Asymptotic Tail Probabilities of Risk Processes in Insurance and Finance

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Heavy- and Light-tailed Distribution Classes I

The long-tailed distribution class

$$\mathcal{L} = \left\{ F \text{ on } (-\infty, \infty) : \lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1 \text{ for all } y \right\}.$$

The subexponential distribution class

$$\mathcal{S} = \left\{ F \text{ on } (-\infty, \infty) : \lim_{x\to\infty} \frac{\overline{F_+^{2*}}(x)}{\overline{F_+}(x)} = 2 \right\}.$$

The class $\mathcal{A}$ is defined as

$$\mathcal{A} = \mathcal{S} \cap \left\{ F \text{ on } (-\infty, \infty) : \limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} < 1 \text{ for some } v > 1 \right\}.$$

Heavy- and Light-tailed Distribution Classes II

The class $\mathcal{R}_{-\alpha}$ for some $0 \leq \alpha \leq \infty$ is defined as

$$\mathcal{R}_{-\alpha} = \left\{ F \text{ on } (-\infty, \infty) : \lim_{x \to \infty} \frac{\overline{F}(vx)}{\overline{F}(x)} = v^{-\alpha} \text{ for all } v > 1 \right\}.$$

The class $\mathcal{L}(\alpha)$ for some $\alpha > 0$ is defined as

$$\mathcal{L}(\alpha) = \left\{ F \text{ on } (-\infty, \infty) : \lim_{x \to \infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = e^{\alpha y} \text{ for all } y \right\}.$$

The class $\mathcal{S}(\alpha)$ for some $\alpha > 0$ is defined as

$$\mathcal{S}(\alpha) = \mathcal{L}(\alpha) \cap \left\{ F \text{ on } (-\infty, \infty) : \lim_{x \to \infty} \frac{\overline{F_+^{2*}}(x)}{\overline{F_+}(x)} \text{ exists and is finite} \right\}.$$

Chapter 3: Model Description

Assume that there is a constant force of interest $r > 0$. We model discounted aggregate claims as the stochastic process

$$D_r(t) = \sum_{k=1}^{\infty} X_k \mathrm{e}^{-r\tau_k} 1_{(\tau_k \le t)}, \qquad t \ge 0,$$

in which we make the following standard assumptions:

- X_1, X_2, ..., are i.i.d., nonnegative, with distribution F;
- $0 < \tau_1 < \tau_2 < \cdots$ are claim arrival times constituting a renewal counting process

  $$N_t = \#\{k = 1, 2, \ldots : \tau_k \le t\}, \qquad t \ge 0,$$

  with renewal function $\lambda_t = \mathbb{E} N_t$;
- $\{X_1, X_2, \ldots\}$ and $\{\tau_1, \tau_2, \cdots\}$ are mutually independent.

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The study of $\mathbb{P}\left(D_r(t) > x\right)$, the tail probability of discounted aggregate claims, is of much practical interest in insurance mathematics.

- It provides an easy and precise approximation when measuring the risk of large losses via Value-at-Risk or Conditional Tail Expectation.
- It usually plays a crucial role in pricing some insurance products.
- $D_r(t)$ is a special case of the stochastic integral

$$Z_t = \int_{0-}^{t} \mathrm{e}^{-R_s} \mathrm{d}P_s, \qquad t \geq 0,$$

where $(R_t)_{t\geq 0}$ and $(P_t)_{t\geq 0}$ are two independent stochastic processes.

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Denote $\Lambda = \{t : \lambda_t > 0\} = \{t : \Pr(\tau_1 \leq t) > 0\}$.

Theorem (3.3)

If $F \in \mathcal{S}$, then the relation

$$\mathbb{P}\left(D_r(t) > x\right) \sim \int_{0-}^{t} \overline{F}(xe^{rs})\mathrm{d}\lambda_s, \qquad as\ x \to \infty, \tag{1}$$

holds uniformly for all $t \in \Lambda_T = \Lambda \cap [0, T]$ for arbitrarily fixed $T \in \Lambda$. That is to say,

$$\lim_{x\to\infty} \sup_{t\in\Lambda_T} \left| \frac{\mathbb{P}\left(D_r(t) > x\right)}{\int_{0-}^{t} \overline{F}(xe^{rs})\mathrm{d}\lambda_s} - 1 \right| = 0.$$

In the next two main results below, we extend the set over which relation (1) holds uniformly to the maximal set Λ.

Theorem (3.4)

If $F \in \mathcal{A}$ and $\mathbb{P}(\tau_1 > \delta) = 1$ for some $\delta > 0$, then relation (1) holds uniformly for all $t \in \Lambda$.

Theorem (3.5)

If $F \in \mathcal{S} \cap \mathcal{R}_{-\alpha}$ for some $0 < \alpha \leq \infty$ and τ_1 is positive, then relation (1) holds uniformly for all $t \in \Lambda$.

Restricting on the compound Poisson risk model, the last theorem of Chapter 3 gives the most explicit expression for the asymptotic tail probability of $D_r(t)$:

Theorem (3.6)

Suppose $(N_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$. If $F \in \mathcal{S}$ and $F_e \in \mathcal{A}$, then the relation

$$\mathbb{P}\left(D_r(t) > x\right) \sim \lambda \int_0^t \overline{F}(x\mathrm{e}^{rs})\mathrm{d}s, \qquad \text{as } x \to \infty,$$

holds uniformly for all $t \in (0, \infty]$.

In Chapter 4, we study the maximum exceedance of a sequence of random variables over a renewal threshold. Precisely, suppose

- $\{(X_n, Y_n), n = 1, 2, \ldots\}$ are i.i.d. with generic random pair (X, Y);
- $\mathbb{E}X = \mu > 0$ and Y follows a distribution F on $(-\infty, \infty)$.

We are interested in the tail behavior of the maximum

$$M = \sup_{n \geq 1} \left(Y_n - \sum_{i=1}^{n-1} X_i \right). \tag{2}$$

This study has been found applicable in many fields, such as corporate finance, insurance risk, production systems. See Robert (2005; *J. Appl. Probab.*), Araman and Glynn (2006; *Ann. Appl. Probab.*), and Palmowski and Zwart (2007; *J. Appl. Probab.*).

Here is the main result of Chapter 4:

Theorem (4.1)

Consider the i.i.d. sequence $\{(X_n, Y_n), n = 1, 2, \ldots\}$ *and the maximum* M *defined in (2). The relation*

$$\lim_{x\to\infty} \frac{\mathbb{P}(M > x)}{\int_x^\infty \overline{F}(y)\mathrm{d}y} = \frac{\alpha}{1 - \mathbb{E}e^{-\alpha X}}$$

holds under one of the following groups of conditions:
(i) $F_e \in \mathcal{L}(\alpha)$ *for some* $\alpha \geq 0$, $\mathbb{E}X^2 < \infty$, *and* $\mathbb{E}e^{-\beta X} < 1$ *for some* $\beta > \alpha$;
(ii) $F_e \in \mathcal{S}(\alpha)$ *for some* $\alpha \geq 0$, $\mathbb{P}(-X > x) = o(\overline{F}(x))$, *and* $\mathbb{E}e^{-\alpha X} < 1$ *provided* $\alpha > 0$.

Consider an incorporated firm. Suppose Z_n, $n = 1, 2, \ldots$, are i.i.d. such that

- Z_n represents its profit during the nth fiscal year;
- ΔZ_n is the money paid out to shareholders if $Z_n > 0$;
- $-\epsilon Z_n$ is the money raised by issuing new equity if $Z_n < 0$;
- $X_n = Z_n - \Delta Z_n^+ + \epsilon Z_n^-$ is the increment of capital amount during the nth fiscal year;
- $R_n = x + \sum_{i=1}^{n-1} X_i + Z_n$ is the capital amount up to the end of the nth fiscal year, with $R_0 = x > 0$ being its initial capital.

The probability that the capital surplus of the firm goes below some critical level b is given by

$$\psi(x,b) = \mathbb{P}\left\{\sup_{n\geq 1}\left(-Z_n - \sum_{i=1}^{n-1} X_i\right) > x - b\right\}.$$

Assume that Z follows the distribution

$$\mathbb{P}(Z \leq z) = \begin{cases} 1 - \frac{\alpha}{\tilde{\alpha}+\alpha}\mathrm{e}^{-\tilde{\alpha}z}, & z \geq 0, \\ \frac{\tilde{\alpha}}{\tilde{\alpha}+\alpha}\mathrm{e}^{\alpha z}, & z < 0, \end{cases}$$

for some $\alpha > \tilde{\alpha} > 0$. Choosing appropriate Δ and ϵ and by Theorem 4.1(i), we have

$$\lim_{x-b\to\infty} \frac{\psi(x,b)}{\mathrm{e}^{-\alpha(x-b)}} = \frac{\tilde{\alpha}^2\epsilon + \alpha\tilde{\alpha}(1-\Delta)\epsilon}{\alpha^2(1-\Delta)\epsilon - \alpha\tilde{\alpha}(1-\Delta)(1-\epsilon) - \tilde{\alpha}^2(1-\epsilon)}.$$

Let $U = (U_t)_{t\geq 0}$ be a stochastic process, with $U_0 = x > 0$, representing the underlying surplus process in a world without economic factors (tax, reinsurance, investment, etc.) of an insurance company. We introduce periodic taxation as well as compensation to the risk model. Given the company survives at time n,

- it pays tax at rate $\gamma \in [0, 1)$ on its net income during the period $(n-1, n]$; or
- it gets compensation at rate $\delta \in [0, 1)$ on its net loss during the period $(n-1, n]$.

In this chapter, we investigate the influence of such taxation and compensation rule on the asymptotic behavior of the ruin probability $\psi_{\gamma,\delta}(x)$.

The underlying loss process without tax and compensation is

$$L_t = x - U_t, \qquad t \geq 0.$$

For each $n = 1, 2, \ldots$, the maximal net loss and the net loss of the company within the period $(n-1, n]$ are, respectively,

$$Y_n = \sup_{n-1 \leq t \leq n} \left(L_t - L_{n-1} \right), \qquad Z_n = L_n - L_{n-1}.$$

After introducing the periodic taxation at rate $0 \leq \gamma < 1$ and compensation at rate $0 \leq \delta < 1$, the loss of the company within the period $(n-1, n]$ becomes

$$X_n = Z_n + \gamma Z_n^- - \delta Z_n^+ = (1-\delta) Z_n^+ - (1-\gamma) Z_n^-.$$

Then, the ruin probability in this situation is equal to

$$\psi_{\gamma,\delta}(x) = \Pr\left(\sup_{n\geq 1}\left(\sum_{k=1}^{n-1} X_k + Y_n\right) > x\right).$$

Assumptions on the loss process L:

- L is a Lévy process with triplet (a, σ^2, ρ) and mean $\mathrm{E}L_1 = -\mu < 0$. Denote $\Pi(\cdot) = (\overline{\rho}(1))^{-1}\rho(\cdot)1_{(1,\infty)}$.
- Choose $\gamma \in [0,1)$ and $\delta \in [0,1)$ such that

$$\mathrm{E}X = (1-\delta)\mu_+ - (1-\gamma)\mu_- < 0. \tag{3}$$

So the insurance company still has positive expected profits under such taxation and compensation and that the ruin is not certain.

Theorem (5.1)

Consider the Lévy insurance model described above. If both Π and Π_e belong to the class $\mathcal{S}$, then for every $0 \leq \gamma < 1$ and $0 \leq \delta < 1$ for which relation (3) holds, we have

$$\psi_{\gamma,\delta}(x) \sim \frac{1}{(1-\gamma)\mu_{-} - (1-\delta)\mu_{+}} \int_{x}^{\infty} \overline{\rho}(y)\mathrm{d}y.$$

Theorem (5.4)

Consider the Lévy insurance model described above. Assume $\mathbb{E}L_1^2 < \infty$ *and* $\Pi \in \mathcal{S}(\alpha)$ *for some* $\alpha > 0$. *If* $0 \le \gamma < 1$ *and* $0 < \delta < 1$ *are such that*

$$\mathbb{E}e^{\alpha'\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)} < 1 \tag{4}$$

for some $\alpha' > \alpha$, *then,*

$$\psi_{\gamma,\delta}(x) \sim \frac{C_\alpha}{1 - \mathbb{E}e^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}}\overline{\rho}(x),$$

where the constant C_α *is defined as*

$$C_\alpha = \lim_{x\to\infty} \frac{\mathbb{P}\left(\sup_{0\le t\le 1} L_t > x\right)}{\overline{\rho}(x)} \in (0, \infty).$$

Corollary (5.7)

Consider the Lévy insurance model described above. Assume

$$L_t = V_t - pt, \qquad t \geq 0,$$

where $p > 0$ *represents the constant premium rate and* $V = (V_t)_{t \geq 0}$ *is a compound Poisson process. Suppose that* F *has a bounded density* $f \in \mathcal{S}d(\alpha)$ *for some* $\alpha > 0$ *and that condition (4) holds. Then,*

$$\psi_{\gamma,\delta}(x) \sim \frac{\lambda C_\alpha}{1 - \mathbb{E}e^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}} \overline{F}(x)$$

with the constant C_α *given by (5.17) on page 88 in the thesis.*

Theorem (5.9)

Consider the Lévy insurance model described above. Assume $\mathbb{E}L_1^2 < \infty$, $\Pi \in \mathcal{L}(\alpha)$ *for some* $\alpha > 0$, *and* $\overline{\Pi}(x) = o\left(\overline{\Pi^{2*}}(x)\right)$. *If* $0 \le \gamma < 1$ *and* $0 < \delta < 1$ *are such that condition (4) holds, then,*

$$\psi_{\gamma,\delta}(x) \sim \frac{1}{1 - \mathbb{E}e^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}}\mathbb{P}(L_1 > x). \tag{5}$$

Corollary (5.11)

Consider the Lévy insurance model described above. Assume

$$L_t = \Gamma_t - pt, \qquad t \geq 0,$$

where $p > 0$ *and* $\Gamma = (\Gamma_t)_{t \geq 0}$ *is a gamma process with parameters* $\alpha, \beta > 0$*. If* $0 \leq \gamma < 1$ *and* $0 < \delta < 1$ *are such that condition (4) holds, then,*

$$\psi_{\gamma,\delta}(x) \sim \frac{\alpha^{\beta-1} (x+p)^{\beta-1} \mathrm{e}^{-\alpha(x+p)}}{\left(1 - \mathbb{E}\mathrm{e}^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}\right) \Gamma(\beta)}.$$

Corollary (5.12)

Consider the Lévy insurance model described above. Assume

$$L_t = V_t - pt, \qquad t \geq 0,$$

where $p > 0$ *represents the constant premium rate and* $V = (V_t)_{t \geq 0}$ *is a compound Poisson process. Suppose that* F *is an exponential distribution with mean* $1/\alpha$*. If* $0 \leq \gamma < 1$ *and* $0 < \delta < 1$ *are such that condition (4) holds, then,*

$$\psi_{\gamma,\delta}(x) \sim \frac{2\sqrt{\lambda/\pi}}{1 - \mathbb{E}e^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}} \int_0^{\frac{\pi}{2}} \Phi\left(\sqrt{2\lambda}\cos\theta - \sqrt{2\alpha(x+p)}\right) \mathrm{d}\theta,$$

where $\Phi(\cdot)$ *is the standard normal distribution.*

In Chapter 6, we study the impact of risky investment on the discounted net loss process in the renewal risk model. Consider the renewal risk model in which the surplus process of an insurance company is

$$U_t = u + pt - S_t, \qquad t \geq 0,$$

where $u > 0$ is the initial surplus level, $p > 0$ is the constant premium rate, and the aggregate claims process is defined as $S_t = \sum_{k=1}^{N_t} X_k$, $t \geq 0$.

Suppose the insurance company is allowed to invest its surplus into two assets: a bond with constant force of interest $r > 0$ and a stock whose price is modeled by an exponential Lévy process. The two assets have their price processes as, respectively,

$$X_0(t) = \mathrm{e}^{rt}, \qquad X_1(t) = \mathrm{e}^{L_t}, \qquad t \geq 0.$$

We assume the so-called constant mix investment strategy. Denote by $\pi \in [0,1]$ the constant fraction of the surplus invested into the stock. Then, the investment process is defined as the solution of the SDE

$$\mathrm{d}X_\pi(t) = X_\pi(t-)\left((1-\pi)r\mathrm{d}t + \pi\mathrm{d}\widehat{L}_t\right), \qquad t > 0,$$

with $X_\pi(0) = 1$. The solution for the above SDE is given by

$$X_\pi(t) = \mathcal{E}\left((1-\pi)r\mathrm{d}t + \pi\mathrm{d}\widehat{L}_t\right) = \mathrm{e}^{L_{\pi,t}}, \qquad t \geq 0,$$

where $\mathcal{E}$ is the stochastic exponential and $(L_{\pi,t})_{t\geq 0}$ is a Lévy process whose triplet is determined by $(L_t)_{t\geq 0}$ and π.

The integrated risk process is defined as the solution to the SDE

$$\mathrm{d}U_{\pi,t} = p\mathrm{d}t - \mathrm{d}S_t + U_{\pi,t-}\left((1-\pi)r\mathrm{d}t + \pi\mathrm{d}\widehat{L}_t\right), \qquad t > 0,$$

with $U_{\pi,0} = u$. Our first result is the following lemma:

Lemma (6.3)

The above SDE has the solution

$$U_{\pi,t} = \mathrm{e}^{L_{\pi,t}}\left(u + \int_0^t \mathrm{e}^{-L_{\pi,v}}\left(p\mathrm{d}v - \mathrm{d}S_v\right)\right), \qquad t \geq 0.$$

The discounted net loss process is defined as

$$V_{\pi,t} = u - \mathrm{e}^{-L_{\pi,t}} U_{\pi,t} = \int_0^t \mathrm{e}^{-L_{\pi,v}} \left(\mathrm{d}S_v - p\mathrm{d}v\right), \qquad t \geq 0.$$

We are interested in the tail behavior of a stationary discounted net loss process. Denote for $k = 1, 2, \ldots$,

$$\begin{aligned} A_{\pi,k} &= \int_{\tau_{k-1}}^{\tau_k} \mathrm{e}^{-\left(L_{\pi,v} - L_{\pi,\tau_{k-1}}\right)} \left(\mathrm{d}S_v - p\mathrm{d}v\right), \\ B_{\pi,k} &= \mathrm{e}^{-\left(L_{\pi,\tau_k} - L_{\pi,\tau_{k-1}}\right)}. \end{aligned}$$

Then $(A_{\pi,k}, B_{\pi,k})$, $k = 1, 2, \ldots$, form a sequence of i.i.d. random pairs with generic random pair

$$(A_\pi, B_\pi) = \left(X\mathrm{e}^{-L_{\pi,\theta}} - p \int_0^\theta \mathrm{e}^{-L_{\pi,v}} \mathrm{d}v, \mathrm{e}^{-L_{\pi,\theta}} \right).$$

Theorem (6.6)

Suppose $\mathbb{E}X < \infty$, $\mathbb{E}L_1 > 0$, *and* $a_\pi < \lambda$.
(i) We have

$$V_{\pi,\tau_k} \xrightarrow{a.s.} V_{\pi,\infty} = \sum_{m=1}^{\infty} A_{\pi,m} \prod_{j=1}^{m-1} B_{\pi,j}, \qquad as\ k \to \infty,$$

where the series of the right-hand side converges absolutely with probability 1. *Moreover,* $V_{\pi,\infty}$ *satisfies the stochastic difference equation*

$$V_{\pi,\infty} \stackrel{d}{=} A_\pi + B_\pi V_{\pi,\infty},$$

where $V_{\pi,\infty}$ *and* (A_π, B_π) *are independent.*
(ii) $V_{\pi,t}$ *converges a.s. to some finite random variable, denoted by* $V^c_{\pi,\infty}$, *if and only if* V_{π,τ_k} *does and*

$$V_{\pi,\infty} = V^c_{\pi,\infty} \qquad a.s.$$

When the claim-size distribution F belongs to the class $\mathcal{R}_{-\alpha}$ for some $\alpha > 0$, by applying the main result of Grey (1994), we obtain the following theorem:

Theorem (6.7)

Let $\kappa = \kappa(\pi) \in (1, \infty)$ be the unique value satisfying $\mathbb{E}e^{-L_{\pi,\kappa}} = 1$. Assume that $F \in \mathcal{R}_{-\alpha}$ with $-\alpha \in (-\kappa(\pi), -1)$. Then,

$$\mathbb{P}\left(V_{\pi,\infty} > x\right) \sim \frac{\mathbb{E}e^{-\alpha L_{\pi,\theta}}}{1 - \mathbb{E}e^{-\alpha L_{\pi,\theta}}} \overline{F}(x).$$