ANALYTIC SOLUTION FOR ELECTROMAGNETIC SCATTERING
BY SPHEROIDS WITH NON-PARALLEL AXES

M.F.R. Cooray and I.R. Ciric
Department of Electrical Engineering
University of Manitoba
Winnipeg, Manitoba, Canada R3T 2N2

ABSTRACT
By expanding the incident and scattered electric fields in terms of an appropriate set of vector spheroidal eigenfunctions, an exact solution is obtained to the problem of scattering of a plane electromagnetic wave by two perfectly conducting prolate spheroids of arbitrary orientation. The boundary conditions are imposed by expressing the field scattered by one spheroid in terms of the spheroidal coordinates attached to the other spheroid, using the rotational-translational addition theorems for vector spheroidal wave functions. The column matrix of the scattered field expansion coefficients is equal to the product of a square matrix, which is independent of the direction and polarization of the incident wave, and the column matrix of the known incident field expansion coefficients. The solution of the associated set of simultaneous linear equations gives the unknown scattered field expansion coefficients, from which one can determine the electromagnetic field at any point in space.

1. INTRODUCTION
Due to the possibility of applying exact analytical methods and to the fact that a large number of real system objects can be modeled by spheroids with appropriate axial ratios, the analysis of electromagnetic wave scattering by spheroids has been of increasing interest during the past few decades.

Bruning and Lo [1] obtained an exact solution to the problem of scattering of a plane electromagnetic wave by a system of two spheres using the translational addition theorems for vector spherical wave functions given by Stein [2] and Cruzan [3]. Sinha and MacPhie [4], and Dalmas and Deleuil [5] developed exact solutions for scattering of a plane electromagnetic wave by two perfectly conducting prolate spheroids with parallel major axes, using different types of vector spheroidal wave functions and the corresponding translational addition theorems [6,7].

In this paper, an exact solution to the problem of scattering of a plane electromagnetic wave by a system of two perfectly conducting prolate spheroids of arbitrary orientation is obtained by applying the rotational-translational addition theorems for vector spheroidal wave functions recently derived by the authors [8], on the basis of the theorems for scalar spheroidal wave functions [9]. The solution is obtained in the form \( S = [G] \, \bar{I} \), where \( S \) is the column matrix of the unknown coefficients in the total scattered field expansion, \( \bar{I} \) is the column matrix of the known coefficients in the incident field expansion, and \([G]\) is the system matrix, whose elements depend only on the scattering system geometry and the frequency of the incident field. By using the transformation \( \xi \rightarrow j\xi \) and \( h \rightarrow -jh \) (or \( F \rightarrow -jF \)), where \( \xi \) is the radial spheroidal coordinate, \( h=kF \), with \( F \) being the semi-interfocal distance and \( k \) the wave number, the solution for oblate spheroids can be obtained directly from that for prolate spheroids.
2. FORMULATION OF THE PROBLEM AND ANALYSIS

Let A and B be two prolate spheroids as shown in Fig. 1, with unprimed coordinates referring to the spheroid A and primed coordinates to the spheroid B. The major axes of A and B are along the z and z' axes of the Cartesian systems Oxz and O'x'y'z', respectively. The system O∥y∥z∥ is parallel to O'x'y'z', and is rotated with respect to Oxz through the Euler angles α, β, γ, with the center O' of B having spherical coordinates d, θ₀, φ₀ with respect to Oxz and d, θ_d, φ_d with respect to O∥y∥z∥. A monochromatic plane electromagnetic wave, which is linearly polarized and has an electric field of unit amplitude, is incident at an angle θ_i with respect to the major axis of A, as shown in Fig. 1, the x-z plane being chosen as the plane of incidence (θ_i=0). The angle between the direction of the incident electric field intensity vector and the direction of the normal to the plane of incidence is the polarization angle γ_k, which is zero for TE polarization and π/2 for TM polarization.

The incident electric field in the unprimed coordinate system, E_{iA}, can be expanded in a series of prolate spheroidal vector wave functions in the form [4,10]

\[ E_{iA} = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \left( p_m^+ M_{mn}^{+1} + p_m^- M_{mn}^{-1} \right) \]

where

\[ p_m^\pm = \frac{2j^{n-1}}{kN_{mn}(h)} S_{mn}(h, \cos \theta_i) \left( \frac{\cos \gamma_k}{\cos \theta_i} \mp j \sin \gamma_k \right) \]

\[ M_{mn}^{+i} = \frac{1}{2} \left[ M_{mn}^{x(i)} \pm j M_{mn}^{y(i)} \right] \]

in which

\[ M_{mn}^{x(i)} = \nabla \psi_{mn}^{(i)} \times \hat{a}, \quad \hat{a} = \hat{x}, \hat{y}, \hat{z} \]

\[ \psi_{mn}^{(i)} = R_{mn}^{(i)}(h, \xi) S_{mn}(h, \eta) e^{jm\phi}, \quad i = 1,2,3,4 \]

\[ R_{mn}^{(i)}(h, \xi), S_{mn}(h, \eta), \text{ and } N_{mn}(h) \text{ are the spheroidal radial function of the } i-\text{th kind, the spheroidal angle function, and the normalization constant of the spheroidal angle function, respectively. The expression of } E_{iA} \text{ for } \gamma_k = 0 \text{ and } \theta_i = \pi/2 \text{ is given in [11]. By arranging the terms in the expansion of } E_{iA} \text{ in the } \phi \text{ sequence } e^{j0}, e^{\pm j\phi}, e^{\pm 2j\phi}, \ldots, \text{ it can be written in matrix form as} \]

\[ E_{iA} = \bar{M}_{iA}^{(1)T} F_A \]

with the bar below a symbol denoting a column matrix and T denoting the transpose of a matrix. The elements of \( \bar{M}_{iA}^{(1)} \) and \( F_A \) are prolate spheroidal vector wave functions of the first kind, expressed in terms of the unprimed spheroidal coordinates \( \xi, \eta, \phi \), and the corresponding known expansion coefficients, respectively,

\[ \bar{M}_{iA}^{(1)T} = [M_{i0}^{T} M_{i1}^{T} M_{i2}^{T} \ldots ] \]

\[ F_A = [F_0^T F_1^T F_2^T \ldots ] \]

where

\[ M_{i0}^{T} = [M_{-1}^{+T} M_{1}^{-T} ] \]

\[ M_{i0}^{T} = [M_{0}^{+T} M_{0}^{-T} M_{+1}^{+T} M_{-(\sigma+1)}^{-T} M_{-(\sigma-1)}^{+T} ] \]

with \( \sigma \geq 1 \).
\[
\mathbf{M}_c^{\pm(1)T} = \begin{bmatrix} \mathbf{M}_{c,\pm(l)}^{\pm(1)} & \mathbf{M}_{c,\pm(l)+1}^{\pm(1)} & \mathbf{M}_{c,\pm(l)+2}^{\pm(1)} & \ldots \end{bmatrix}
\]
and
\[
\mathbf{P}_0^T = \begin{bmatrix} \mathbf{P}_{-1}^T & \mathbf{P}_1^T \\
\mathbf{P}_0^T = \begin{bmatrix} \mathbf{P}_{\sigma-1}^T & \mathbf{P}_{\sigma+1}^T & \mathbf{P}_{(\sigma+1)}^T & \ldots \end{bmatrix}, \quad \sigma \geq 1,
\]
with
\[
\mathbf{P}_x^{\pm T} = \begin{bmatrix} \mathbf{P}_{x,\pm(l)}^{\pm} & \mathbf{P}_{x,\pm(l)+1}^{\pm} & \mathbf{P}_{x,\pm(l)+2}^{\pm} & \ldots \end{bmatrix}
\]
Since the direction of the incident wave vector \( \mathbf{k} \) with respect to the unprimed system is specified by the angular spherical coordinates \( \theta_i, \phi_i = 0 \), as shown in Fig. 1,
\[
\mathbf{k} = -k (\sin \theta_i \hat{x} + \cos \theta_i \hat{z})
\]
If the angular spherical coordinates \( \theta_i, \phi_i \) give the direction of \( \mathbf{k} \) with respect to the primed system, then we have
\[
\mathbf{k} = -k (\sin \theta_i \cos \phi_i \hat{\mathbf{x}}' + \sin \theta_i \sin \phi_i \hat{\mathbf{y}}' + \cos \theta_i \hat{\mathbf{z}}')
\]
The unit vectors \( \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}} \) in the unprimed system can be expressed in terms of \( \hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}' \) in the primed system as
\[
\hat{\mathbf{a}} = c_{ax}' \hat{\mathbf{x}}' + c_{ay}' \hat{\mathbf{y}}' + c_{az}' \hat{\mathbf{z}}', \quad \hat{\mathbf{a}} = \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}
\]
where
\[
\begin{align*}
c_{ax}' &= \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma \\
c_{xy}' &= -(\cos \alpha \cos \beta \sin \gamma + \sin \alpha \cos \gamma) \\
c_{xz}' &= \cos \alpha \sin \beta \\
c_{yx}' &= \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma \\
c_{yy}' &= \cos \alpha \cos \gamma - \sin \alpha \cos \beta \sin \gamma \\
c_{yx}' &= \sin \alpha \sin \beta \\
c_{zx}' &= -\sin \beta \cos \gamma \\
c_{zy}' &= \sin \beta \sin \gamma \\
c_{zz}' &= \cos \beta
\end{align*}
\]
with \( \alpha, \beta, \gamma \) being the Euler angles, as defined in [12]. Substituting \( \hat{\mathbf{x}} \) and \( \hat{\mathbf{z}} \) from (14) in (12) and identifying the corresponding coefficients of \( \hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}' \) with those in (13) gives
\[
\begin{align*}
\sin \theta_i \cos \phi_i &= c_{ax}' \sin \theta_i + c_{zx}' \cos \theta_i \\
\sin \theta_i \sin \phi_i &= c_{xy}' \sin \theta_i + c_{zy}' \cos \theta_i \\
\cos \phi_i &= c_{ax}' \sin \theta_i + c_{zx}' \cos \theta_i
\end{align*}
\]
from which \( \theta_i \) and \( \phi_i \) can be evaluated, since \( \theta_i \) and \( c_{ax}', c_{ay}', c_{az}' \) \( (a=x,y,z) \) are known.
The incident electric field in the primed coordinate system, \( \mathbf{E}_{ib} \), can now be written as
\[
\mathbf{E}_{ib} = \mathbf{E}_{ib}^{TE} \cos \gamma_k + \mathbf{E}_{ib}^{TM} \sin \gamma_k
\]
where
\[ E_{ib}^{TE} = \hat{y} e^{-jk \cdot r} \]  
\[ E_{ib}^{TM} = (-\cos \theta_i \hat{x} + \sin \theta_i \hat{z}) e^{-jk \cdot r} \]  

From the relationship between the vectors \( \mathbf{r}, \mathbf{r}', \) and \( \mathbf{d}, \) we get

\[ e^{-jk \cdot r} = e^{-jk \cdot d} e^{-jk \cdot r'} \]  

Taking first the gradient on both sides of (20), and then the cross product with \( \hat{x} \) gives

\[ \hat{x} e^{-jk \cdot r} = (jk \cos \theta_i)^{-1} e^{-jk \cdot d} \nabla (e^{-jk \cdot r'}) \times \hat{x} \]  

Substituting \( \hat{x} \) from (14) and applying the expansion [10]

\[ e^{-jk \cdot r'} = 2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{j^n}{N_{mn}(h')} S_{mn}(h', \cos \theta_i) e^{-jm \phi_i} \psi_{mn}^{(1)}(h; \xi', \eta', \phi) \]  

gives

\[ E_{ib}^{TE} = 2 (jk \cos \theta_i)^{-1} e^{-jk \cdot d} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{j^n}{N_{mn}(h')} S_{mn}(h', \cos \theta_i) e^{-jm \phi_i} \]

\[ \cdot [c_{xx} M_{mn}^{x(i)}(h'; \mathbf{r}') + c_{xy} M_{mn}^{x(i)}(h'; \mathbf{r}') + c_{xz} M_{mn}^{x(i)}(h'; \mathbf{r}')] \]  

with \( \mathbf{r}' \) denoting the coordinate triad \((\xi', \eta', \phi)\). Using the vector functions (see (3)-(5))

\[ M_{mn}^{z(i')} = \frac{1}{2} [M_{mn}^{x(i)} \pm j M_{mn}^{y(i)}], \quad i = 1, 2, 3, 4 \]  

(23) can be rewritten as

\[ E_{ib}^{TE} = e^{-jk \cdot d} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q_{mn}^{TE} [(c_{xx} - jc_{xy}) M_{mn}^{z(i')}(h'; \mathbf{r}') + (c_{xx} + jc_{xy}) M_{mn}^{z(i')}(h'; \mathbf{r}')] \]

where

\[ q_{mn}^{TE} = \frac{2j^{n-1}}{kN_{mn}(h')} S_{mn}(h', \cos \theta_i) e^{-jm \phi_i} \]  

This expansion can be used only when \( \theta_i \neq \pi/2 \). To obtain an expansion for the case \( \theta_i = \pi/2, \) we take the gradient on both sides of (20) and then the cross product with \( \hat{z}, \) which yields

\[ E_{ib}^{TE} = -(jk \sin \theta_i)^{-1} e^{-jk \cdot d} \nabla (e^{-jk \cdot r'}) \times \hat{z} \]  

Substituting \( e^{-jk \cdot r'} \) from (22) and \( \hat{z} \) from (14), and then using (24), we have

\[ E_{ib}^{TE} = e^{-jk \cdot d} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q_{mn}^{TE} [(c_{xx} - jc_{xy}) M_{mn}^{z(i')}(h'; \mathbf{r}') + (c_{xx} + jc_{xy}) M_{mn}^{z(i')}(h'; \mathbf{r}')] \]

with

\[ q_{mn}^{TE} = \frac{2j^{n-1}}{kN_{mn}(h')} S_{mn}(h', \cos \theta_i) e^{-jm \phi_i} \]
which is valid for $\sin \theta_i \neq 0$. Taking now the gradient on both sides of (20) and then the cross product with $\hat{y}$, we get

$$(-\cos \theta_i \hat{x} + \sin \theta_i \hat{z}) e^{-jk \cdot r} = -jk^{-1} e^{-jk \cdot d} \nabla (e^{-jk \cdot r}) \times \hat{y}$$

Substituting $e^{-jk \cdot r'}$ from (22) and $\hat{y}$ from (14), and then using (24), gives

$$E_{TB} = e^{-jk \cdot d} \sum_{m=\infty}^{\infty} \sum_{n=|m|}^{\infty} q_{mn}^{TM} \left[ (c_{yx} \mp jc_{yy}) M_{mn}^{(1)'} (h'; r') 
+ (c_{yx} \pm jc_{yy}) M_{mn}^{(1)''} (h'; r') + c_{yz} M_{mn}^{x(1)} (h'; r') \right]$$

where

$$q_{mn}^{TM} = \frac{2j^{n-1}}{kN_{mn}(h')} S_{mn}(h', \cos \theta_i') e^{-jm \phi_i}$$

Thus the incident field $E_{tb}$ in (17) can be expanded in a series of prolate spheroidal vector wave functions in the form

$$E_{tb} = e^{-jk \cdot d} \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \left( p_{mn}^{x} M_{mn}^{x(1)} + p_{mn}^{z} M_{mn}^{z(1)} \right)$$

where

$$p_{mn}^{x} = \frac{2j^{n-1}}{kN_{mn}(h')} S_{mn}(h', \cos \theta_i') e^{-jm \phi_i} \left[ (c_{xx} \mp jc_{xy}) \frac{\cos \gamma_k}{\cos \theta_i} \right]$$

$$\text{for } \theta_i \neq \frac{\pi}{2}$$

$$+ \left[ (c_{xx} \mp jc_{xy}) \frac{\cos \gamma_k}{\sin \theta_i} \right] \text{ for } \theta_i \neq 0, \pi$$

$$p_{mn}^{z} = \frac{2j^{n-1}}{kN_{mn}(h')} S_{mn}(h', \cos \theta_i') e^{-jm \phi_i} \left[ c_{yz} \sin \gamma_k \right]$$

$$\text{for } \theta_i \neq \frac{\pi}{2}$$

$$+ \left[ c_{zz} \frac{\cos \gamma_k}{\sin \theta_i} \right] \text{ for } \theta_i \neq 0, \pi$$

It is interesting to note that if $h' = h$ and the coordinate system $O'x'y'z'$ is brought to coincide with $Oxyz$, then the expressions in (33)–(35) reduce to those corresponding to the incident field $E_A$ (see (1)–(2)). If the terms in the series expansion of $E_{tb}$ are arranged in the $\phi'$ sequence $e^{j0}, e^{\pm j\psi}, e^{\pm 2j\psi}$ . . . , then we can write this expansion also in matrix form as

$$E_{tb} = M_{tb}^{(1)T} I_B$$

where $M_{tb}^{(1)}$ and $I_B$ are column matrices whose elements are prolate spheroidal vector wave
functions of the first kind, expressed in terms of the primed spheroidal coordinates $\xi', \eta', \phi'$, and the corresponding known expansion coefficients, respectively,

$$M_{i0}^{(1)T} = [M_{i0}^T \ M_{i1}^T \ M_{i2}^T \ldots], \quad H_0^T = [P_0^T \ P_1^T \ P_2^T \ldots] \ e^{-jkd} \quad (37)$$

in which

$$M_{i0}^T = [M_{i0}^{(+1)T} \ M_{i1}^{(-1)T} \ M_{i2}^{(1)T}] \quad (38)$$

with

$$M_{i0}^{(+1)T} = [M_{i0}^{(1)T} \ M_{i1}^{(1)T} \ M_{i2}^{(1)T} \ldots] \quad (39)$$

and

$$P_0^T = [P_{i0}^{(+1)T} \ P_{i1}^{(-1)T} \ P_{i2}^{(0)T}] \quad (40)$$

with

$$P_{i0}^{(+1)T} = [P_{i0}^{(1)T} \ P_{i1}^{(1)T} \ P_{i2}^{(1)T} \ldots] \quad (41)$$

The spheroid B scatters an electromagnetic field which corresponds to a non-plane wave whose electric field intensity $E_{sB}$ can be expanded in the form [4]

$$E_{sB} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \beta_{mn}^{+} M_{m}^{(4)T} + \beta_{m+1,n+1}^{+} M_{m+1}^{(4)T} \right) + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \beta_{mn}^{-} M_{m}^{(4)T} + \beta_{m+1,n+1}^{-} M_{m+1}^{(4)T} \right)$$

$$+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \beta_{m+1,n+1}^{+} M_{m+1}^{(4)T} + \beta_{m+1,n+1}^{-} M_{m+1}^{(4)T} \right) \quad (42)$$

If the terms in the expansion of $E_{sB}$ are arranged in the same $\phi'$ sequence as in the expansion of $E_{sB}$, then we can write

$$E_{sB} = M_{sB}^{(4)T} \ \beta \quad (43)$$

where $M_{sB}^{(4)}$ and $\beta$ are column matrices whose elements are prolate spheroidal vector wave functions of the fourth kind, expressed in terms of the primed spheroidal coordinates, and the corresponding unknown expansion coefficients, respectively,

$$M_{sB}^{(4)T} = [M_{s0}^T \ M_{s1}^T \ M_{s2}^T \ldots], \quad \beta^T = [\beta_0^T \ \beta_1^T \ \beta_2^T \ldots] \quad (44)$$

in which

$$M_{s0}^T = [M_{s0}^{(+4)T} \ M_{s0}^{(-4)T}] \quad (45)$$

with
\[ M_{\tau}^{(4)}T = \begin{bmatrix} M_{t,\tau t}^{(4)} & M_{t,\tau t+1}^{(4)} & M_{t,\tau t+2}^{(4)} \\ \vdots & \vdots & \vdots \end{bmatrix} \]
\[ M_{\tau}^{(4)}T = \begin{bmatrix} M_{t,\tau t}^{(4)} & M_{t,\tau t+1}^{(4)} & M_{t,\tau t+2}^{(4)} \\ \vdots & \vdots & \vdots \end{bmatrix} \]

and
\[ B_0^T = \begin{bmatrix} \beta_1^T & \beta_0^T \end{bmatrix} \]
\[ B_\sigma^T = \begin{bmatrix} \beta_2^T & \beta_1^T & \beta_{-1}^T & \beta_{\sigma-1}^T & B_{T}^T \end{bmatrix}, \quad \sigma \geq 1 \]

with
\[ \beta_1^T = \begin{bmatrix} \beta_{1,\tau t}^T & \beta_{1,\tau t+1}^T & \beta_{1,\tau t+2}^T \end{bmatrix} \]
\[ \beta_2^T = \begin{bmatrix} \beta_{2,\tau t}^T & \beta_{2,\tau t+1}^T & \beta_{2,\tau t+2}^T \end{bmatrix} \]

The boundary conditions at the spheroid A surface are imposed by expressing the scattered field \( E_{SB} \), which acts as a secondary incident field for the spheroid A, in terms of vector wave functions of the first kind in unprimed coordinates, by using the appropriate rotational-translational addition theorem for vector spheroidal wave functions [8]:

\[
M_{mm}^{(4)}(h;r') = \sum_{\nu=0}^{\infty} \sum_{\mu=\nu}^{\infty} (4)Q_{\mu
u}^{mn}(\alpha,\beta,\gamma;\mathbf{d}) \left[ C_1 M_{\mu
u}^{(1)}(h;r) + C_2 M_{\mu
u}^{(1)}(h;r) + C_3 M_{\mu
u}^{(1)}(h;r) \right], \quad r \leq d
\]

\[ M_{mn}^{(4)}(h;r') = \sum_{\nu=0}^{\infty} \sum_{\mu=\nu}^{\infty} (4)Q_{\mu
u}^{mn}(\alpha,\beta,\gamma;\mathbf{d}) \left[ C_4 M_{\mu
u}^{(1)}(h;r) + C_5 M_{\mu
u}^{(1)}(h;r) \right], \quad r \leq d
\]

where \( r \) and \( r' \) represent the coordinate triads \((\xi,\eta,\phi)\) and \((\xi',\eta',\phi')\), respectively, and

\[ C_1' = \frac{1}{2} [(c_{xx'} + c_{yy'}) + j(c_{xy'} - c_{yx'})] \]
\[ C_2' = \frac{1}{2} [(c_{xx'} - c_{yy'}) + j(c_{xy'} + c_{yx'})] \]
\[ C_3' = \frac{1}{2} (c_{zx'} + jc_{zy'}) \]
\[ C_4' = c_{xx'} - jc_{yy'} \]
\[ C_5' = c_{zz'} \]

with the asterisk denoting the complex conjugate. \((4)Q_{\mu
u}^{mn}(\alpha,\beta,\gamma;\mathbf{d})\) are the rotational-translational coefficients in the expansion of scalar spheroidal wave functions of the fourth kind in primed coordinates in terms of scalar spheroidal wave functions of the first kind in unprimed coordinates, for \( r \leq d \). Considering the translation from the system \( O_{x'y'z'} \) to \( O_{x'\parallel y'\parallel z'\parallel} \), and then the rotation of the system \( O_{x'\parallel y'\parallel z'\parallel} \) about O through the Euler angles...
we can derive the expression of these coefficients for \( r \leq d \) and \( i = 1, 2, 3, 4 \), in the form

\[
\mathcal{Q}^{m_0}_{\mu \nu} (\alpha, \beta, \gamma; \delta) = \sum_{s=|m_0|, |m_0|+1}^{\infty} \sum_{l=|\mu|, |\mu|+1}^{\infty} j^{s-n+l} \frac{N_{\mu l}}{N_{\mu \nu} (\gamma)} \frac{d_{\mu l}^{\nu \gamma} (\gamma)}{d_{\mu \nu} (\gamma)}
\]

\[
\cdot \sum_{c=-l}^{l} R^c_{\mu l} (-\gamma, -\beta, -\alpha) (i) b_{cl}^{m_0} (d)
\]

with

\[
(i) b_{cl}^{m_0} (d) = (-1)^c \sum_{p=p_0}^{\infty} (-1)^c \beta^{j+p-s} (2l+1) a (m, s - c, l | p) \cdot \psi^{(i)}_{m-c, p} (d)
\]

in which \( a (m, s - c, l | p) \) are the linearization expansion coefficients \([2,3]\),

\[
p_0 = \max \{1 - s, 1 - m - c\},
\]

and

\[
\cdot \psi^{(i)}_{m-c, p} (d) = z_{m-c, p}^{(i)} (kd) P^m_{-c} (\cos \theta_d) e^{i (m-c) \phi_d}
\]

where \( z_{m-c, p}^{(i)} \), \( i = 1, 2, 3, 4 \), are the spherical Bessel functions \( j_p, n_p, h_p^{(1)}, \) and \( h_p^{(2)} \), respectively, and \( P^m_{-c} \) is the associated Legendre function of the first kind.

By arranging the terms in the series expansions (49)-(51), in the \( \phi \) sequence \( e^{i0}, e^{i2}, e^{i4}, \ldots \), we can express the outgoing vector wave functions in primed coordinates \( \mathbf{M}_{SB}^{(4)} \) in terms of incoming vector wave functions in unprimed coordinates \( \mathbf{M}_{BA}^{(1)} \) in the form

\[
\mathbf{M}_{SB}^{(4)} = [\Gamma] \mathbf{M}_{BA}^{(1)}
\]

The structure and elements of the matrix \([\Gamma]\) are defined elsewhere. The transpose of \( \mathbf{M}_{BA}^{(1)} \) is

\[
\mathbf{M}_{BA}^{(1)T} = \begin{bmatrix}
\mathbf{M}_{BA,0}^{(1)T} & \mathbf{M}_{BA,1}^{(1)T} & \mathbf{M}_{BA,2}^{(1)T} & \ldots
\end{bmatrix}
\]

where

\[
\mathbf{M}_{BA,0}^{(1)T} = \begin{bmatrix}
\mathbf{M}_{BA,0}^{(1)T} & \mathbf{M}_{BA,1}^{(1)T} & \mathbf{M}_{BA,2}^{(1)T}
\end{bmatrix}
\]

\[
\mathbf{M}_{BA,0}^{(1)T} = \begin{bmatrix}
\mathbf{M}_{BA,0}^{(1)T} & \mathbf{M}_{BA,1}^{(1)T} & \mathbf{M}_{BA,2}^{(1)T} & \ldots
\end{bmatrix}, \quad \sigma \geq 1
\]

Denoting the secondary incident field by \( \mathbf{E}_{SB}^{(4)} \), taking the transpose of both sides of (56) and then substituting \( \mathbf{M}_{SB}^{(4)T} \) in (43) gives

\[
\mathbf{E}_{SB}^{(4)} = \mathbf{M}_{BA}^{(1)T} [\Gamma]^T \beta
\]

Due to the two electric fields \( \mathbf{E}_A \) and \( \mathbf{E}_{SB}^{(4)} \) incident on the spheroid \( A \), it scatters an electric field \( \mathbf{E}_{SA} \), which can also be expanded in a series of prolate spheroidal vector wave functions and expressed in a matrix form as

\[
\mathbf{E}_{SA} = \mathbf{M}_{SA}^{(4)T} \alpha
\]

\( \mathbf{M}_{SA}^{(4)} \) and \( \alpha \) are column matrices whose elements are prolate spheroidal vector wave functions of the fourth kind, expressed in terms of unprimed spheroidal coordinates, and the corresponding unknown expansion coefficients, respectively. The elements of \( \mathbf{M}_{SA}^{(4)} \) and \( \alpha \) are obtained from those of \( \mathbf{M}_{SB}^{(4)} \) and \( \beta \) respectively, by evaluating the corresponding vector wave functions with respect to the unprimed system and replacing \( \beta \) by \( \alpha \).
The total electric field seen from the spheroid A, is thus given by

\[ E_A = E_{iA} + E_{sBA} + E_{sA} \]

\[ = M^{(1)T}_{iA} I_A + M^{(1)T}_{BA} [\Gamma']^T \beta + M^{(4)T}_{sA} \alpha \]  

(61)

Similarly, the total electric field seen from the spheroid B can be expressed in terms of appropriate prolate spheroidal vector wave functions in the primed coordinate system as

\[ E_B = E_{iB} + E_{sAB} + E_{sB} \]

\[ = M^{(1)T}_{iB} I_B + M^{(1)T}_{AB} [\Gamma']^T \alpha + M^{(4)T}_{sB} \beta \]  

(62)

The structure and elements of the matrix \( [\Gamma'] \) are defined elsewhere. The elements of \( M^{(1)T}_{AB} \) can be obtained from the corresponding elements of the matrix \( M^{(1)T}_{BA} \) by expressing the corresponding vector wave functions in primed coordinates.

3. IMPOSING THE BOUNDARY CONDITIONS

On the surface of each perfectly conducting prolate spheroid \( \xi = \xi_A \) and \( \xi = \xi_B \), the tangential components (\( \eta \) and \( \phi \)) of the total electric field intensity must be equal to zero. Thus we get from (61) and (62)

\[ \left( M^{(1)T}_{iA} I_A + M^{(1)T}_{BA} [\Gamma']^T \beta + M^{(4)T}_{sA} \alpha \right) \xi = \xi_A = 0 \]  

(63)

\[ \left( M^{(1)T}_{iB} I_B + M^{(1)T}_{AB} [\Gamma']^T \alpha + M^{(4)T}_{sB} \beta \right) \xi = \xi_B = 0 \]  

(64)

Taking the scalar product of both sides of (63) and (64) by \( \left\{ \hat{\eta}, \hat{\phi} \right\} \) \( S_{m, m+1 \pm N} (k, \eta) \) \( e^{\pm j (m \pm 1) \Phi} \) and \( \left\{ \hat{\eta}, \hat{\phi} \right\} \) \( S_{m, m+1 \pm N} (k', \eta') \) \( e^{\pm j (m \pm 1) \Phi'} \), respectively, for \( m = \ldots -2, -1, 0, 1, 2, \ldots \), \( N = 0, 1, 2, \ldots \), integrating correspondingly over the surfaces of the two spheroids, and using the orthogonality properties of spheroidal wave functions, gives \([13,4]\)

\[ \left[ [Q_A] \quad [R_{BA}] [\Gamma']^T \right] \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] = - \left[ [R_A] I_A \right] \]  

with \( [Q_A], [Q_B], [R_{BA}], [R_{AB}], [R_A], \) and \( [R_B] \) defined elsewhere. Eq. (65) can now be rewritten in the form

\[ \mathbf{S} = [G] \mathbf{I} \]  

(66)

where

\[ \mathbf{S} = \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] \quad \mathbf{I} = \left[ \begin{array}{c} I_A \\ I_B \end{array} \right] \]  

(67)

\[ [G] = - \left( \left[ [Q_A] \quad [R_{BA}] [\Gamma']^T \right] \right)^{-1} \left[ \begin{array}{c} [R_A] \\ [0] \end{array} \right] \]  

(68)

\( [G] \) is the system matrix, which depends only on the geometry of the scattering system and the frequency of the incident wave.
The matrix form (66) gives the coefficients in the expansions of the electric fields scattered by the two spheroids A and B. Knowing these coefficients one can calculate the resultant electric field at any point as

\[ E = E_i + E_{SA} + E_{SB} \]

4. CONCLUSIONS

An exact solution to the problem of electromagnetic scattering by a system of two spheroids of arbitrary orientation, using the rotational-translational addition theorems for vector spheroidal wave functions [8], has been obtained for the first time. By expanding the resultant field seen from a coordinate system associated with each spheroid in terms of an appropriate set of vector spheroidal eigenfunctions, the imposition of the exact boundary conditions yields finally the field scattered by the two spheroids. In an accompanying paper radar cross sections are computed for systems of two conducting spheroids in various configurations. The formulation presented in this paper for perfectly conducting spheroids can be extended to dielectric spheroids and also to excitations which are not plane waves.

REFERENCES


Fig. 1. System of two prolate spheroids of arbitrary orientation.