ITERATIVE TECHNIQUE FOR SCATTERING BY
A LINEAR ARRAY OF SPHERES

A-K. HAMID, I. R. CIRIC, and M. HAMID
Department of Electrical and Computer Engineering
University of Manitoba, Winnipeg, Manitoba, R3T 2N2 Canada.

INTRODUCTION

Solution to scattering of a plane electromagnetic wave by a system of spheres is important in a variety of practical applications. For example, the propagation of electromagnetic waves through rain and hail can be analyzed by modeling these complex scatterers by collections of spheres. The formulation for a system of dielectric spheres can be applied to the analysis of the gain and sidelobes for aperture antennas loaded by such spheres [1]. Exact analytic solutions to the problem of scattering by a system of conducting or dielectric spheres have been obtained in [2,3] by using the translational addition theorem for vector spherical wave functions. The required computer time and memory to invert the resulting system matrix increases rapidly with the number of spheres. In addition, numerical results for certain sphere dimensions and separations are difficult to obtain by this analytical method due to the associated ill-conditioned matrices.

In this paper a novel iterative procedure is proposed for the solution to the scattering by a system of conducting spheres. This approach requires the solution of the field scattered by each sphere assumed to be alone in the incident field, which acts as an incident field on the other spheres. Therefore, the first order scattered field results from the excitation of each sphere by the incident field only, while the second order scattered field results from the excitation of each sphere by the sum of all first order scattered fields. Hence this iterative process continues until the solution converges. One of the advantages of employing this approach is that the proposed solution does not require matrix inversion and therefore the desired scattered field coefficients are obtained after each iteration and used in the subsequent iteration. Numerical results are plotted for the normalized backscattering and bistatic cross section patterns for various electrical separations, radii, angles of incidence, and also compared with published results [2] to demonstrate the efficiency of the method.

ITERATIVE SOLUTION

Let a uniform electromagnetic plane wave of arbitrary incidence and with unit electric field intensity be incident on a system of N conducting spheres centered on the z-axis. The pth sphere has a radius \( a_p \) \((p=1,2,...,N)\) and its center is at \( z=d_p \) with local Cartesian coordinates \((x_p,y_p,z_p)\). The separation distance between the centers of the pth and qth spheres is denoted by \( d_{pq} \). The incident wave is assumed to be polarized in the \( \hat{y} \) direction and the incident electric field has the form

\[
\vec{E}^i = e^{i \vec{k} \cdot \vec{r}} \hat{y}
\]  

(1)

where \( k \) is the wavenumber for the medium surrounding the spheres. The incident
The electric field can be expressed in terms of the spherical wave functions of the first kind \( \bar{M}_{mn}^{(1)} \) and \( \bar{N}_{mn}^{(1)} \), as

\[
\bar{E}^i(r_p, \theta_p, \phi_p) = \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n} [P_p(m,n) \bar{N}_{mn}^{(1)}(r_p, \theta_p, \phi_p) + Q_p(m,n) \bar{M}_{mn}^{(1)}(r_p, \theta_p, \phi_p) ]
\]

(2)

The incident field expansion coefficients \( P_p \) and \( Q_p \) being given in [2]. The electric field scattered from the pth sphere is expanded in terms of the spherical wave functions of the third kind as

\[
\bar{E}^s_p(r_p, \theta_p, \phi_p) = \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n} [A_{pE}(m,n) \bar{N}_{mn}^{(3)}(r_p, \theta_p, \phi_p) + A_{pM}(m,n) \bar{M}_{mn}^{(3)}(r_p, \theta_p, \phi_p) ]
\]

(3)

where \( A_{pE} \) and \( A_{pM} \) are the unknown field scattering coefficients, while \( \bar{M}_{mn}^{(3)} \) and \( \bar{N}_{mn}^{(3)} \) are the spherical vector wave functions that represent outgoing spherical waves.

The first order field scattered by the pth sphere results from the excitation of this sphere by the incident plane wave alone. The corresponding coefficients in (3) are obtained in the form [3]

\[
\begin{bmatrix}
\bar{A}_{pE_1} \\
\bar{A}_{pM_1}
\end{bmatrix} = \begin{bmatrix}
[v_p] & 0 \\
0 & [u_p]
\end{bmatrix} \begin{bmatrix}
\bar{P}_p \\
\bar{Q}_p
\end{bmatrix}
\]

(4)

where \( \bar{A}_{pE_1}, \bar{A}_{pM_1}, \bar{P}_p, \bar{Q}_p \) are column matrices, while \([v_p], [u_p]\) are diagonal submatrices.

The second order field scattered results from the excitation of the pth sphere by the scattered field from the remaining N-1 spheres due to the initial plane wave incident field. The total electric field at the surface of the pth sphere is equal to the sum of all the first order fields scattered from the remaining spheres plus the second order field scattered from the pth sphere, i.e.,

\[
\bar{r}_p \times \left\{ \sum_{q=1}^{N} \bar{E}^s_q (r_q, \theta_q, \phi_q) + \bar{E}^s_p (r_p, \theta_p, \phi_p) \right\} \bigg|_{r_p=a_p} = 0 \quad p=1,2,\ldots,N
\]

(5)

To enforce this boundary condition we express the outgoing vector spherical wave functions in the coordinates attached to each sphere q into incoming vector wave functions in terms of the coordinates attached to the pth sphere. Hence by applying the addition theorem and the orthogonality properties of the spherical wave functions, we obtain

\[
A_{pE_2}(m,n) = v_n(k_a) \left\{ \sum_{q=1}^{N} \sum_{\nu=1}^{\infty} [A_{mn}^{\nu}(d_{pq}) A_{qE_1}(m,\nu) + B_{mn}^{\nu}(d_{pq}) A_{qM_1}(m,\nu)] \right\}
\]

(6)

\[
A_{pM_2}(m,n) = u_n(k_a) \left\{ \sum_{q=1}^{N} \sum_{\nu=1}^{\infty} [A_{mn}^{\nu}(d_{pq}) A_{qM_1}(m,\nu) + B_{mn}^{\nu}(d_{pq}) A_{qE_1}(m,\nu)] \right\}
\]

(7)

where \( A_{pE_1} \) and \( A_{pM_1} \) are given by (4), \( A_{pE_2} \) and \( A_{pM_2} \) are the second order field scattering coefficients, \( A_{mn}^{\nu} \) and \( B_{mn}^{\nu} \) are the translation coefficients in the addition
theorem. Equations (6) and (7) may be written in a matrix form as
\[ \bar{A}_{p_2} = T \bar{A}_{q_1}, \quad p \neq q \] (8)
with
\[ T = \begin{bmatrix} [v_p] & 0 \\ 0 & [u_p] \end{bmatrix} \sum_{q=1}^{N} \begin{bmatrix} A_{pq} \\ B_{pq} \end{bmatrix} \] (9)
where \([A_{pq}]\) and \([B_{pq}]\) are square submatrices whose elements are \(A_{mn}^{\alpha} \) and \(B_{mn}^{\alpha} \).

The general expression for the \(i\)th order field scattering coefficients can be written as
\[ \bar{A}_{p_i} = T \bar{A}_{q_{i-1}}, \quad i = 2,3,\ldots, p \neq q \] (10)
Once the scattering coefficients are determined, the total scattered field from all spheres due to the \(i\)th order fields scattered can be written in the form
\[ \vec{E}^s = \sum_{p=1}^{N} \sum_{m=-n}^{n} \sum_{n=1}^{\infty} \sum_{i=1,2,\ldots} \left[ A_{pE_i(m,n)}^{(3)}(r_p,\theta_p,\phi_p) + A_{pM_i(m,n)}^{(3)}(r_p,\theta_p,\phi_p) \right] \] (11)

RESULTS AND DISCUSSIONS

Typical numerical results are computed and presented graphically for the normalized bistatic cross section patterns versus the scattering angle \(\theta\) as shown in Fig. 1, for a linear array of three and eight spheres and with endfire incidence. Fig. 2 presents the normalized backscattering cross section patterns as a function of the aspect angle \(\alpha\) for an array of three and five spheres of the same geometry. It can be seen from the above cases that, even for touching spheres, a number of only 4 iterations are sufficient in order to obtain very accurate results, i.e. at least with three decimal digits.

REFERENCES

Fig. 1. Normalized bistatic cross section patterns versus $\theta$ for a system of (a) three and (b) eight identical spheres with $k_a=0.5$, $k_d=1.0$.

Fig. 2. Normalized backscattering cross section patterns versus $\alpha$ for a system of (a) three and (b) five identical spheres with $k_a=0.5$, $k_d=1.0$. 