LATE-TIME CURRENT RESPONSE OF A CONDUCTING CYLINDER EXCITED BY A LINE CURRENT

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INTRODUCTION

Transient response of a perfectly conducting circular cylinder to a step-function or an impulsive electromagnetic plane wave has been already thoroughly studied by analytical and numerical methods [1]. Recently the problem of early-time current response of a perfectly conducting circular cylinder to cylindrical electromagnetic waves has also been solved by the authors [2],[3]. A general analytical procedure consists in obtaining first the frequency domain solution to the problem under investigation and then using the inverse Laplace transform for deriving the time domain solution.

In this paper, we consider the late-time current response of a perfectly conducting circular cylinder to cylindrical electromagnetic waves generated by a parallel filament carrying a unit-step current. The analysis is based on the frequency domain eigenfunction solution of the induced current density on the cylinder surface. The exact inverse Laplace transform of the current density can not be obtained due to the complicated form of the eigenfunctions. For the late-time current response, i.e., for small values of the Laplace transform variable, the surface current density can be represented by the first few terms in the series solution. The corresponding inverse Laplace transform is computed by closing the contour in the complex plane, approximating the resultant branch cut contributions and adding the contributions due to the first few poles for each term retained. Unlike the solution for the early-time response obtained in earlier papers, a single asymptotic expression which is valid for both the illuminated and shadow region is derived. The analytical expressions obtained in this paper, along with those for the early-time response, are useful in determining the response of two or more cylinders to plane electromagnetic waves.

FORMULATION AND ANALYSIS

The geometry of the problem is shown in Fig. 1 where $a$ is the radius of the cylinder and $r_0$ the distance from the line source to the cylinder axis. The surrounding medium is assumed to be free space. For a line source carrying a unit-step current, the Laplace transform of the electric field for $r < r_0$ in the absence of the cylinder can be written as

$$E_z^l(r,\phi,k) = \frac{j}{4} \mu_0 \sum_{n=0}^{\infty} \varepsilon_n J_n(kr)H_n^{(1)}(kr_0) \cos n\phi$$  \hspace{1cm} (1)

where $\mu_0$ is the permeability of free space, $r,\phi$ are the cylindrical coordinates of the field point, $k=js/c$ with $s$ being the Laplace transform variable and $c$ the speed of light.
$J_n$ and $H_n^{(1)}$ are the Bessel and Hankel functions of the first kind, respectively. The Laplace transform of the scattered electric field can be derived as

$$E_z^s(r,\phi,k) = \frac{j}{4\epsilon_0} \sum_{n=0}^{\infty} \epsilon_n \frac{J_n(ka)H_n^{(1)}(kr_0)H_n^{(1)}(kr)\cos n\phi}{H_n^{(1)}(ka)}$$

where $\epsilon_0=1$ and $\epsilon_n=2$, for $n \neq 0$. The surface current density is equal to the tangential component of the magnetic field intensity on the cylinder surface

$$J_z(\phi,t) = H_\phi^i(a,\phi,t) + H_\phi^s(a,\phi,t)$$

where $H_\phi^i$ and $H_\phi^s$ are the incident and scattered magnetic fields, respectively. The Laplace transform of $H_\phi^s(a,\phi,t)$ can be obtained from (2) and $H_\phi^s(a,\phi,t)$ can be written as

$$H_\phi^s(a,\phi,t) = -\frac{1}{8\pi} \sum_{n=0}^{\infty} \epsilon_n \cos n\phi L_n(t)$$

where

$$L_n(t) = \int_{j\sigma_0}^{j\sigma_0+\infty} M_n(k,t)dk$$

in which

$$M_n(k,t) = e^{-jkt} \frac{J_n(ka)H_n^{(1)}(kr_0)H_n^{(1)}(ka)}{H_n^{(1)}(ka)}$$

In order to evaluate the integral in (5), we choose the contour as shown in Fig. 2. Hence

$$L_n(t) = -2\pi j \sum R_{nl} \left[ \int_{C_{r_0}} + \int_{C_\epsilon} + \int_{L_1} + \int_{L_2} \right] M_n(k,t)dk$$

where $R_{nl} = [e^{-j\zeta_n\tau}H_n^{(1)}(\zeta_n r_0)J_n(\zeta_n)]/a$ with $r_0'=r_0/a$ and $\tau=ct/a$ is the residue of $M_n$ at the simple pole $\zeta_{nl}$ at which $H_n^{(1)}(\zeta_{nl})=0$. It can be shown that, when $t > (r_0+a)/c$, \lim_{R\rightarrow\infty} \int M_n(k,t)dk=0$, and

$$\lim_{e\rightarrow 0} \int_{C_\epsilon} M_n(k,t)dk = -\frac{4}{a\epsilon_n} \frac{1}{r_0'^n}$$

Exploiting appropriate analytic continuation of Bessel functions, the contributions from the line integrals along the branch cut can be evaluated as

$$\int_{L_1} M_n(k,t)dk + \int_{L_2} M_n(k,t)dk = (-1)^{n+1} \frac{4}{a} B_n^s(t)$$

where

$$B_n^s(t) = \int_0^\infty e^{-z\tau} I_n(z) \left\{ \frac{nI_n(zr_0')}{z} + \frac{K_n(zr_0')/z-I_n(zr_0')K_n(z)K_{n+1}(z)-\pi^2I_n(z)I_{n+1}(z)}{K_n^2(z)+\pi^2I_n^2(z)} \right\} dz$$

where $I_n$ and $K_n$ are the modified Bessel functions of the first and second kinds, respectively. Now using Eqs. (4)-(10), we obtain
\[ H_\phi^i(a, \phi, t) = -\frac{1}{2\pi a} \frac{r_0^2(\cos \phi - 1)}{r_0^2 - 2r_0^2\cos \phi + 1} - \frac{1}{2\pi a} \sum_{n=0}^{\infty} (-1)^n \varepsilon_n \cos n\phi B_n^i(t) \]

\[ -\frac{1}{2a} \sum_{n=0}^{[n/2]} \sum_{l=1}^{[n/2]} \text{Im} \left[ e^{-j\zeta_{nl} \tau} H_n^{(1)}(\zeta_{nl} r_0') J_n(\zeta_{nl}) \right] \cos n\phi \]

where \( [n/2] \) represents the integer part of \( n/2 \). Following the same procedure we obtain

\[ H_\phi^i(a, \phi, t) = -\frac{1}{2\pi a} \frac{r_0^2(\cos \phi - 1)}{r_0^2 - 2r_0^2\cos \phi + 1} + \frac{1}{2\pi a} \sum_{n=0}^{\infty} (-1)^n \varepsilon_n \cos n\phi B_n^i(t) \]

where

\[ B_n^i(t) = \int e^{-z^2} I_n(zr_0') [nI_n(z) + I_{n+1}(z)] dz \]

From Eqs. (3), (11) and (12), we finally obtain

\[ J_z(\phi, t) = -\frac{1}{2\pi a} \frac{r_0^2 - 1}{r_0^2 - 2r_0^2\cos \phi + 1} - \frac{1}{2\pi a} \sum_{n=0}^{\infty} (-1)^n \varepsilon_n \cos n\phi B_n(t) \]

\[ -\frac{1}{2a} \sum_{n=0}^{[n/2]} \sum_{l=1}^{[n/2]} \text{Im} \left[ e^{-j\zeta_{nl} \tau} H_n^{(1)}(\zeta_{nl} r_0') J_n(\zeta_{nl}) \right] \cos n\phi \]

where \( B_n(t) = B_n^s(t) - B_n^i(t) \).

**NUMERICAL RESULTS AND DISCUSSION**

The induced current density is computed for \( r_0' = 2 \), when \( \tau > r_0' + 1 \). The series of continuous-spectrum integral terms is found to converge rapidly and retention of only the first 3 terms provide adequate accuracy. In the integral terms, contributions from the neighborhood of the singularity at \( z = 0 \) are calculated analytically by using small argument approximations of the required Bessel functions, while the upper integration limit is truncated for the remaining numerical integration because all significant contributions from the integrand are found to occur for \( z < 10 \). All the significant contributions to the discrete-spectrum residue series are provided by the \( l = 1 \) layer of \( \zeta_{nl} \) and only the first 6 terms are retained for summation. The current response is as shown in Fig. 3, where \( J = -aJ_z \) and \( \tau = c t / a \). It should be noted that the magnitude of the current decreases as \( \phi \) increases from 0 to \( \pi \). For a given value of \( \phi \), \( J \) increases with time and gradually approaches a limit which corresponds to the first term in Eq. (14).

**REFERENCES**

Fig. 1. Cross section of the line source parallel to a conducting cylinder.

Fig. 2. The integration contour in $k$-plane.

Fig. 3. Current response for $r_0/a = 2$. 