A LUNEBERG-KLINE EXPANSION SOLUTION FOR THE TRANSIENT SCATTERING OF CYLINDRICAL WAVES BY A CIRCULAR CYLINDER

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INTRODUCTION

Analysis of the transient electromagnetic fields in the presence of conducting cylinders is important from both theoretical and practical point of view. The transient response of a perfectly conducting circular cylinder excited by a plane electromagnetic wave has already been studied [1,2]. Recently, the early-time response of a perfectly conducting circular cylinder to cylindrical electromagnetic waves were studied by the authors [3,4]. As given in [3] and [4], the results obtained for the shadow region using different approaches are in good agreement, while the results for the illuminated region can only be approximated to the physical optics solution. In this paper we use the Luneberg-Kline expansion to obtain a more accurate solution for the illuminated region.

FORMULATION AND ANALYSIS

The geometry of the problem is shown in Fig. 1 where \( a \) is the radius of the cylinder and \( r_0 \) the distance from the line source to the cylinder axis. The surrounding medium is assumed to be free space. For a line source carrying a unit-step current, the Laplace transform of the electric field \( E_f(r,\theta,s) \) in the absence of the cylinder can be written as

\[
E_f(r,\theta,s) = -\frac{j}{4} \mu_0 H_0^{(1)}(jkR) \tag{1}
\]

where \( r, \theta \) are the cylindrical coordinates of the field point, \( k=s/c \) with \( s \) being the Laplace transform variable and \( c \) the speed of light, \( \mu_0, R, \) and \( H_0^{(1)} \) are the permeability of free space, the distance from the line source to the field point, and the Hankel function of the first kind, respectively. For the early-time response, we consider the case of \( s \rightarrow \infty \). Thus \( E_f \) can be written in the asymptotic form as [5]

\[
E_f(r,\theta,s) = \frac{\mu_0}{2 \sqrt{2\pi kR}} e^{-kR} \left[ 1 - \frac{1}{8} \frac{1}{kR} + \frac{9}{128} \frac{1}{(kR)^2} - \cdots \right] \tag{2}
\]

The Laplace transform of the scattered electric field satisfies the wave equation

\[
\nabla^2 E_s^f(r,\theta,s) - k^2 E_s^f(r,\theta,s) = 0 \tag{3}
\]

Now we assume that \( E_s^f \) has the asymptotic expansion (for \( k \rightarrow \infty \)) of the form

\[
E_s^f(r,\theta,s) = \frac{\mu_0}{2 \sqrt{2\pi k}} e^{-k\Psi(r,\theta)} \sum_{n=0}^{\infty} \frac{v_n(r,\theta)}{k^n} \tag{4}
\]

in which \( \Psi(r,\theta) \) and \( v_n(r,\theta) \) are the unknown functions to be determined. The boundary condition requires that the total tangential electric field be equal to zero at the surface of the cylinder, which can be written as

\[
E_s^f(a,\theta,s) + E_s^f(a,\theta,s) = 0 \tag{5}
\]
Substituting (2) and (4) in (5) gives

\[ \Psi(a, \theta) = R_0 \]  

\[ \nu_0(a, \theta) = -R_0^{-1/2} \quad \nu_1(a, \theta) = \frac{1}{8} R_0^{-3/2} \quad \nu_2(a, \theta) = -\frac{9}{128} R_0^{-5/2}, \ldots \]  

where \( R_0 = (r_0^2 + a^2 - 2r_0 a \cos \phi)^{1/2} \) with \( \phi \) as shown in Fig. 1. Then we substitute (4) into (3) to obtain

\[ \left\{ \begin{array}{l}
(\nabla \Psi)^2 = 1 \\
2 \nabla \Psi \cdot \nabla \nu + \nu_\alpha \nabla^2 \Psi = \nabla^2 \nu_{n-1} \quad n = 0, 1, 2, \ldots ; \quad \nu_{-1} = 0
\end{array} \right. \]  

Equation (8) can be solved in the caustic coordinate system to give [6]

\[ \nu_n(l, \psi) = \nu_n(l_0, \psi) \left( \frac{l_0}{l} \right)^{1/2} + \frac{1}{2l^{1/2}} \int l^{1/2} \nabla^2 \nu_{n-1}(l', \psi) dl' \]  

where \( l \) and \( \psi \) are the caustic coordinates shown in Fig. 1 and \( l_0 = \frac{a}{2} \cos \alpha \) is the distance along the reflected ray from the caustic to the cylinder surface, with \( \alpha \) as shown in Fig. 1.

It can be shown that the Laplacian operator in the caustic coordinates is

\[ \nabla^2 = \left( 1 + \frac{A}{l^2} \right) \frac{\partial^2}{\partial l^2} + \left( \frac{1}{l} \right)^2 \frac{\partial}{\partial l} \left( \frac{C - A}{l^3} \frac{\partial}{\partial l} \right) - \frac{2B}{l^2} \frac{\partial^2}{\partial l \partial \psi} + \frac{1}{l^2} \frac{\partial^2}{\partial \psi^2} + \frac{B}{l^3} \frac{\partial}{\partial \psi} \]  

where

\[ A = \frac{a^2}{4} \sin^2 \alpha \left( 1 + \frac{d \phi}{d \psi} \right)^2, \quad B = \frac{a}{2} \sin \alpha \left( 1 + \frac{d \phi}{d \psi} \right) \]

\[ C = a \left[ -\frac{1}{2} \cos \alpha \left( \frac{d \phi}{d \psi} \right)^2 + \cos \alpha (1 - \frac{d \phi}{d \psi}) + \frac{1}{2} \sin \alpha \frac{d^2 \phi}{d \psi^2} \right] \]  

The phase function \( \Psi(r, \theta) \) can be written as

\[ \Psi(r, \theta) = R_0 + l - \frac{a}{2} \cos \alpha \]  

From Eqs. (7)-(11), expressions for \( \nu_0, \nu_1, \) and \( \nu_2 \) are obtained. The sum of the incident and scattered fields gives the total electric field

\[ E_z(r, \theta) = \frac{\mu_0}{2} \frac{1}{\sqrt{2} \pi k} \left\{ \left[ \frac{1}{R_0^{1/2}} e^{-kR_0} - \frac{1}{R_0^{1/2}} \left( \frac{l_0}{l} \right)^{1/2} e^{-k(R_0 + l - \frac{a}{2} \cos \alpha)} \right] \right. \]

\[ + \frac{1}{k} \left[ \left[ \frac{1}{R_0^{1/2}} \frac{l_0}{l} \right]^{1/2} e^{-k(R_0 + l - \frac{a}{2} \cos \alpha)} \right] \right. \]

\[ - \frac{1}{k} \left[ \left[ \frac{1}{R_0^{1/2}} \frac{l_0}{l} \right]^{1/2} e^{-k(R_0 + l - \frac{a}{2} \cos \alpha)} \right] \right. \]

\[ \frac{1}{k^2} \left[ \left[ \frac{1}{R_0^{1/2}} \frac{l_0}{l} \right]^{1/2} e^{-k(R_0 + l - \frac{a}{2} \cos \alpha)} \right] \right. \]

\[ + \frac{1}{k^2} \left[ \frac{9}{128} \frac{1}{R_0^{5/2}} e^{-k(R_0 + l - \frac{a}{2} \cos \alpha)} \right] + \cdots \} \]  

The surface current density is equal to the tangential component of the magnetic field intensity on the surface of the cylinder, \( J_z(\phi, s) = \frac{1}{\mu_0 s} \frac{\partial E_z}{\partial r} \bigg|_{r=0} \). After a very long and
tedious derivation, we finally obtain the Laplace transform of the current density up to the power of $s^{-5/2}$ as

$$J_z(\phi, s) = -\frac{1}{2c\sqrt{2\pi}} e^{-\frac{\sqrt{R}}{c}} \left\{ \frac{2\cos \alpha}{R_0^{1/2}} \left( \frac{c}{s} \right)^{1/2} + \frac{1}{4\cos \alpha} \frac{1}{R_0^{3/2}} (3\cos^2 \alpha - 2 + \frac{R_0}{l_0}) \left( \frac{c}{s} \right)^{3/2} \right\}$$

$$+ \frac{D}{512 \cos \alpha} \frac{R_0^{1/2}}{(l_0 + R_0)^3} \left( \frac{c}{s} \right)^{5/2}$$

(14)

where

$$D = -3(5\cos^2 \alpha - 4)\cos^3 \alpha (R_0/a)^{-3} - 2(45\cos^2 \alpha - 32)\cos^2 \alpha (R_0/a)^{-2}$$

$$- 4(45\cos^2 \alpha + 8 - \frac{24}{\cos^2 \alpha}) \cos \alpha (R_0/a)^{-1} - 8(15\cos^2 \alpha + 8 + \frac{40}{\cos^2 \alpha})$$

$$+ 320(1 - \frac{40}{\cos^2 \alpha}) \frac{1}{\cos \alpha} (R_0/a) + 512(3 - \frac{4}{\cos^2 \alpha}) \frac{1}{\cos^2 \alpha} (R_0/a)^2$$

The inverse Laplace transform of Eq. (14) gives the time dependent solution as

$$J_z(\phi, t) = -\frac{1}{\sqrt{2\pi}} \left\{ \frac{\cos \alpha}{(R_0/a)^{1/2}} \frac{1}{t^{1/2}} + \frac{1}{4\cos \alpha} \frac{1}{(R_0/a)^{3/2}} [3\cos^2 \alpha - 2 + \frac{2}{\cos \alpha} (R_0/a)] \frac{1}{t^{3/2}} \right\}$$

$$+ \frac{D}{96} \frac{1}{\cos \alpha} \frac{(R_0/a)^{1/2}}{(\cos \alpha + 2R_0/a)^3} \frac{1}{t^{3/2}} u(\tau)$$

(15)

where $\tau = \frac{c}{a} (t - R_0/c)$.

NUMERICAL RESULTS AND DISCUSSION

Computed results for the early-time currents in the illuminated region, when $r_0 = 5\alpha$ and $r_0 = 10\alpha$ are shown in Fig. 2 (a) and (b), respectively, where $\tau$ is the normalized local time, counted from the arrival of the wave front. The results are compared with the physical optics solution obtained in [3]. It can be seen that as time increases, the results obtained on the basis of the Luneberg-Kline expansion decrease more rapidly than those of the physical optics solution. For small times the results corresponding to the two different approaches are in better agreement. It should be mentioned that the first term in (15) is the solution obtained in [4] by using the saddle point method, which is approximately the physical optics solution. Due to the nature of the series in (15) we expect accurate results for points which are not close to the shadow boundary and for times such that the subsequent terms in (15) are decreasing in magnitude.

REFERENCES


Fig. 1. Coordinate system

Fig. 2. Early-time current distribution; series expansion, physical optics.