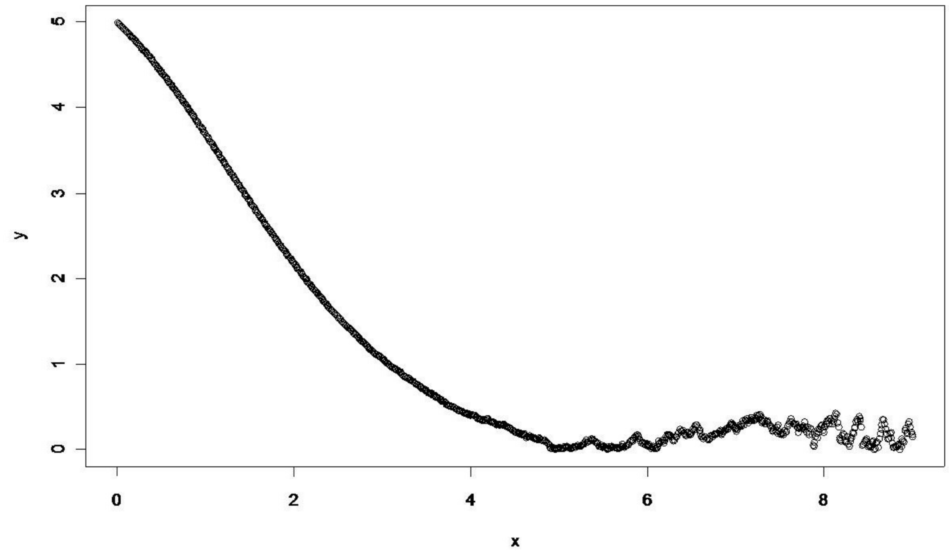

Figure 4.1: y-axis represents for each fixed x the LHS substracts the RHS in (4.3) when Y is Exp(1) distributed.

(ii) Heavy-tailed Case:

Let (X, Y) have the following marginal distributions

$$F_X(x) = \frac{1}{2}\mathrm{e}^x 1_{\{x<0\}} + \left(1 - \frac{1}{2}\mathrm{e}^{-x/2}\right) 1_{\{x\geq 0\}}, \qquad F_Y(y) = \left(1 - \mathrm{e}^{-\sqrt{y}}\right) 1_{\{x\geq 0\}}.$$

Here Y follows a heavy-tailed Weibull distribution with shape parameter $\tau = 1/2$. It is easy to see that the equilibrium distribution of F_Y is subexponential (see Example 1.4.7 of Embrechts *et al.* (1997)), $\mathbb{E}X = 1/2 > 0$, and $\mathbb{P}(-X > x) = o\left(\overline{F_Y}(x)\right)$. Hence, all conditions of Theorem 4.1(ii) are satisfied. We use the same Farlie-Gumbel-Morgenstern copula as above to model the dependence structure between X and Y. The simulation result is shown in Figure 4.2 below.

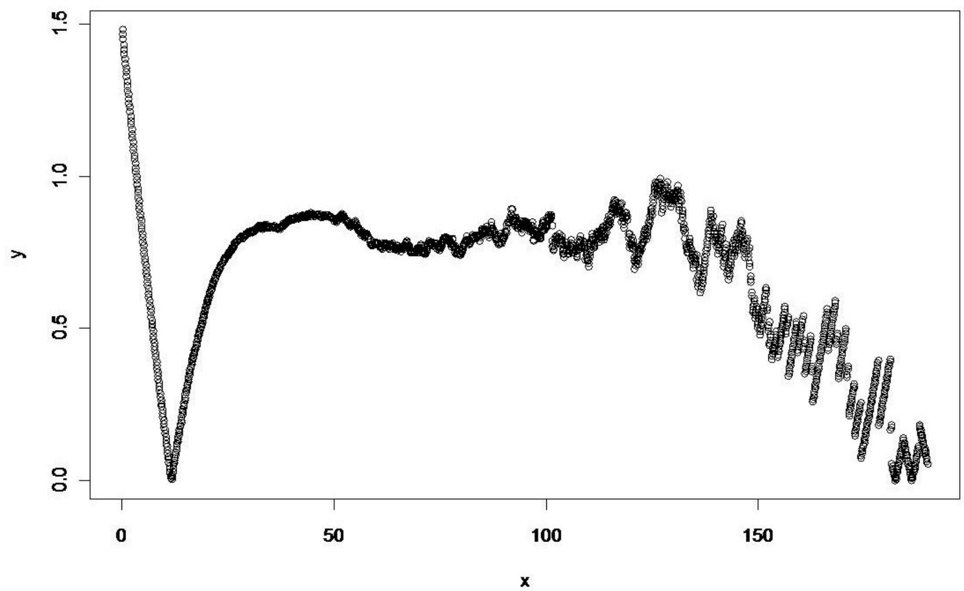

Figure 4.2: y-axis represents for each fixed x the LHS substracts the RHS in (4.3) when Y is heavy-tailed Weibull distributed.

CHAPTER 5
THE LÉVY INSURANCE RISK MODEL UNDER TAXATION

In this chapter, we use a general Lévy process to model the underlying surplus process of an insurance company in a world without economic factors. This so-called Lévy risk model has recently attracted a lot of attention in the insurance literature. We are particularly interested in how to capture the impact of tax payments on the ruin probability. In a series of recent papers by Albrecher and his coauthors, it is assumed that taxes are paid at a certain fixed rate immediately when the surplus of the company is at a running maximum. In reality, however, taxes are usually paid periodically (e.g. monthly, semi-annually, or annually). Therefore, we introduce periodic taxation under which the company pays tax at a fixed rate on its net income during each period. As main results, we derive for the ruin probability several explicit asymptotic relations, in which the prefactor varies with the tax rate, reflecting the impact of tax payments. This chapter is based on the joint research paper Hao and Tang (2009).

5.1 Introduction

As mentioned in Chapter 4, the ruin probability of an insurance company is the probability that its surplus process falls below 0 at some time. Let $U = (U_t)_{t\geq 0}$ be a stochastic process, with $U_0 = x > 0$, representing the underlying surplus process in a world without economic factors (tax, reinsurance, investment, etc.) of an insurance company. Assuming that U is a compound Poisson process with positive drift and that taxes are paid at a fixed rate $\gamma \in [0,1)$ whenever U is at a running maximum (called the loss-carry-forward taxation), Albrecher and Hipp (2007) and Albrecher *et al.* (2009) proved the following strikingly simple relationship between $\psi_\gamma(x)$ and $\psi_0(x)$, the ruin probabilities with and without tax:

$$\psi_\gamma(x) = 1 - (1 - \psi_0(x))^{1/(1-\gamma)} . \tag{5.1}$$

Albrecher *et al.* (2008b) further showed that the tax identity (5.1) still holds for a spectrally negative Lévy surplus process U under the loss-carry-forward taxation. Also, Albrecher *et al.* (2008a) proved a similar tax identity for a dual surplus process U with general inter-innovation times and exponential innovation sizes under the same type of taxation.

All these papers cited above assume the loss-carry-forward taxation. In reality, however, taxes are usually paid periodically (e.g. monthly, semi-annually, or annually). Furthermore, if the surplus process contains a diffusion part, then the moments of running maxima do not form any continuous time interval. In this case, the loss-carry-forward type taxation is rather unrealistic, as was also commented by Albrecher and Hipp (2007). Figure 5.1 below shows how the loss-carry-forward taxation affects the underlying surplus process in the compound Poisson model.

In this chapter, we introduce periodic taxation as well as reinsurance to the risk model. Precisely, we assume that at each discrete moment $n = 1, 2, \ldots$, the company, given that it survives at that moment, pays tax at rate $\gamma \in [0,1)$ on its net income during the period $(n-1, n]$ and it gets paid by reinsurance at rate $\delta \in [0,1)$ on its net loss during the period $(n-1, n]$. We are interested in the influence of such

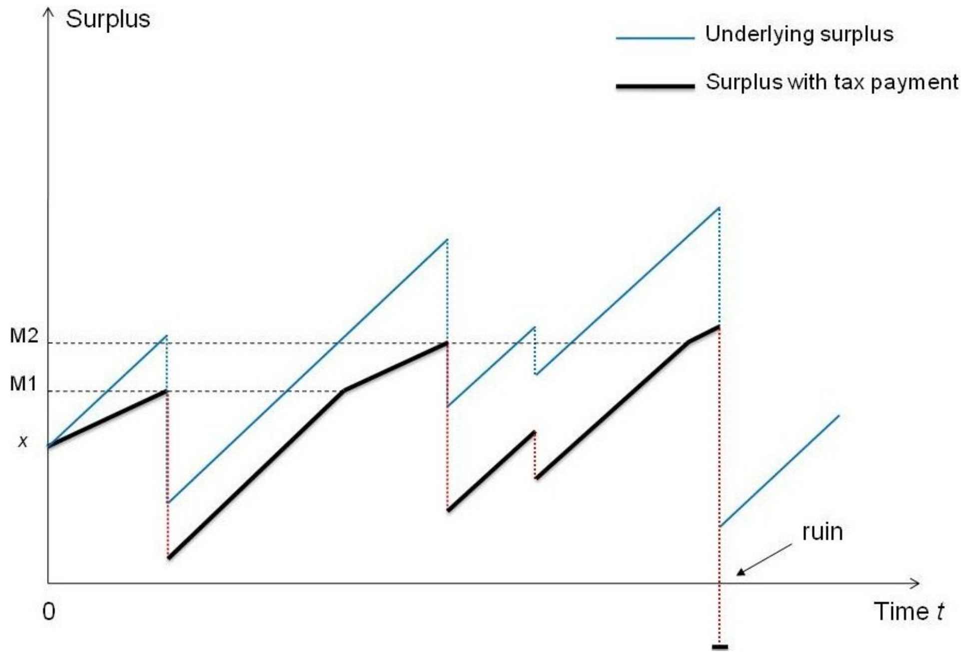

Figure 5.1: Loss-carry-forward taxation

a scheme of taxation rule and reinsurance policy on the asymptotic behavior of the ruin probability. Figure 5.2 below shows how the periodic taxation affects the same underlying surplus process as that in Figure 5.1.

Let us briefly compare these two types of taxation. Under the loss-carry-forward taxation, as long as the surplus does not hit its historical peak, the insurance company can legally evade any tax payment possibly for a long time, even if it makes profits every single period during that time. While under the periodic taxation, the insurance company has to pay tax whenever it survives and its net income is positive in that period. Hence, the latter imposes a more strict taxation rule and produces more significant impact on the ruin probability than the former does. This will be demonstrated in Section 5.2.

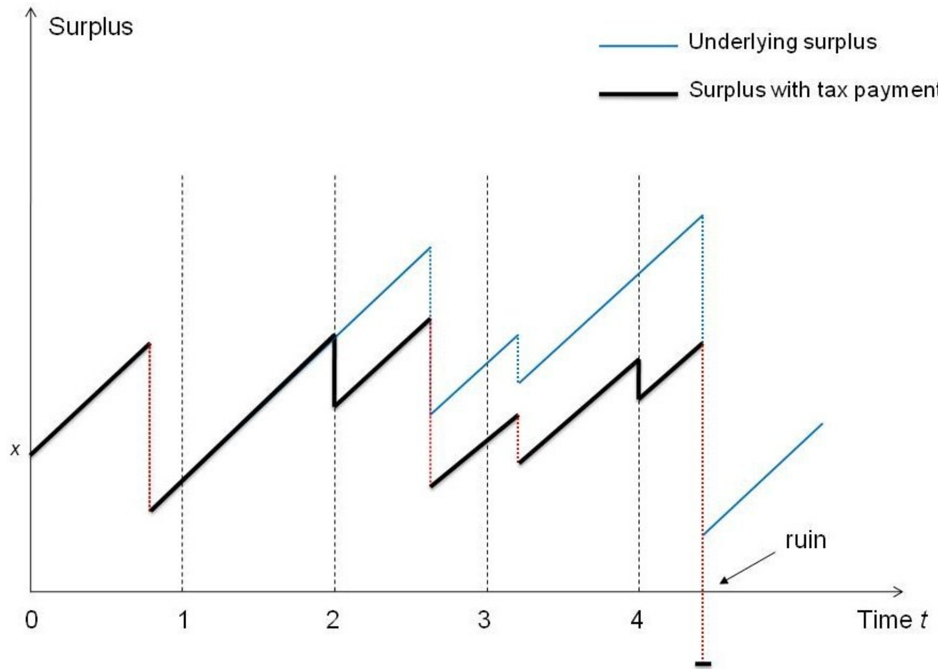

Figure 5.2: Periodic taxation

It is convenient for us to look at the loss process before tax and reinsurance,

$$L_t = x - U_t, \qquad t \geq 0.$$

For each $n = 1, 2, \ldots$, the maximal net loss and the net loss within the period $(n-1, n]$ are, respectively,

$$Y_n = \sup_{n-1 \leq t \leq n} \left(L_t - L_{n-1} \right), \qquad Z_n = L_n - L_{n-1}.$$

After introducing the periodic taxation at rate $0 \leq \gamma < 1$ and reinsurance at rate $0 \leq \delta < 1$, the loss of the company within the period $(n-1, n]$ becomes

$$X_n = Z_n + \gamma Z_n^- - \delta Z_n^+ = (1-\delta) Z_n^+ - (1-\gamma) Z_n^-.$$

Then, it is easy to see that the ruin probability in this situation is equal to

$$\psi_{\gamma,\delta}(x) = \mathbb{P}\left(\sup_{n\geq 1}\left(\sum_{k=1}^{n-1} X_k + Y_n\right) > x\right). \tag{5.2}$$

Notice that we have used $\psi_\gamma(x)$ (with only one subscript) for the ruin probability under the loss-carry-forward taxation and used $\psi_{\gamma,\delta}(x)$ (with two subscripts) for the ruin probability under the periodic taxation and reinsurance. We shall let the notation speak for itself.

In this chapter, we shall assume that the loss process L is a Lévy process with mean $\mathbb{E}L_1 = -\mu < 0$ (so that it converges to $-\infty$ almost surely). Consequently, the random pairs (X_n, Y_n), $n = 1, 2, \ldots$, appearing in (5.2) are i.i.d. copies of the random pair

$$(X, Y) \stackrel{d}{=} \left((1-\delta)L_1^+ - (1-\gamma)L_1^-, \sup_{0\leq t\leq 1} L_t\right). \tag{5.3}$$

Write $\mu_+ = \mathbb{E}L_1^+$ and $\mu_- = \mathbb{E}L_1^-$, which are assumed to be finite. Throughout this chapter, we always choose $\gamma \in [0,1)$ and $\delta \in [0,1)$ such that

$$\mathbb{E}X = (1-\delta)\mu_+ - (1-\gamma)\mu_- < 0, \tag{5.4}$$

so that the insurance company still has positive expected profits under such a scheme of taxation rule and reinsurance policy and that the ruin is not certain.

For a Lévy measure ρ, write $\overline{\rho}(x) = \rho\left((x,\infty)\right)$ for $x \geq 0$. When $\overline{\rho}(1) > 0$, introduce $\Pi(\cdot) = \left(\overline{\rho}(1)\right)^{-1} \rho(\cdot) 1_{(1,\infty)}$, which is a proper probability measure on $(1,\infty)$. We shall assume that the Lévy measure ρ has a tail $\overline{\rho}$ asymptotic to a subexponential tail, a convolution-equivalent tail, and an exponential-like tail. These are natural assumptions when studying the tail probability of the Lévy process. In risk theory, these assumptions have recently been used by e.g. Klüppelberg *et al.* (2004) and Doney and Kyprianou (2006).

In the rest of this chapter we present our main results in Sections 5.2-5.4 for the cases that the Lévy measure ρ of the loss process L has a subexponential tail, a convolution-equivalent tail, and an exponential-like tail, respectively.

5.2 The Case of Subexponential Tails

In our first main result below we look at the case that the Lévy measure ρ has a subexponential tail.

Theorem 5.1. *Consider the Lévy insurance model introduced in Section 5.1. If both* Π *and* Π_e *belong to the class* $\mathcal{S}$ *(which are satisfied when* $\Pi \in \mathcal{S}^*$*), then for every* $0 \leq \gamma < 1$ *and* $0 \leq \delta < 1$ *for which relation (5.4) holds, we have*

$$\psi_{\gamma,\delta}(x) \sim \frac{1}{(1-\gamma)\mu_- - (1-\delta)\mu_+} \int_x^\infty \overline{\rho}(y)\mathrm{d}y. \tag{5.5}$$

Klüppelberg *et al.* (2004) systematically studied the asymptotic behavior of the ruin probability in the Lévy insurance model without tax or reinsurance. Restricting to the case that L is spectrally positive with Lévy measure ρ such that $\Pi \in \mathcal{S}^*$, we see that Theorem 6.2(i) of Klüppelberg *et al.* (2004) corresponds to our Theorem 5.1 with $\gamma = \delta = 0$.

Clearly, the tax identity (5.1) under the loss-carry-forward taxation implies that

$$\psi_\gamma(x) \sim \frac{1}{1-\gamma}\psi_0(x); \tag{5.6}$$

see also Albrecher and Hipp (2007). While under our periodic taxation, substituting $\delta = 0$ to (5.5) yields that

$$\psi_{\gamma,0}(x) \sim \frac{1}{1-\gamma\frac{\mu_-}{\mu_- - \mu_+}}\psi_{0,0}(x). \tag{5.7}$$

Note that $\psi_0(x)$ in (5.6) and $\psi_{0,0}(x)$ in (5.7) are identical. The coefficients in relations (5.6) and (5.7) respectively capture the impact of the two taxation rules on the asymptotic behavior of the ruin probability. Now that $\mu_-/(\mu_- - \mu_+) > 1$ in (5.7), comparing (5.6) with (5.7) we conclude that periodic taxation produces more significant impact on the ruin probability than the loss-carry-forward taxation does.

To prove Theorem 5.1, we need the following two lemmas:

Lemma 5.2. *Let L be a Lévy process with Lévy measure ρ such that $\Pi \in \mathcal{S}$. Then,*

$$\mathbb{P}\left(\sup_{0\le t\le 1} L_t > x\right) \sim \overline{\rho}(x). \tag{5.8}$$

Lemma 5.3 (Theorem 1 of Palmowski and Zwart (2007))**.** *Let random pairs (X_n, Y_n), $n = 1, 2, \ldots,$ be i.i.d. copies of a random pair (X, Y). Denote $M = X \vee Y$. If $-\infty < \mathbb{E}X < 0$, $\mathbb{E}M < \infty$, and $\int_x^\infty \mathbb{P}(M > y)\,\mathrm{d}y$ is asymptotic to a subexponential tail, then,*

$$\mathbb{P}\left(\sup_{n\ge 1}\left(\sum_{k=1}^{n-1} X_k + Y_n\right) > x\right) \sim \frac{1}{|\mathbb{E}X|}\int_x^\infty \mathbb{P}(M > y)\,\mathrm{d}y.$$

Lemma 5.2 is an implication of Theorem 3.1 of Rosiński and Samorodnitsky (1993) (see the example of Lévy motion on their page 1006).

Proof of Theorem 5.1. Recall (5.2), where the random pairs (X_n, Y_n), $n = 1, 2, \ldots,$ are i.i.d. copies of the random pair (X, Y) given in (5.3). Use the notation $M = X \vee Y$

in Lemma 5.3. Since $\Pi \in \mathcal{S}$, from Lemma 5.2 we have

$$\mathbb{P}\left(Y > x\right) \sim \overline{\rho}(x). \tag{5.9}$$

It is clear that $Y \geq L_1^+ \geq X^+$. Hence by (5.9) and $\Pi_I \in \mathcal{S}$,

$$\int_x^\infty \mathbb{P}\left(M > y\right) \mathrm{d}y = \int_x^\infty \mathbb{P}\left(Y > y\right) \mathrm{d}y \sim \int_x^\infty \overline{\rho}(y)\mathrm{d}y,$$

a subexponential tail. Then by Lemma 5.3, we obtain (5.5). □

5.3 The Case of Convolution-equivalent Tails

Next, we consider the case that the Lévy measure ρ has a light tail such that $\Pi \in \mathcal{S}(\alpha)$ for some $\alpha > 0$.

Theorem 5.4. *Consider the Lévy insurance model introduced in Section 5.1. Assume $\mathbb{E}L_1^2 < \infty$ and $\Pi \in \mathcal{S}(\alpha)$ for some $\alpha > 0$. If $0 \leq \gamma < 1$ and $0 < \delta < 1$ are such that*

$$\mathbb{E}\mathrm{e}^{\alpha'\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)} < 1 \tag{5.10}$$

for some $\alpha' > \alpha$, then,

$$\psi_{\gamma,\delta}(x) \sim \frac{c_\alpha}{1 - \mathbb{E}\mathrm{e}^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}}\overline{\rho}(x), \tag{5.11}$$

where the constant c_α is defined as

$$c_\alpha = \lim_{x\to\infty} \frac{\mathbb{P}\left(\sup_{0\leq t\leq 1} L_t > x\right)}{\overline{\rho}(x)} \in (0, \infty). \tag{5.12}$$

The existence of the limit c_α in (5.12) was proved by Braverman and Samorodnitsky (1995); see the following lemma. Condition (5.10) is feasible because $\mathbb{E}\mathrm{e}^{\alpha L_1^+} < \infty$ and $0 < \delta < 1$.

Lemma 5.5 (Theorem 3.1 of Braverman and Samorodnitsky (1995))**.** *Let L be a Lévy process with Lévy measure ρ such that $\Pi \in \mathcal{S}(\alpha)$ for some $\alpha > 0$. Then,*

$$\lim_{x\to\infty} \frac{\mathbb{P}\left(\sup_{0\le t\le 1} L_t > x\right)}{\overline{\rho}(x)} = c \qquad \textit{for some } c \in (0,\infty).$$

By Lemma 2.21, relation (5.11) may be rewritten as

$$\psi_{\gamma,\delta}(x) \sim \frac{\alpha c_\alpha}{1 - \mathbb{E}\mathrm{e}^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}} \int_x^\infty \overline{\rho}(y)\mathrm{d}y. \tag{5.13}$$

With the understanding that $c_0 = 1$ by relation (5.8) and that the coefficient in the right-hand side of relation (5.13) converges to $((1-\gamma)\mu_- - (1-\delta)\mu_+)^{-1}$ as $\alpha \to 0$, relation (5.5) in Theorem 5.1 indicates that relation (5.13) still holds when $\alpha = 0$.

Recalling the light-tailed case of Theorem 4.1(i) in Chapter 4, we give the proof of Theorem 5.4:

Proof of Theorem 5.4. Use the notation in (5.3). By relation (5.12) and closure of the class $\mathcal{S}(\alpha)$ under tail equivalence, the distribution of Y also belongs to the class $\mathcal{S}(\alpha)$. The moment conditions on X required in Theorem 4.1(i) are clearly satisfied. Then, using Theorem 4.1(i) we immediately obtain (5.11). □

To apply Theorem 5.4, a direct problem is how to determine the constant c_α in (5.12). This has been a very difficult problem for a Lévy process L whose Lévy measure ρ has a convolution-equivalent tail. For related discussions see Albin and Sundén (2009) and references therein. The following lemma gives an expression for c_α:

Lemma 5.6. *Let L be a Lévy process with Lévy measure ρ such that $\Pi \in \mathcal{S}(\alpha)$ for some $\alpha > 0$. Then for all $t > 0$,*

$$\lim_{x\to\infty} \frac{\mathbb{P}\left(L_t > x\right)}{\overline{\rho}(x)} = t\mathbb{E}\mathrm{e}^{\alpha L_t} := h(t).$$

There is a unique probability distribution G on $[0,1]$ satisfying $\int_0^1 t^{-1}G(\mathrm{d}t) < \infty$ with moments given by

$$\mu_n(G) = \frac{v_n(n+1)!}{\int_0^1 h(t)\mathrm{d}t}, \qquad n = 1, 2, \ldots,$$

where

$$v_n = \int_{0<t_1\le\cdots\le t_{n+1}\le 1} t_1 \mathbb{E}\mathrm{e}^{\alpha \min_{1\le k\le n+1} L_{t_k}} \mathrm{d}t_1 \cdots \mathrm{d}t_{n+1}. \tag{5.14}$$

Finally,

$$\lim_{x\to\infty} \frac{\mathbb{P}\left(\sup_{0\le t\le 1} L_t > x\right)}{\overline{\rho}(x)} = \int_0^1 t^{-1}G(\mathrm{d}t) \int_0^1 h(t)\mathrm{d}t := c_\alpha. \tag{5.15}$$

Lemma 5.6 is a combination of Proposition 1.3 and Theorem 2.1 of Braverman (1997). Here we need to point out that the constants v_n defined by Braverman (1997) are not correct. This is due to a calculation error in his Lemma 3.1. Indeed, under his assumptions and notation, instead of his relation (3.1) we should have

$$\mathbb{P}\left(\Sigma_k > x, 1 \le k \le n\right) \sim \overline{F_1}(x)\mathbb{E}\mathrm{e}^{\alpha\left(\min_{1\le k\le n} \Sigma_k - X_1\right)},$$

where $\Sigma_k = \sum_{i=1}^k X_i, 1 \le k \le n$. Therefore, to qualify his Theorem 2.1, the constants v_n should be given by our (5.14) above. However, we remark that the expression for c_α given in (5.15) is far from being explicit and can not be evaluated unless L is a subordinator.

To pursue a more explicit expression for c_α, we then restrict the Lévy process L to a compound Poisson process with negative drift.

Corollary 5.7. *Consider the Lévy insurance model introduced in Section 5.1. Assume*

$$L_t = V_t - pt, \qquad t \geq 0, \tag{5.16}$$

where $p > 0$ represents the constant premium rate and $V = (V_t)_{t\geq 0}$ is a compound Poisson process as given in Section 2.1, i.e., $V_t = \sum_{k=1}^{N_t} \xi_k$, where $N = (N_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$ and $\xi_1, \xi_2, \ldots$ are i.i.d. random variables independent of N and with common distribution F on $(-\infty, \infty)$. Suppose that F has a bounded density $f \in \mathcal{S}d(\alpha)$ for some $\alpha > 0$ and that condition (5.10) holds. Then,

$$\psi_{\gamma,\delta}(x) \sim \frac{\lambda c_\alpha}{1 - \mathbb{E}\mathrm{e}^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}} \overline{F}(x)$$

with the constant c_α given by

$$c_\alpha = \mathrm{e}^{\lambda\left(\mathbb{E}\mathrm{e}^{\alpha\xi}-1\right)-\alpha p} + \alpha \int_0^1 \left(\int_0^t \mathbb{P}\left(\sum_{k=1}^{N_t} \xi_k \leq ps \right) \mathrm{d}s \right) \frac{1-t}{t} \mathrm{e}^{\lambda(1-t)\left(\mathbb{E}\mathrm{e}^{\alpha\xi}-1\right)-\alpha p(1-t)} \mathrm{d}t. \tag{5.17}$$

For example, if $F = IG(\mu, \nu)$ with density given by (2.18), i.e.,

$$f(x) = \left(\frac{\nu}{2\pi x^3}\right)^{1/2} \exp\left\{\frac{-\nu(x-\mu)^2}{2\mu^2 x}\right\}, \qquad \mu, \nu, x > 0,$$

then we can appropriately choose the constants p, γ, and δ such that condition (5.10) is satisfied.

While the expression for c_α defined in (5.17) is still not completely explicit, with the only unknown part $\int_0^t \mathbb{P}\left(\sum_{k=1}^{N_t} \xi_k \leq ps\right) \mathrm{d}s$ for $0 < t \leq 1$, it is simple enough for simulations, especially when ξ follows an inverse Gaussian distribution.

Next we prove Corollary 5.7. Let $F(\cdot, t)$ be the distribution of aggregate claims,

$$F(x,t) = \mathbb{P}\left(\sum_{k=1}^{N_t} \xi_k \leq x\right),$$

and let $f(\cdot, t)$ be its density. Write $Y_t = \sup_{0\le s\le t} L_s$. Then $Y_1 = Y$. The lemma below is a restatement of Theorems 2.1 and 2.2 of Asmussen (2000):

Lemma 5.8. *For the compound Poisson model (5.16), we have*

$$\mathbb{P}\left(Y_t \le 0\right) = \frac{1}{t}\int_0^t F\left(ps, t\right) \mathrm{d}s, \qquad t > 0,$$

and

$$1 - \mathbb{P}\left(Y_T > x\right) = F\left(x + pT, T\right) - \int_0^T \mathbb{P}\left(Y_{T-t} \le 0\right) f\left(x + pt, t\right) \mathrm{d}t, \qquad T > 0.$$

Proof of Corollary 5.7. By Theorem 5.4, it suffices to verify (5.17). By Lemma 5.8, we have

$$\mathbb{P}\left(Y > x\right) = \overline{F}\left(x + p, 1\right) + \int_0^1 \mathbb{P}\left(Y_{1-t} \le 0\right) f\left(x + pt, t\right) \mathrm{d}t. \tag{5.18}$$

Since $f \in \mathcal{S}d(\alpha)$ for $\alpha > 0$ implies $F \in \mathcal{S}(\alpha)$, we apply the dominated convergence theorem justified by Lemma 2.25 to obtain that

$$\begin{aligned}\overline{F}\left(x + p, 1\right) &= \sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{k=1}^{n} \xi_k > x + p\right) \mathbb{P}\left(N_1 = n\right) \\ &\sim \lambda \mathrm{e}^{\lambda\left(\mathbb{E}\mathrm{e}^{\alpha\xi} - 1\right)} \overline{F}(x + p) \sim \lambda \mathrm{e}^{\lambda\left(\mathbb{E}\mathrm{e}^{\alpha\xi} - 1\right) - \alpha p} \overline{F}(x).\end{aligned} \tag{5.19}$$

Similarly, by Lemma 2.26, for each fixed $t \in (0, 1]$,

$$f\left(x + pt, t\right) \sim \sum_{n=1}^{\infty} \frac{\left(\lambda t\right)^n}{n!} \mathrm{e}^{-\lambda t} \cdot n \left(\mathbb{E}\mathrm{e}^{\alpha\xi}\right)^{n-1} f(x + pt) \sim \lambda t \mathrm{e}^{\lambda t\left(\mathbb{E}\mathrm{e}^{\alpha\xi} - 1\right) - \alpha pt} f(x). \tag{5.20}$$

Substitute (5.20) into the integral in (5.18). In order to apply the dominated convergence theorem here, we notice that, by Lemma 2.26 again, there exists some $K > 0$

such that for all $x \geq 0$ for which $f(x) > 0$ and for all $t \in (0,1]$,

$$\begin{aligned}\frac{f(x+pt,t)}{f(x)} &= \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t} \cdot \frac{f^{n\star}(x+pt)}{f(x+pt)} \cdot \frac{f(x+pt)}{f(x)} \\ &\leq \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t} \cdot K \left(\mathbb{E}\mathrm{e}^{\alpha\xi} + 1\right)^n \leq K \mathrm{e}^{\lambda t \mathbb{E}\mathrm{e}^{\alpha\xi}},\end{aligned}$$

where in the last but one step we used the local uniformity of the convergence of $f(x+y)/f(x)$ to $\mathrm{e}^{-\alpha y}$. Hence,

$$\int_0^1 \mathbb{P}(Y_{1-t} \leq 0) \frac{f(x+pt,t)}{f(x)} \mathrm{d}t \leq K \int_0^1 \mathrm{e}^{\lambda t \mathbb{E}\mathrm{e}^{\alpha\xi}} \mathrm{d}t < \infty.$$

Then, using Lemma 5.8 and the dominated convergence theorem,

$$\begin{aligned}&\int_0^1 \mathbb{P}(Y_{1-t} \leq 0) f(x+pt,t)\,\mathrm{d}t \\ &\sim f(x) \int_0^1 \left(\frac{1}{1-t} \int_0^{1-t} F(ps, 1-t)\,\mathrm{d}s\right) \lambda t \mathrm{e}^{\lambda t\left(\mathbb{E}\mathrm{e}^{\alpha\xi}-1\right)-\alpha pt} \mathrm{d}t. \qquad (5.21)\end{aligned}$$

Plugging (5.19) and (5.21) into (5.18) and using the facts that $\overline{\rho}(\cdot) = \lambda \overline{F}(\cdot)$ and $f(x)/\overline{F}(x) \to \alpha$, we obtain (5.17). □

5.4 The Case of Exponential-like Tails

Finally, we consider the case that the Lévy measure ρ has a light tail such that $\Pi \in \mathcal{L}(\alpha) \backslash \mathcal{S}(\alpha)$ for some $\alpha > 0$.

Theorem 5.9. *Consider the Lévy insurance model introduced in Section 5.1. Assume* $\mathbb{E}L_1^2 < \infty$, $\Pi \in \mathcal{L}(\alpha)$ *for some* $\alpha > 0$, *and* $\overline{\Pi}(x) = o\left(\overline{\Pi^{2*}}(x)\right)$. *If* $0 \leq \gamma < 1$ *and* $0 < \delta < 1$ *are such that condition (5.10) holds, then,*

$$\psi_{\gamma,\delta}(x) \sim \frac{1}{1 - \mathbb{E}\mathrm{e}^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}} \mathbb{P}(L_1 > x). \qquad (5.22)$$

We need the following result, which is a combination of Theorem 3.3 and Corollary 6.2 of Albin and Sundén (2009):

Lemma 5.10. *Let L be a Lévy process with Lévy measure ρ such that $\Pi \in \mathcal{L}(\alpha)$ for some $\alpha > 0$ and $\overline{\Pi}(x) = o\left(\overline{\Pi^{2*}}(x)\right)$. Then for all $t > 0$, the distribution of L_t belongs to $\mathcal{L}(\alpha)\backslash\mathcal{S}(\alpha)$ and*

$$\mathbb{P}\left(\sup_{0\le s\le t} L_s > x\right) \sim \mathbb{P}\left(L_t > x\right).$$

Note that the conditions on the Lévy measure ρ in Lemma 5.10 are fulfilled if $\overline{\rho}$ is asymptotic to the tail of an exponential distribution, an Erlang distribution, or, more generally, a Gamma distribution.

Proof of Theorem 5.9. Use the notation in (5.3). By Lemma 5.10 we know that the distribution of L_1 belongs to $\mathcal{L}(\alpha)\backslash\mathcal{S}(\alpha)$ and

$$\mathbb{P}\left(Y > x\right) \sim \mathbb{P}\left(L_1 > x\right).$$

Hence, the distribution of Y belongs to $\mathcal{L}(\alpha)\backslash\mathcal{S}(\alpha)$ as well. Finally, using Theorem 4.1(i) again we obtain relation (5.22). □

The asymptotic relation (5.22) is in terms of the tail of L_1 instead of the tail of the Lévy measure ρ. In case the tail of L_1 is unknown, relation (5.22) is not completely explicit. We are going to show two special, but important, cases of Theorem 5.9 in which a completely explicit asymptotic relation for the ruin probability is given.

First, we consider a gamma process $U = (U_t)_{t\ge0}$ as given in Section 2.1. For the gamma process U, it is easy to verify that $\overline{\Pi}(x) = o\left(\overline{\Pi^{2*}}(x)\right)$. By Theorem 5.9, we immediately have the following:

Corollary 5.11. *Consider the Lévy insurance model introduced in Section 5.1. Assume*

$$L_t = \Gamma_t - pt, \qquad t \geq 0,$$

where $p > 0$ *and* $\Gamma = (\Gamma_t)_{t\geq 0}$ *is a gamma process as given in Section 2.1 with parameters* $\alpha, \beta > 0$*, i.e.,* Γ *starts from* 0 *with stationary and independent increments and* Γ_1 *has the gamma(*α*,*β*) distribution with density* $f(x) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} \mathrm{e}^{-\alpha x}$ *for* $\alpha, \beta, x > 0$*. If* $0 \leq \gamma < 1$ *and* $0 < \delta < 1$ *are such that condition (5.10) holds, then,*

$$\psi_{\gamma,\delta}(x) \sim \frac{\alpha^{\beta-1} (x+p)^{\beta-1} \mathrm{e}^{-\alpha(x+p)}}{\left(1 - \mathbb{E}\mathrm{e}^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}\right)\Gamma(\beta)}.$$

Next, we consider a compound Poisson process with negative drift again.

Corollary 5.12. *Consider the Lévy insurance model introduced in Section 5.1. Assume*

$$L_t = V_t - pt, \qquad t \geq 0,$$

where $p > 0$ *represents the constant premium rate and* $V = (V_t)_{t\geq 0}$ *is a compound Poisson process as given in Section 2.1. Suppose that* F *is an exponential distribution with mean* $1/\alpha$*. If* $0 \leq \gamma < 1$ *and* $0 < \delta < 1$ *are such that condition (5.10) holds, then,*

$$\psi_{\gamma,\delta}(x) \sim \frac{2\sqrt{\lambda/\pi}}{1 - \mathbb{E}\mathrm{e}^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}} \int_0^{\pi/2} \Phi\left(\sqrt{2\lambda}\cos\theta - \sqrt{2\alpha(x+p)}\right) \mathrm{d}\theta, \tag{5.23}$$

where $\Phi(\cdot)$ *is the standard normal distribution.*

The following is an elementary result:

Lemma 5.13. *Let* $f(\cdot):[-1,1]\to(0,\infty)$ *be a nonincreasing function right continuous at* 1 *and let* $c>0$ *be a constant. Then,*

$$\int_0^{\pi} \mathrm{e}^{cx\cos\theta} f(\cos\theta)\mathrm{d}\theta \sim \int_0^{\pi/2} \mathrm{e}^{cx\cos\theta} f(\cos\theta)\mathrm{d}\theta \sim f(1)\int_0^{\pi/2} \mathrm{e}^{cx\cos\theta}\mathrm{d}\theta. \tag{5.24}$$

Proof. For every $0<\epsilon\le\pi/2$, we split the first integral in (5.24) into two parts as $\int_0^\epsilon+\int_\epsilon^\pi$. It is easy to see that the second part is asymptotically negligible, as

$$\frac{\int_\epsilon^\pi \mathrm{e}^{cx\cos\theta} f(\cos\theta)\mathrm{d}\theta}{\int_0^\epsilon \mathrm{e}^{cx\cos\theta} f(\cos\theta)\mathrm{d}\theta} \le \frac{\pi f(-1)\mathrm{e}^{cx\cos\epsilon}}{f(1)\int_0^{\epsilon/2} \mathrm{e}^{cx\cos\theta}\mathrm{d}\theta} \to 0.$$

Hence, the first relation of (5.24) holds and, moreover,

$$\int_0^{\pi/2} \mathrm{e}^{cx\cos\theta} f(\cos\theta)\mathrm{d}\theta \sim \int_0^{\epsilon} \mathrm{e}^{cx\cos\theta} f(\cos\theta)\mathrm{d}\theta. \tag{5.25}$$

Note that, since f is right continuous at 1, if in (5.25) $\epsilon>0$ is chosen to be sufficiently close to 0, then $f(\cos\theta)$ is sufficiently close to $f(1)$. Therefore, by (5.25) and the arbitrariness of $\epsilon>0$, we obtain the second relation of (5.24). □

Proof of Corollary 5.12. Clearly, $\mathbb{E}L_1^2<\infty$, $\Pi\in\mathcal{L}(\alpha)$, and $\overline{\Pi}(x)=o\left(\overline{\Pi^{2*}}(x)\right)$. Therefore by Theorem 5.9, we only need to focus on derivation of the tail probability $\mathbb{P}(L_1>x)$. Since the n-fold convolution of an exponential distribution with mean $1/\alpha$ is a gamma distribution with parameters (α,n), we have

$$\begin{aligned}\mathbb{P}(L_1>x) &= \int_{x+p}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha^n}{(n-1)!}y^{n-1}\mathrm{e}^{-\alpha y}\cdot\frac{\lambda^n}{n!}\mathrm{e}^{-\lambda}\mathrm{d}y \\ &= \sqrt{\alpha\lambda}\mathrm{e}^{-\lambda}\int_{x+p}^{\infty}\sum_{n=0}^{\infty}\frac{\left(\sqrt{\alpha\lambda y}\right)^{2n+1}}{n!(n+1)!}y^{-1/2}\mathrm{e}^{-\alpha y}\mathrm{d}y. \end{aligned}\tag{5.26}$$

The last series in the above is of the structure of the modified Bessel function of order 1; that is,

$$\sum_{n=0}^{\infty}\frac{\left(\sqrt{\alpha\lambda y}\right)^{2n+1}}{n!(n+1)!} = \frac{1}{\pi}\int_0^{\pi}\mathrm{e}^{2\sqrt{\alpha\lambda y}\cos\theta}\cos\theta\mathrm{d}\theta. \tag{5.27}$$

Using Lemma 5.13 twice, as $y \to \infty$,

$$\int_0^{\pi} \mathrm{e}^{2\sqrt{\alpha\lambda y}\cos\theta} \cos\theta \mathrm{d}\theta \sim \int_0^{\pi/2} \mathrm{e}^{2\sqrt{\alpha\lambda y}\cos\theta} \mathrm{d}\theta \sim \mathrm{e}^{\lambda} \int_0^{\pi/2} \mathrm{e}^{2\sqrt{\alpha\lambda y}\cos\theta - \lambda\cos^2\theta} \mathrm{d}\theta.$$

Substituting this into (5.27) then substituting (5.27) into (5.26), we obtain that

$$\begin{aligned} \mathbb{P}\left(L_1 > x\right) &\sim \frac{\sqrt{\alpha\lambda}}{\pi} \mathrm{e}^{-\lambda} \int_{x+p}^{\infty} \left(\mathrm{e}^{\lambda} \int_0^{\pi/2} \mathrm{e}^{2\sqrt{\alpha\lambda y}\cos\theta - \lambda\cos^2\theta} \mathrm{d}\theta \right) y^{-1/2} \mathrm{e}^{-\alpha y} \mathrm{d}y \\ &\sim \frac{2\sqrt{\alpha\lambda}}{\pi} \int_0^{\pi/2} \left(\int_{\sqrt{x+p}}^{\infty} \mathrm{e}^{-\alpha u^2 + 2u\sqrt{\alpha\lambda}\cos\theta - \lambda\cos^2\theta} \mathrm{d}u \right) \mathrm{d}\theta \\ &= 2\sqrt{\frac{\lambda}{\pi}} \int_0^{\pi/2} \Phi\left(\sqrt{2\lambda}\cos\theta - \sqrt{2\alpha\left(x+p\right)} \right) \mathrm{d}\theta. \end{aligned}$$

Substituting this into (5.22) yields (5.23). □

CHAPTER 6
THE RENEWAL RISK MODEL WITH RISKY INVESTMENT

In this chapter, we study the tail behavior of the stochastically discounted net loss process in the renewal risk model with risky investment. Consider an insurance company who invests its surplus into a portfolio consisting of both a riskless bond and a risky stock. Suppose the price process of the bond grows with a constant force of interest, while the price process of the stock is modeled by an exponential Lévy process. The study of such a risk model has become a hot topic in the past decade. Paulsen (2008) gave a comprehensive review on ruin in this model when assets earn investment income. Assuming a constant mix investment strategy, i.e., the proportions of surplus invested into the riskless and risky assets remain constant, Klüppelberg and Kostadinova (2008) investigated the tail behavior of the stationary distribution for the discounted net loss process in the compound Poisson risk model. We extend their main result by deriving an asymptotic formula for the tail probability of the stochastically discounted net loss process under more general assumptions.

6.1 Introduction

Consider the renewal risk model in which the surplus process of an insurance company is modeled by

$$U_t = u + pt - S_t, \qquad t \geq 0,$$

where $u > 0$ is the initial surplus level, $p > 0$ is the constant premium rate, and $S = (S_t)_{t\geq 0}$ defines the aggregate claims process. Assume $S_t = \sum_{k=1}^{N_t} X_k,\ t \geq 0,$

where, the same as in Chapter 3, claim sizes $X_k, k = 1, 2, \ldots$, constitute a sequence of i.i.d. nonnegative random variables with generic random variable X with distribution F and finite mean, while their arrival times τ_k, $k = 1, 2, \ldots$, independent of X_k, $k = 1, 2, \ldots$, constitute a renewal counting process

$$N_t = \#\{k = 1, 2, \ldots : \tau_k \leq t\}, \qquad t \geq 0.$$

Hence, the inter-arrival times $\theta_1 = \tau_1, \theta_k = \tau_k - \tau_{k-1}, k = 2, 3, \ldots$, constitute a sequence of i.i.d., nonnegative random variables with common not-degenerate-at-zero distribution G, which is assumed to be nonlattice and have mean $1/\lambda$.

Following Klüppelberg and Kostadinova (2008), suppose the insurance company is allowed to invest its surplus into two assets: a bond with constant force of interest $r > 0$ and a stock whose price is modeled by an exponential Lévy process. The two assets have their price processes as, respectively,

$$P_t^{(0)} = \mathrm{e}^{rt}, \qquad P_t^{(1)} = \mathrm{e}^{L_t}, \qquad t \geq 0, \tag{6.1}$$

where $L = (L_t)_{t \geq 0}$ is a Lévy process with characteristic exponent given by

$$\Psi(s) = \mathrm{i}as + \frac{\sigma^2}{2}s^2 + \int_{-\infty}^{\infty} \left(1 - \mathrm{e}^{\mathrm{i}sx} + \mathrm{i}sx1_{\{|x| \leq 1\}}\right) \rho(\mathrm{d}x)$$

with $a \in (-\infty, \infty)$, $\sigma \geq 0$, and Lévy measure ρ on $(-\infty, \infty)$ satisfying $\rho(\{0\}) = 0$ and $\int_{-\infty}^{\infty} (x^2 \wedge 1) \rho(\mathrm{d}x) < \infty$. See Section 2.1 for more details of Lévy processes.

We assume the so-called constant mix investment strategy, i.e., the proportions of surplus invested into the bond and stock remain constant through time. The advantages of this investment strategy are discussed in Emmer *et al.* (2001) and

Emmer and Klüppelberg (2004). As commented by Emmer *et al.* (2001, Section 2), this strategy is actually dynamic in the sense that it requires at every instance of time a rebalancing of the investment portfolio depending on the corresponding instantaneous price changes. From now on, we denote by $\pi \in [0,1]$ the fraction of the surplus invested into the stock.

For the two price processes in (6.1), by Lemma 2.15 we write their corresponding stochastic differential equations (SDEs) as

$$\mathrm{d}P_t^{(0)} = rP_t^{(0)}\mathrm{d}t, \qquad t \geq 0,$$

with $P_0^{(0)} = 1$, and

$$\begin{aligned}\mathrm{d}P_t^{(1)} &= P_{t-}^{(1)}\mathrm{d}\widehat{L}_t \\ &= P_{t-}^{(1)}\left(\mathrm{d}L_t + \frac{\sigma^2}{2}\mathrm{d}t + \mathrm{e}^{\Delta L_t} - 1 - \Delta L_t\right), \qquad t > 0,\end{aligned}$$

with $P_0^{(1)} = 1$, where $\Delta L_t = L_t - L_{t-}$ denotes the jump of the process L at time $t > 0$. As explained in Subsection 2.2.3, the process $\widehat{L}$ is such that $\mathrm{e}^{L_t} = \mathcal{E}\left(\widehat{L}_t\right)$, $t \geq 0$, where $\mathcal{E}$ denotes the stochastic exponential of a process. From the above two SDEs we define the investment process:

Definition 6.1. *For $\pi \in [0,1]$ the investment process is defined as the solution of the SDE*

$$\mathrm{d}P_t^{(\pi)} = P_{t-}^{(\pi)}\left((1-\pi)r\mathrm{d}t + \pi\mathrm{d}\widehat{L}_t\right), \qquad t > 0, \tag{6.2}$$

with $P_0^{(\pi)} = 1$.

Lemma 6.2 (Lemma 2.5 of Emmer and Klüppleberg (2004)). *The SDE (6.2) has the solution*

$$P_t^{(\pi)} = \mathcal{E}\left((1-\pi)r\mathrm{d}t + \pi\mathrm{d}\widehat{L}_t\right) = \mathrm{e}^{L_{\pi,t}}, \qquad t \geq 0,$$

where $L_{\pi,t}$ is a Lévy process with characteristic triplet $(a_\pi, \sigma_\pi^2, \rho_\pi)$ given by

$$\begin{aligned} a_\pi &= a\pi - (1-\pi)\left(r + \frac{\sigma^2}{2}\pi\right) \\ &\quad - \int_{-\infty}^{\infty} \left(\log\left(1+\pi\left(\mathrm{e}^x - 1\right)\right) 1_{\{|\log(1+\pi(\mathrm{e}^x-1))|\leq 1\}} - \pi x 1_{\{|x|\leq 1\}}\right) \rho\left(\mathrm{d}x\right), \\ \sigma_\pi^2 &= \pi^2\sigma^2, \\ \rho_\pi(A) &= \rho\left(\{x : \log\left(1+\pi\left(\mathrm{e}^x-1\right)\right) \in A\}\right) \text{ for any Borel set } A. \end{aligned}$$

In particular, when $\pi = 0$ or 1, $L_{\pi,t}$ reduces to rt or L_t, respectively. We assume $\varphi_\pi(1) < \infty$ throughout this chapter so that $\mathbb{E}\mathrm{e}^{-L_{\pi,t}} = \mathrm{e}^{t\varphi_\pi(1)} < \infty$ is finite for all $t \geq 0$. By Lemma 6.7 below, the assumption $\varphi_\pi(1) < \infty$ is implied by $\varphi(1) < \infty$.

6.2 The Integrated Risk Process

Suppose the insurance company invests all its surplus into the market introduced in Section 6.1 following the constant mix investment strategy. Its risk process, called integrated risk process, becomes the solution to the SDE

$$\mathrm{d}U_{\pi,t} = p\mathrm{d}t - \mathrm{d}S_t + U_{\pi,t-}\left((1-\pi)r\mathrm{d}t + \pi\mathrm{d}\widehat{L}_t\right), \qquad t > 0, \tag{6.3}$$

with $U_{\pi,0} = u$. The following lemma gives the solution for SDE (6.3). The solution can be obtained by using the method of integrating factors; see for example Bichteler (2002). For the sake of self-containedness, we still give the proof here.

Lemma 6.3. *The SDE (6.3) has the solution*

$$U_{\pi,t} = \mathrm{e}^{L_{\pi,t}} \left(u + \int_{0-}^{t} \mathrm{e}^{-L_{\pi,v}} \left(p\mathrm{d}v - \mathrm{d}S_v \right) \right), \qquad t \geq 0. \tag{6.4}$$

Proof. Define

$$Z_t = \int_{0-}^{t} \mathrm{e}^{-L_{\pi,v-}} \left(p\mathrm{d}v - \mathrm{d}S_v \right) = \int_{0-}^{t} \mathrm{e}^{-L_{\pi,v}} \left(p\mathrm{d}v - \mathrm{d}S_v \right),$$

where the equality holds because the processes L_π and S are independent and hence have no common jumps almost surely; see Proposition 5.3 of Cont and Tankov (2004). Since $P^{(\pi)}$ is an exponential Lévy process and Z is the integral with respect to a finite variation process, under the moment conditions mentioned in the end of Section 6.1 both processes $P^{(\pi)}$ and Z are semimartingales; see Section 2.2 for details. Hence, the integration by parts formula gives

$$\mathrm{d}\left(P_t^{(\pi)} Z_t \right) = P_{t-}^{(\pi)} \mathrm{d}Z_t + Z_{t-} \mathrm{d}P_t^{(\pi)} + \mathrm{d}\left[P^{(\pi)}, Z \right]_t, \qquad t > 0,$$

where $\left[P^{(\pi)}, Z \right]$ denotes the quadratic covariation process of $P^{(\pi)}$ and Z. Since S is a finite variation process and so is Z and, once again, the processes L_π and S have no common jumps almost surely, by the properties introduced in Subsection 2.2.2 we obtain

$$\left[P^{(\pi)}, Z \right]_t \equiv 0, \qquad t \geq 0.$$

Hence, for $t \geq 0$,

$$\begin{aligned} \mathrm{d}\left(P_t^{(\pi)} Z_t \right) &= P_{t-}^{(\pi)} \mathrm{d}Z_t + Z_{t-} \mathrm{d}P_t^{(\pi)} \\ &= P_{t-}^{(\pi)} \mathrm{e}^{-L_{\pi,t-}} \left(p\mathrm{d}t - \mathrm{d}S_t \right) + \mathrm{d}P_t^{(\pi)} \int_{0-}^{t-} \mathrm{e}^{-L_{\pi,v}} \left(p\mathrm{d}v - \mathrm{d}S_v \right) \\ &= p\mathrm{d}t - \mathrm{d}S_t + \mathrm{d}P_t^{(\pi)} \int_{0-}^{t-} \mathrm{e}^{-L_{\pi,v}} \left(p\mathrm{d}v - \mathrm{d}S_v \right). \end{aligned}$$

From the equality above, (6.4), and (6.2), for $t > 0$,

$$
\begin{aligned}
\mathrm{d}U_{\pi,t} &= u\mathrm{d}P_t^{(\pi)} + \mathrm{d}\left(P_t^{(\pi)} Z_t\right) \\
&= u\mathrm{d}P_t^{(\pi)} + p\mathrm{d}t - \mathrm{d}S_t + \mathrm{d}P_t^{(\pi)} \int_{0-}^{t-} \mathrm{e}^{-L_{\pi,v}} \left(p\mathrm{d}v - \mathrm{d}S_v\right) \\
&= p\mathrm{d}t - \mathrm{d}S_t + P_{t-}^{(\pi)} \left(u + \int_{0-}^{t-} \mathrm{e}^{-L_{\pi,v}} \left(p\mathrm{d}v - \mathrm{d}S_v\right)\right) \frac{\mathrm{d}P_t^{(\pi)}}{P_{t-}^{(\pi)}} \\
&= p\mathrm{d}t - \mathrm{d}S_t + U_{\pi,t-}\left((1-\pi)r\mathrm{d}t + \pi\mathrm{d}\widehat{L}_t\right).
\end{aligned}
$$

This verifies (6.3). $\square$

6.3 The Discounted Net Loss Process

Definition 6.4. *The discounted net loss process is defined as*

$$
V_{\pi,t} = u - \mathrm{e}^{-L_{\pi,t}} U_{\pi,t} = \int_{0-}^{t} \mathrm{e}^{-L_{\pi,v}} \left(\mathrm{d}S_v - p\mathrm{d}v\right), \qquad t \geq 0, \tag{6.5}
$$

where $U_{\pi,t}$ is given in (6.4).

We are interested in the tail behavior of the stationary discounted net loss process. We study V_π via its natural discretization at claim-arrival times, V_{π,τ_k}, $k = 0, 1, \ldots$, where $\tau_0 = 0$. For $k = 1, 2, \ldots$, write

$$
\begin{aligned}
A_{\pi,k} &= X_k \mathrm{e}^{-\left(L_{\pi,\tau_k} - L_{\pi,\tau_{k-1}}\right)} - p\int_{\tau_{k-1}}^{\tau_k} \mathrm{e}^{-\left(L_{\pi,v} - L_{\pi,\tau_{k-1}}\right)} \mathrm{d}v, \\
B_{\pi,k} &= \mathrm{e}^{-\left(L_{\pi,\tau_k} - L_{\pi,\tau_{k-1}}\right)}.
\end{aligned}
$$

Then $(A_{\pi,k}, B_{\pi,k})$, $k = 1, 2, \ldots$, form a sequence of i.i.d. random pairs with generic random pair

$$
(A_\pi, B_\pi) = \left(X\mathrm{e}^{-L_{\pi,\theta}} - p\int_0^\theta \mathrm{e}^{-L_{\pi,v}} \mathrm{d}v, \mathrm{e}^{-L_{\pi,\theta}}\right). \tag{6.6}
$$

It is obvious that

$$V_{\pi,0} = 0, \qquad V_{\pi,\tau_k} = \sum_{m=1}^{k} A_{\pi,m} \prod_{j=1}^{m-1} B_{\pi,j}, \qquad k = 1, 2, \dots,$$

where the product over an empty set of indices produces a value 1.

Denoting by $-\eta$ the left abscissa of convergence of $g(s) = \mathbb{E}\mathrm{e}^{-s\theta}$, we give the following theorem:

Theorem 6.5. *Consider the renewal risk model with risky investment introduced in Section 6.1. Suppose $\mathbb{E}X < \infty$, $\mathbb{E}L_1 > 0$, and the Laplace exponent of L_π satisfies $\varphi_\pi(1) < \eta$.*

(i) We have

$$V_{\pi,\tau_k} \xrightarrow{a.s.} V_{\pi,\infty} = \sum_{m=1}^{\infty} A_{\pi,m} \prod_{j=1}^{m-1} B_{\pi,j}, \qquad \text{as } k \to \infty, \tag{6.7}$$

where the series of the right-hand side converges absolutely with probability 1. Moreover, $V_{\pi,\infty}$ satisfies the stochastic difference equation

$$V_{\pi,\infty} \stackrel{d}{=} A_\pi + B_\pi V_{\pi,\infty}, \tag{6.8}$$

where $V_{\pi,\infty}$ and (A_π, B_π) are independent.

(ii) $V_{\pi,t}$ almost surely converges to some finite random variable $V^c_{\pi,\infty}$ if and only if V_{π,τ_k} almost surely converges to some finite random variable and $V_{\pi,\infty}$. Furthermore,

$$V_{\pi,\infty} \stackrel{a.s.}{=} V^c_{\pi,\infty}. \tag{6.9}$$

To give the proof of Theorem 6.5, we need some lemmas. The following lemma, which was proved by Ross (1983), holds for a general renewal counting process:

Lemma 6.6 (Proposition 3.4.5 of Ross (1983))**.** *Consider the renewal counting process $(N_t)_{t\geq 0}$ given in (3.1) whose i.i.d. inter-arrival times follow a common non-lattice distribution G with finite mean. As $t \to \infty$, $t - \tau_{N_t}$ converges in distribution to G_e, the equilibrium distribution of G.*

Lemma 6.7 (Lemma A.1 of Klüppelberg and Kostadinova (2008))**.** *Consider L and L_π introduced above. We have*

(i) If $\mathbb{E}L_1 < \infty$, then $\mathbb{E}L_{\pi,1} < \infty$.

(ii) If $\mathbb{E}L_1 > 0$, then $\mathbb{E}L_{\pi,1} > 0$.

(iii) If $\varphi(s) = \log \mathbb{E}\mathrm{e}^{-sL_1} < \infty$, then $\varphi_\pi(s) = \log \mathbb{E}\mathrm{e}^{-sL_{\pi,1}} < \infty$.

For $a > 0$, denote $\log^+ a = \max\{0, \log a\}$. We have the following lemma:

Lemma 6.8. *Assume $\mathbb{E}X < \infty$, $\mathbb{E}L_1 > 0$, and $\varphi_\pi(1) < \eta$. Then for A_π and B_π defined in (6.6), we have*

$$\mathbb{E}\log^+|A_\pi| < \infty \qquad \textit{and} \qquad \mathbb{E}\log B_\pi < 0.$$

Proof. By Lemma 6.7(i), we have $0 < \mathbb{E}L_{\pi,1} < \infty$. It is clear that

$$\mathbb{E}\log B_\pi = -\mathbb{E}L_{\pi,\theta} = -\mathbb{E}L_{\pi,1}\mathbb{E}\theta < 0.$$

For the proof of $\mathbb{E}\log^+|A_\pi| < \infty$, we use the elementary inequality $\log x < x$ for all $x > 0$. Then,

$$\begin{aligned}\mathbb{E}\log^+|A_\pi| &= \mathbb{E}\log^+\left|X\mathrm{e}^{-L_{\pi,\theta}} - p\int_0^\theta \mathrm{e}^{-L_{\pi,v}}\mathrm{d}v\right| \\ &\leq \mathbb{E}\left|X\mathrm{e}^{-L_{\pi,\theta}} - p\int_0^\theta \mathrm{e}^{-L_{\pi,v}}\mathrm{d}v\right| \\ &\leq \mathbb{E}X\mathbb{E}\mathrm{e}^{-L_{\pi,\theta}} + p\mathbb{E}\int_0^\theta \mathrm{e}^{-L_{\pi,v}}\mathrm{d}v.\end{aligned}$$

In the first term, $\mathbb{E}X < \infty$ and, since $\varphi_\pi(1) < \eta$,

$$\mathbb{E}\mathrm{e}^{-L_{\pi,\theta}} = \int_{0-}^{\infty} \mathbb{E}\mathrm{e}^{-L_{\pi,v}} G(\mathrm{d}v) = \int_{0-}^{\infty} \mathrm{e}^{v\varphi_\pi(1)} G(\mathrm{d}v) < \infty.$$

In the second term, using $\varphi_\pi(1) < \eta$ again,

$$\begin{aligned}\mathbb{E}\int_0^{\theta} \mathrm{e}^{-L_{\pi,v}} \mathrm{d}v &= \int_{0-}^{\infty}\int_0^t \mathbb{E}\mathrm{e}^{-L_{\pi,v}} \mathrm{d}v G(\mathrm{d}v) = \int_{0-}^{\infty}\int_0^t \mathrm{e}^{v\varphi_\pi(1)} \mathrm{d}v G(\mathrm{d}t) \\ &= \begin{cases} \int_{0-}^{\infty} tG(\mathrm{d}t) = \mathbb{E}\theta = 1/\lambda < \infty & \text{if } \varphi_\pi(1) = 0; \\ \frac{1}{\varphi_\pi(1)}\left(\int_{0-}^{\infty} \mathrm{e}^{t\varphi_\pi(1)} G(\mathrm{d}t) - 1\right) < \infty & \text{if } \varphi_\pi(1) \neq 0. \end{cases}\end{aligned}$$

This ends the proof. □

Now we are ready to give the proof of Theorem 6.5.

Proof of Theorem 6.5. (i) In order to prove (6.7) and (6.8) we introduce random variables $\widehat{V}_{\pi,\tau_k}$, $k = 0, 1, \ldots$, such that $\widehat{V}_{\pi,0} = 0$ and

$$\widehat{V}_{\pi,\tau_k} = A_{\pi,k} + B_{\pi,k}\widehat{V}_{\pi,\tau_{k-1}} = \sum_{m=1}^{k} A_{\pi,m} \prod_{j=m+1}^{k} B_{\pi,j}, \qquad k = 1, 2, \ldots.$$

We observe that for every $k = 1, 2, \ldots$,

$$\{(A_{\pi,j}, B_{\pi,j}), j = 1, 2, \ldots k\} \stackrel{d}{=} \{(A_{\pi,k-j+1}, B_{\pi,k-j+1}), j = 1, 2, \ldots k\},$$

which implies that

$$\sum_{m=1}^{k} A_{\pi,m} \prod_{j=1}^{m-1} B_{\pi,j} \stackrel{d}{=} \sum_{m=1}^{k} A_{\pi,m} \prod_{j=m+1}^{k} B_{\pi,j}.$$

Hence, $V_{\pi,\tau_k} \stackrel{\mathrm{d}}{=} \widehat{V}_{\pi,\tau_k}$ holds for every $k = 1, 2, \ldots$. Applying Proposition 8.4.3 of Embrechts *et al.* (1997) on $\widehat{V}_{\pi,\tau_k}$ we obtain (6.7) and (6.8) immediately; see also Vervaat (1979). The conditions in that proposition are guaranteed by Lemma 6.8.

(ii) For every $t \geq 0$,

$$V_{\pi,t} = V_{\pi,\tau_{N_t}} - p\mathrm{e}^{-L_{\pi,\tau_{N_t}}} \int_{\tau_{N_t}}^{t} \mathrm{e}^{-\left(L_{\pi,v} - L_{\pi,\tau_{N_t}}\right)} \mathrm{d}v,$$

where in the last line the integral is independent of $V_{\pi,\tau_{N_t}}$. Since $N_t \overset{\text{a.s.}}{\to} \infty$ as $t \to \infty$, we know from (i) that $V_{\pi,\tau_{N_t}} \overset{\text{a.s.}}{\to} V_{\pi,\infty}$ as $t \to \infty$. Moreover, as $\mathbb{E}L_1 > 0$, by Lemma 6.7(ii) we have that $\mathbb{E}L_{\pi,1} > 0$ and hence $\mathrm{e}^{-L_{\pi,\tau_{N_t}}} \overset{\text{a.s.}}{\to} 0$ as $t \to \infty$. Finally,

$$\int_{\tau_{N_t}}^{t} \mathrm{e}^{-\left(L_{\pi,v} - L_{\pi,\tau_{N_t}}\right)} \mathrm{d}v \overset{d}{=} \int_{0}^{t-\tau_{N_t}} \mathrm{e}^{-L_{\pi,v}} \mathrm{d}v.$$

As $t - \tau_{N_t}$ converges in distribution to G_e as $t \to \infty$ by Lemma 6.6, the last integral almost surely converges to a finite random variable. Then relation (6.9) follows immediately. □

6.4 Claims with Extended-regularly-varying Tails

In this section, we assume that $F \in \mathrm{ERV}$. Recalling $\lambda_t = \mathbb{E}N_t$ and $\Lambda = \{t : \lambda_t > 0\} \cup \{\infty\}$, we give an explicit expression for the asymptotic tail probability of $V_{\pi,T}$ for all $T \in \Lambda$ in the following theorem:

Theorem 6.9. *Consider $V_{\pi,t}$ defined in (6.5). Suppose $F \in \mathrm{ERV}(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$, $\mathbb{E}L_1 > 0$, and $\varphi_\pi(\beta + \varepsilon) < 0$ for some $\varepsilon > 0$. Then, it holds for every $T \in \Lambda$ that*

$$\mathbb{P}\left(V_{\pi,T} > x\right) \sim \int_{0-}^{T} \overline{F}\left(x\mathrm{e}^{L_{\pi,t}}\right) \mathrm{d}\lambda_t. \tag{6.10}$$

When $\alpha = \beta$ and $T = \infty$, it can be derived from relation (6.10) that

$$\mathbb{P}\left(V_{\pi,\infty} > x\right) \sim \frac{\mathbb{E}\mathrm{e}^{\varphi_\pi(\alpha)\theta}}{1 - \mathbb{E}\mathrm{e}^{\varphi_\pi(\alpha)\theta}} \overline{F}(x). \tag{6.11}$$

Klüppelberg and Kostadinova (2008) obtained relation (6.11) for the special case that $(N_t)_{t\geq 0}$ is a Poisson process by applying a key result of Grey (1994); see Theorem 4.6(a) of Klüppelberg and Kostadinova (2008). See also Heyde and Wang (2009) for a result of the finite-time ruin probability similar to (6.11) but for $(N_t)_{t\geq 0}$ being a Poisson process.

To prove Theorem 6.9, we first prepare two lemmas. The following lemma was obtained by Wang and Tang (2006):

Lemma 6.10. *Let* $\{X_k, k = 1, 2, \ldots\}$ *be a sequence of i.i.d. nonnegative random variables with common distribution* F *on* $[0, \infty)$ *and* $\{\omega_k, k = 1, 2, \ldots\}$ *another sequence of positive random variables. Suppose the two sequences are mutually independent. If* $F \in \mathrm{ERV}(-\alpha, -\beta)$ *for some* $0 < \alpha \leq \beta < \infty$ *and*

$$\mathbb{E}\left(\sum_{k=1}^{\infty} \omega_k^u\right)^v < \infty$$

for some $0 < u < \min\{1, \alpha\}$ *and* $v > \beta/u$. *Then, it holds that*

$$\mathbb{P}\left(\sum_{k=1}^{\infty} \omega_k X_k > x\right) \sim \sum_{k=1}^{\infty} \mathbb{P}\left(\omega_k X_k > x\right).$$

The next lemma, obtained by Maulik and Zwart (2006), concerns the exponential functional of a Lévy process:

Lemma 6.11. *Let* $(L_t)_{t\geq 0}$ *be a Lévy process with Laplace exponent* $\varphi(\cdot)$ *and* $W = \int_0^\infty \mathrm{e}^{-L_t}\mathrm{d}t$.

(i) $W < \infty$ *almost surely if and only if* $L_t \stackrel{a.s.}{\to} \infty$ *as* $t \to \infty$;

(ii) If $s > 0$ *and* $\varphi(s) < 0$, *then* $\mathbb{E}W^s < \infty$.

Now we are ready to give the proof of Theorem 6.9.

Proof of Theorem 6.9. Throughout this proof, $0 < t \leq T$ is understood as $0 < t < \infty$ when $T = \infty$.

First we derive an upper asymptotic bound for $\mathbb{P}(V_{\pi,T} > x)$. It is clear from the definition of $V_{\pi,T}$ in (6.5) that

$$\mathbb{P}(V_{\pi,T} > x) \leq \mathbb{P}\left(\sum_{k=1}^{\infty} X_k \mathrm{e}^{-L_{\pi,\tau_k}} 1_{(\tau_k \leq T)} > x\right).$$

To apply Lemma 6.10, we see that, for all u and v such that $0 < u < \min\{1, \alpha\}$ and $\beta < uv < \beta + \varepsilon$,

$$\mathbb{E}\left(\sum_{k=1}^{\infty} \mathrm{e}^{-uL_{\pi,\tau_k}} 1_{(\tau_k \leq T)}\right)^v \leq \mathbb{E}\left(\sum_{k=1}^{\infty} ck^{-2}c^{-1}k^2 \mathrm{e}^{-uL_{\pi,\tau_k}}\right)^v, \tag{6.12}$$

where c is the constant such that $c\sum_{k=1}^{\infty} k^{-2} = 1$. By Jensen's inequality, it holds almost surely that

$$\left(\sum_{k=1}^{\infty} ck^{-2}c^{-1}k^2 \mathrm{e}^{-uL_{\pi,\tau_k}}\right)^v \leq \sum_{k=1}^{\infty} ck^{-2}\left(c^{-1}k^2 \mathrm{e}^{-uL_{\pi,\tau_k}}\right)^v.$$

Since $\varphi_\pi(0) = 0$, $\varphi_\pi(\beta + \varepsilon) < 0$, and $\varphi_\pi(\cdot)$ is strictly convex, $\varphi_\pi(s) < 0$ for all $s \in (0, \beta + \varepsilon]$. Following (6.12), we have

$$\begin{aligned}
\mathbb{E}\left(\sum_{k=1}^{\infty} \mathrm{e}^{-uL_{\pi,\tau_k}} 1_{(\tau_k \leq T)}\right)^v &\leq c^{1-v} \sum_{k=1}^{\infty} k^{2v-2} \mathbb{E}\mathrm{e}^{-uvL_{\pi,\tau_k}} \\
&= c^{1-v} \sum_{k=1}^{\infty} k^{2v-2} \mathrm{e}^{\varphi_\pi(uv)k} < \infty.
\end{aligned}$$

Therefore, by Lemma 6.10,

$$\mathbb{P}(V_{\pi,T} > x) \lesssim \sum_{k=1}^{\infty} \mathbb{P}\left(X_k \mathrm{e}^{-L_{\pi,\tau_k}} 1_{(\tau_k \leq T)} > x\right) = \int_{0-}^{T} \overline{F}\left(x\mathrm{e}^{L_{\pi,t}}\right) \mathrm{d}\lambda_t.$$

Then we derive the corresponding lower asymptotic bound for $\mathbb{P}\left(V_{\pi,T} > x\right)$. Since $\mathbb{E}L_1 > 0$, by Lemmas 6.7(ii) and 6.11(i) we know that $W_\pi = \int_0^\infty \mathrm{e}^{-L_{\pi,t}}\mathrm{d}t$ is a finite random variable. Likewise, from (6.5) we have

$$\mathbb{P}\left(V_{\pi,T} > x\right) \geq \mathbb{P}\left(\sum_{k=1}^{\infty} X_k \mathrm{e}^{-L_{\pi,\tau_k}} 1_{(\tau_k \leq T)} > x + pW_\pi\right).$$

For arbitrarily fixed $\varepsilon_1 > 0$, it holds that

$$\begin{aligned}\mathbb{P}\left(V_{\pi,T} > x\right) &\geq \mathbb{P}\left(\sum_{k=1}^{\infty} X_k \mathrm{e}^{-L_{\pi,\tau_k}} 1_{(\tau_k \leq T)} > (1+\varepsilon_1)x\right) - \mathbb{P}\left(pW_\pi > \varepsilon_1 x\right) \\ &= I_1(x) - I_2(x). \end{aligned} \tag{6.13}$$

Similarly to the above,

$$I_1(x) \sim \int_0^T \overline{F}\left((1+\varepsilon_1)x\mathrm{e}^{L_{\pi,t}}\right) \mathrm{d}\lambda_t. \tag{6.14}$$

Arbitrarily choose some $\varepsilon_2 > 0$ such that the inequality

$$\frac{\overline{F}((1+\varepsilon_1)x)}{\overline{F}(x)} \geq (1+\varepsilon_1)^{-\beta-1}$$

holds for all $x > 1/\varepsilon_2$. Then, uniformly for all $t > 0$,

$$\begin{aligned}\overline{F}\left((1+\varepsilon_1)x\mathrm{e}^{L_{\pi,t}}\right) &\geq \mathbb{P}\left(X\mathrm{e}^{-L_{\pi,t}} > (1+\varepsilon_1)x, \mathrm{e}^{-L_{\pi,t}} \leq \varepsilon_2 x\right) \\ &\gtrsim (1+\varepsilon_1)^{-\beta-1}\mathbb{P}\left(X\mathrm{e}^{-L_{\pi,t}} > x, \mathrm{e}^{-L_{\pi,t}} \leq \varepsilon_2 x\right) \\ &\geq (1+\varepsilon_1)^{-\beta-1}\left(\overline{F}\left(x\mathrm{e}^{L_{\pi,t}}\right) - \mathbb{P}\left(\mathrm{e}^{-L_{\pi,t}} > \varepsilon_2 x\right)\right) \\ &\geq (1+\varepsilon_1)^{-\beta-1}\left(\overline{F}\left(x\mathrm{e}^{L_{\pi,t}}\right) - \frac{\mathrm{e}^{\varphi_\pi(\beta+\varepsilon)t}}{(\varepsilon_2 x)^{\beta+\varepsilon}}\right). \end{aligned}$$

It follows from (6.14) that

$$\begin{aligned} I_1(x) &\gtrsim (1+\varepsilon_1)^{-\beta-1}\left(\int_0^T \overline{F}\left(x\mathrm{e}^{L_{\pi,t}}\right)\mathrm{d}\lambda_t - \int_0^T \frac{\mathrm{e}^{\varphi_\pi(\beta+\varepsilon)t}}{(\varepsilon_2 x)^{\beta+\varepsilon}}\mathrm{d}\lambda_t\right) \\ &= (1+\varepsilon_1)^{-\beta-1}\int_0^T \overline{F}\left(x\mathrm{e}^{L_{\pi,t}}\right)\mathrm{d}\lambda_t - o\left(\overline{F}(x)\right). \end{aligned}$$

By Lemma 6.11(ii), $\mathbb{E}W_{\pi}^{\beta+\varepsilon}<\infty$. Hence,

$$I_2(x)=o\left(\overline{F}(x)\right).$$

Substitute these estimates into (6.13) to obtain that

$$\mathbb{P}\left(V_{\pi,T}>x\right)\gtrsim(1+\varepsilon_1)^{-\beta-1}\int_0^T\overline{F}\left(x\mathrm{e}^{L_{\pi,t}}\right)\mathrm{d}\lambda_t-o\left(\overline{F}(x)\right).$$

Since it holds for arbitrarily fixed $M>0$ that

$$\begin{aligned}\int_0^T\overline{F}\left(x\mathrm{e}^{L_{\pi,t}}\right)\mathrm{d}\lambda_t&\geq\int_0^T\mathbb{P}\left(X\mathrm{e}^{-L_{\pi,t}}>x,\sup_{0<t\leq T}L_{\pi,t}\leq M\right)\mathrm{d}\lambda_t\\&\geq\lambda_T\overline{F}\left(x\mathrm{e}^{M}\right)\mathbb{P}\left(\sup_{0<t\leq T}L_{\pi,t}\leq M\right)\\&\asymp\overline{F}(x),\end{aligned}$$

where $a(x)\asymp b(x)$ means that $\limsup_{x\to\infty}a(x)/b(x)<\infty$ and $\limsup_{x\to\infty}b(x)/a(x)<\infty$, it follows that

$$\mathbb{P}\left(V_{\pi,T}>x\right)\gtrsim(1+\varepsilon_1)^{-\beta-1}\int_0^T\overline{F}\left(x\mathrm{e}^{L_{\pi,t}}\right)\mathrm{d}\lambda_t.$$

Since $\varepsilon_1>0$ can be arbitrarily small, it follows that

$$\mathbb{P}\left(V_{\pi,T}>x\right)\gtrsim\int_0^T\overline{F}\left(x\mathrm{e}^{L_{\pi,t}}\right)\mathrm{d}\lambda_t.$$

This ends the proof of Theorem 6.9. □

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