Series Expansions for Matrix Inverses

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Abstract – Various convergent matrix series formulae for the inverse of positive definite matrices are derived starting from series expansions of a simple matrix exponential function. The structure of these series assures a stable numerical computation since successive terms of the series are obtained through a multiplication with a matrix whose condition becomes better and better as the computation progresses.

Keywords – matrix inverse; solution of linear systems.

INTRODUCTION

Matrix series expansions were previously presented [1] for the solution of first order vector differential equations or for the solution of linear systems whose matrices are well-conditioned or are preliminarily preconditioned.

MATRICES SERIES FORMULAE

Consider a nonsingular matrix \( C \in \mathbb{R}^{n \times n} \) and a simple matrix function \( v^{C^{-1}} \) where \( I \) is the identity matrix of order \( n \) and \( v \) is a real variable, \( v \in (0, 2) \), which can be expanded in series as

\[
v^{C^{-1}} = \left[1 - (1 - v)\right]^{-1} = I + \frac{(1 - v)}{1!}(I - C) + \frac{(1-v)^2}{2!}(I - C)(2I - C) + \cdots
\]

This power series is convergent and integrating it both sides from \( v \) to \( v = 1 \) we have

\[
\frac{v^{C^{-1}}}{C} = (1 - v)I + \frac{(1 - v)^2}{2!}(I - C) + \frac{(1 - v)^3}{3!}(I - C)(2I - C) + \cdots
\]  

or

\[
\frac{1}{C}(I - v^{C^{-1}}) = (1 - v)I + \sum_{k=1}^{\infty} \frac{(1 - v)^{k+1}}{k+1}(I - C) \left( I - \frac{C}{2} \right) \cdots \left( I - \frac{C}{k} \right)
\]

For a positive definite \( C \) and \( v \to 0 \) this gives

\[
C^{-1} = I + \sum_{k=1}^{\infty} \frac{1}{k+1}(I - C) \left( I - \frac{C}{2} \right) \cdots \left( I - \frac{C}{k} \right)
\]  

Expression (4) was derived in [1] using a different approach.

Integrating again (3) from \( v \) to \( v = 1 \) we obtain for \( v \to 0 \)

\[
(I + C)^{-1} = \frac{1}{2} I + \sum_{k=1}^{\infty} \frac{1}{(k+2)(k+3) \cdots (k+p+1)}(I - C) \left( I - \frac{C}{2} \right) \cdots \left( I - \frac{C}{k} \right)
\]  

under the condition that \( I + C \) is positive definite. This particular formula can also be obtained directly by replacing \( C \) with \( I + C \) in (4). Repeated integrations lead to a general formula, namely,

\[
\left( I + \frac{C}{p} \right)^{-1} = \frac{p}{p+1} I + \frac{p+1}{p+2} \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2) \cdots (k+p+1)}(I - C) \left( I - \frac{C}{2} \right) \cdots \left( I - \frac{C}{k} \right)
\]  

valid for any \( C \), positive definite or not, as long as \( I + C/p \) is positive definite. We notice that a sufficiently big \( p \) makes \( I + C/p \) positive definite even if \( C \) is not. This allows to use this formula for solving systems of linear algebraic equations as shown in the next Section. Obviously, for same \( C \) the convergence of the series increases rapidly as \( p \) increases.

An expression containing the same infinite series as in (6) can be derived by replacing \( C \) with \( pl + C \) in (4) but formula (6) is the simplest to be used for numerical computations. Replacing \( C \) by \( pC/q \) we obtain a more general formula, i.e.,

\[
\left( I + \frac{C}{q} \right)^{-1} = \frac{p}{q+1} I + \frac{q}{q+2} \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2) \cdots (k+p+1)}(I - \frac{pC}{q}) \left( I - \frac{C}{2} \right) \cdots \left( I - \frac{C}{k} \right)
\]

which is valid for any positive or negative number \( q \) (\( q \neq 0 \), not necessarily an integer, as long as \( I + C/q \) is positive definite. The series in (6) and (7) are highly convergent and for a given \( p \) they have same coefficients independently of the value of \( q \). Notice that for \( q = 1, 2, \ldots, p - 1 \) and a given \( p \) the series in (7) are much more convergent than the series in (6) with \( p \) replaced by \( 1, 2, \ldots, p - 1 \), respectively. We remark that each term of the series in (4) – (7) is obtained from the previous one by multiplication with a better and better conditioned matrix which tends to the identity matrix as \( k \) increases.
APPLICATION TO THE SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

For a system $Ax = b$, where $A \in \mathbb{R}^{n \times n}$ is a nonsingular positive definite matrix and $b \in \mathbb{R}^n$ a column vector, the solution $x = A^{-1}b$ can be expressed using the formulae (4) – (7) for the inverse. With (6), for example,

$$A^{-1} = \frac{p}{p+1}I + \frac{p}{p+1}\sum_{k=p+1}^{\infty} \frac{1}{(k+1)(k+2)\cdots(k+p+1)}(I-C)\left(I - \frac{C}{k}\right)$$  \hspace{1cm} (8)

for any $p = 1, 2, \ldots$, where $C = p(A-I)$. In terms of $A$ (8) becomes

$$A^{-1} = \frac{p}{p+1}I + \frac{p}{p+1}\sum_{k=p+1}^{\infty} \frac{1}{(k+1)(k+2)\cdots(k+p+1)}(I-C)\left(I - \frac{C}{k}\right)$$  \hspace{1cm} (9)

or, replacing $A$ with $Ar/p$, where $r$ is an arbitrary positive number,

$$A^{-1} = \frac{r}{p+1}I + \frac{r}{p+1}\sum_{k=p+1}^{\infty} \frac{1}{(k+1)(k+2)\cdots(k+p+1)}(I-C)\left(I - \frac{C}{k}\right)$$  \hspace{1cm} (10)

For $p = 0$ and $r = 1$, (10) is the special case (4). To generate numerical results for the solution $x = A^{-1}b$ some values for the parameters $p$ and $r$ are chosen and the infinite series are truncated to a number $N$ of terms depending on the desired accuracy. With (10), for example,

$$x = \frac{r}{p+1}b + \sum_{k=1}^{N} c_k b_k$$  \hspace{1cm} (11)

where

$$c_k = \frac{r}{k+p+1}, \quad b_k = \left(I - \frac{ra}{p+1}\right)\left(I - \frac{ra}{p+2}\right)\cdots\left(I - \frac{ra}{p+k}\right), \quad b_0 = b$$  \hspace{1cm} (12)

Each new vector $b_k$ is calculated by multiplying $A$ with the previous one, then multiplying the resulting vector with $r/(p + k)$ and subtracting from $b_{k-1}$.

The matrix inverses in (4) – (10) can be expressed with only the odd or the even terms in the respective series. For (4) this is done by using the expansion [3]

$$I + \sum_{k=1}^{\infty} \frac{(-1)^k}{k+1}(I-C)\left(I - \frac{C}{2}\right)\cdots\left(I - \frac{C}{k}\right) = (\ln 2)s(C)$$  \hspace{1cm} (13)

where $s(C)$ is a rapidly convergent series,

$$s(C) = I + \frac{C(\ln 2)}{2!} + \frac{(\ln 2)^2}{3!}C + \cdots$$  \hspace{1cm} (14)

From (13) and (4) multiplied by $\alpha, \alpha \in (-1, 1)$, we have

$$\alpha C^{-1} = (1+\alpha)I - (\ln 2)s(\alpha C)$$  \hspace{1cm} (15)

If $\alpha = -1$ we obtain

$$C^{-1} = (\ln 2)s(C) + \sum_{k=1}^{\infty} \frac{1}{(k+1)(I-C)\left(I - \frac{C}{2}\right)\cdots\left(I - \frac{C}{2k-1}\right)}$$  \hspace{1cm} (16)

If $\alpha = 1$ we have

$$C^{-1} = 2I - (\ln 2)s(C) + 2\sum_{k=1}^{\infty} \frac{1}{2k+1}(I-C)\left(I - \frac{C}{2}\right)\cdots\left(I - \frac{C}{2k}\right)$$  \hspace{1cm} (17)

For any other $\alpha, |\alpha| < 1$, the series in (15) is alternating but the absolute value of its coefficients is not monotone decreasing. Its convergence increases as $|\alpha|$ decreases. As in the case of the ordinary number series, (16) and (17) can be further expressed with only their odd or even terms by adding the corresponding alternating series. The latter series have coefficients whose absolute values are monotonically decreasing and, therefore, a good convergence. The procedure can be repeated, each time the number of terms in the truncated original series in (4) being halved. Similarly, this can also be worked out for the series in (8) – (10).

**Note.** The matrix product in (4) – (10) and (15) – (17) can be expressed as a matrix polynomial using the relationship with the Stirling numbers of the first kind $S_{x+1}^{(n)}$ [4]

$$\left(I - \frac{C}{k}\right)\left(I - \frac{C}{2}\right)\cdots\left(I - \frac{C}{2k}\right) = (-1)^k \frac{1}{k!} \sum_{n=0}^{k} S_{x+1}^{(n)}\left(C^{n-1}\right)$$  \hspace{1cm} (18)

While each new term in the series of (4) – (10) and (15) – (17) is calculated through a multiplication with a matrix that becomes more and more well-conditioned as $k$ increases, the computation with the expression in (18) would require successive multiplications with the same original matrix and, for each $k$, a new polynomial is to be constructed and new Stirling numbers have to be generated. The formulae presented in this paper are, therefore, simpler and more efficient to be used for numerical computations.

**CONCLUSION**

The common feature of the formulae presented consists in the fact that each new term in the series involved is obtained through a multiplication with a matrix which is better conditioned than the one used for the previous term. This gives a good stability to the computational process. $(I + C/q)^{-1}$ in (7) can be looked at as being a continuous function of the real variable $q$ containing a very rapidly convergent infinite series. A direct implementation of (7) and numerical results for the solution of general systems of linear algebraic equations will be presented in an extended version of the paper. The matrix identities presented in this paper are, obviously, also valid when $C$ or $A$ are just real scalars and $I$ is replaced by $1$, resulting in novel series expressions or summation formulae. For $C = 0$, (7) reduces to a well known summation formula [4].

**REFERENCES**


