SINGLE INTEGRAL EQUATION WITH UNIQUE SOLUTION FOR WAVE SCATTERING BY A SYSTEM OF DIELECTRIC CYLINDERS

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Introduction

Surface integral equation models for electromagnetic wave scattering by dielectric cylinders are well known. The most common of these involve distributions of electric and magnetic surface current constrained by a pair of coupled integral equations. Less well known, but of great practical interest, are integral equations that have been developed for the solution of this same problem in terms of a single unknown current distribution. An excellent summary can be found in [1]. The first of these so called single integral equations had represented the scattered field only in terms of an electric current distribution, and had consequently suffered from non-uniqueness at frequencies corresponding to eigenvalues of the associated interior problems. More recently, single integral equations with unique solution at all frequencies have been proposed, but numerical results had not been presented [1]-[3]. Unique solution of a single integral equation is achieved by representing the scattered field in terms of a combined electric and magnetic current distribution having a fixed ratio.

In this paper, a single integral equation with unique solution, that was first developed within the context of acoustical scattering [1], is formulated for the problem of transverse magnetic wave scattering by a system of dielectric cylinders. Numerical results are obtained for the case of a single lossless homogeneous dielectric cylinder, and the effectiveness of the method is examined with respect to the ratio of magnetic and electric current densities.

Formulation

The Physical Problem: Consider a system of homogeneous dielectric cylinders of arbitrary cross section which are arranged parallel to the z-axis. The system is illuminated by an incident transverse magnetic ($H_z=0$) plane wave. The electric field only has an $E_x$ component which satisfies a homogeneous Helmholtz equation in each homogeneous dielectric subregion,

\[
\begin{align*}
(\nabla^2 + k_0^2)E_x(r) &= 0, \quad r \in V_0, \\
(\nabla^2 + k_i^2)E_x(r) &= 0, \quad r \in V_i, \quad i = 1, 2, \ldots n,
\end{align*}
\]

where the subdomain $V_0$ is the union of the surrounding free-space and all free-space voids within the $n$ multiply-connected dielectric cylinders $V_i$, and $k_0^2 = \omega^2 \varepsilon_0 \mu_0$, $k_i^2 = \omega^2 \varepsilon_i \mu_i$. A harmonic time dependence, $\exp(j \omega t)$, is assumed and suppressed. The field and its normal derivative are continuous across each interface and thus satisfy the boundary conditions

\[
\begin{align*}
(E_x(r))_+ &= (E_x(r))_-, \quad r \in S_i, \\
\rho_i \left( \frac{\partial}{\partial n} E_x(r) \right)_+ &= \left( \frac{\partial}{\partial n} E_x(r) \right)_-, \quad r \in S_i,
\end{align*}
\]

where $\rho_i = \mu_i / \mu_0$, and the unit vector normal to the contour $S_i$ of the subregion $V_i$ is directed from $V_i$ (i.e. the '-' side) to $V_0$ (i.e. the '+' side).
**Integral Representations:** We begin by expressing the true field $E_z$ as the sum of an incident field $E_{z}^{inc}$ and a scattered field $E_z^{sc}$ in the collective free-space region $V_0$. Then we define a particular field $E_0$ that is equal to the true scattered field in $V_0$, but is unrestrained in $V_i$. We represent $E_0$ and its limit in terms of a distribution of electric and magnetic surface currents as

$$E_0(r) = (a E_z^e + b E_z^m) \{J_0\}, \quad r \notin S_0,$$

(5)

$$\left( E_0(r) \right)_+ = \left( \frac{b}{2} - 1 + a E_z^e + b E_z^m \right) \{J_0\}, \quad r \in S_{0+},$$

(6)

where $a$ and $b$ are constants to be appropriately chosen, the superscripts $e$ and $m$ specify the electric and magnetic nature of the field source, respectively, $J_0$ is the single unknown current density function and $\{ \}$ indicates the action of the operator to the left of it and the respective field source, as shown in (9), (10) [5], with the operator subscripts indicating both the contour of integration and the index of the associated wave number; $I$ is the identity operator and the contour $\overline{S_0}$ is the union of all the contours $S_i$, $i = 1, 2, \ldots, n$.

Next, we define the particular field $E_i$, which is equal to the true field in $V_i$ and identically equal to zero elsewhere, that is

$$E_z(r) = E_i(r) = E_z^e \{J_i^e\} + E_z^m \{J_i^m\}, \quad r \in V_i, r \notin S_i,$$

(7)

$$0 = (E_i(r))_+ = \left( \frac{1}{2} + E_z^e \right) \{J_i^e\} + E_z^m \{J_i^m\}, \quad r \in S_i.$$  

(8)

In (5) through (8), the weakly singular operator $E_z^e$ is defined as

$$E_z^e \{J\} = \frac{-\omega h_0}{4} \int_{S_0} J(r') H^{(2)}_0(k_0 R) dl', \quad \alpha = 0, i,$$

(9)

while the Cauchy-singular operator $E_z^m$ is defined as the principal value of the integral

$$E_z^m \{J\} = \frac{-i}{4} \int_{S_0} J(r') H^{(2)}_1(k_0 R) k_0 (\hat{n'} \cdot \hat{R}) dl'. \quad \alpha = 0, i,$$

(10)

where $\hat{R} = R/R$, $R = r - r'$, $R = |R|$, $r$ and $r'$ are the position vectors of the field point and source point, respectively, $\hat{n'}$ is the unit vector normal to the surface of integration and directed from $V_i$ to $V_0$, and $H^{(2)}_0$, $H^{(2)}_1$ are the zero and first order Hankel functions of the second kind, respectively.

Using the null field constraint (8) and the boundary conditions (3) and (4), the sources of the interior field, $J_i^e$ and $J_i^m$, can be expressed in terms of the incident and scattered fields as

$$J_i^e = 0 - (H_i^e)_+ - H_i^{inc},$$

(11)

$$J_i^m = 0 - (E_i^m)_+ - E_i^{inc},$$

(12)

where $H_i^{inc}$ is the scalar component of the incident magnetic field tangent to the contour $S_i$ and where the following magnetic field representation is used,

$$\left( H_0(r) \right)_+ = \left( \frac{a}{2} - 1 + a H_0^e + b H_0^m \right) \{J_0\}, \quad r \in S_{0+},$$

(13)
in which the Cauchy-singular operator \( H_0 \) is given by
\[
H_0 \{ J_0 \} = \frac{-i}{4} \int_{S_0} J_0(\mathbf{r}') H^{(2)}_0(k_0 R) k_0 (\mathbf{n} \cdot \mathbf{R}') d\mathbf{l}'
\]
(14)
and the hyper singular operator \( H_0'' \) is defined as
\[
H_0'' \{ J_0 \} = \frac{-1}{4 \omega \mu_0} \int_{S_0} J_0(\mathbf{r}') \frac{\partial^2}{\partial n' \partial n'} H^{(2)}_0(k_0 R) d\mathbf{l}'.
\]
(15)
Substituting (13) and (6) into (11) and (12), respectively, and exploiting the commutative relationship adapted from [4],
\[
E_i' H_i'' = \frac{1}{4} I - (E_i')^2,
\]
(16)
we rewrite (8) as the combined source single integral equation
\[
\left( a \left[ \frac{1}{2} I + E_i'' \right] E_0'' + E_i' \left[ \frac{1}{2} I + H_0'' \right] \right) J_0
+ b \left[ \frac{1}{4} (1 + \rho_i) I + \frac{1}{2} (E_i'' + E_0'') + E_i'' (E_0'' - \rho_i E_i''') + E_i' \left( H_i'' - \rho_i H_i''' \right) \right] J_0
= -\left( \frac{1}{2} I + E_i'' \right) \{ E_i^{inc}(\mathbf{r}) \} - E_i' \{ H_i^{inc}(\mathbf{r}) \}, \quad \mathbf{r} \in S, \quad i = 1, 2, \ldots, n.
\]
(17)
Having solved for \( J_0 \), the scattered field in \( V_0 \) is given by (5), while (13) and (6) are used in (11), (12) to obtain the electric and magnetic current sources needed to calculate the fields in each dielectric subregion by (7).

A uniqueness theorem valid also for (17) has been proved in [1] and outlined in [2] which can be stated as: assuming that the free-space medium has some finite loss, then solutions of (17) are unique if and only if the real part of the complex ratio \( b/a \) is negative.

**Numerical Results**

The monostatic radar cross section (RCS) of a lossless dielectric circular cylinder (relative permittivity 4.0 and radius \( r \)) is calculated in the vicinity of \( k_0 r = 6.38 \), which corresponds to a resonant frequency of the unstable single integral equation formulated only in terms of electric current and designated as the MVM equation in [1]. The resonant behavior of the MVM equation is plotted in Figs. 1a) and 1b), together with the results of several combined source formulations. These results represent the minimum perturbations — from the expected slowly varying RCS — which were found by adjusting the magnitude of the ratio \( b/a \) while holding its phase angle constant. It is obvious that just any arbitrary combination of electric and magnetic current is not necessarily an improvement to the singular MVM equation. Indeed, \( b/a \) phase angles of \( 225^\circ \) and \( 270^\circ \) produce greater distortion than the MVM formulation. However, for the other phase angles tested, we noted moderate to significant improvement, in particular for the \( 45^\circ \) case. Our sampled phase angles suggest that better results may still be found in the range of \( 0^\circ \) to \( 90^\circ \).

The uniqueness theorem developed in [1] suggests that unique solutions (i.e. no perturbation) are guaranteed if the real portion of the complex ratio \( b/a \) is less than zero, otherwise irregular frequencies may be present. Though improved, our numerical results are not free of perturbations; however, there are several reasons why this might be the case. First, the uniqueness condition of [1] requires that the free-space medium be lossy (i.e. \( k_0 \) is complex with a negative imaginary
part). Our lossless example may lead to a weaker case for uniqueness. Secondly, our unknown current source is considered constant over each surface segment. This may lead to a poor, or even an altogether wrong, approximation of the magnetic charge which is modeled implicitly by (15). That is, the uniqueness theorem developed for the integral equation must be enhanced to consider the convergence of the matrix equation, particularly in the lossless case.

![Graph](image)

Fig. 1. Backscattered RCS of a lossless dielectric cylinder of relative permittivity 4.0 illuminated by a transverse magnetic plane wave close to an internal resonance, for various combined source ratios $b/a$ in the presented single integral equation.

**Conclusion**

A combined source single integral equation for transverse magnetic wave scattering by a system of dielectric cylinders has been derived. This integral equation has, in theory, a unique solution if and only if the real part of the complex ratio $b/a$ is less than zero. The numerical solution of the monostatic backscattering RCS of a lossless dielectric cylinder has shown improved results over those from a previously presented formulation. The perturbation near the eigen frequency can be dampened to some degree, but the extent of the correction is very sensitive to the selection of $b/a$, and an optimum value is not predicted by the uniqueness theorem but must be found experimentally.

**References**


