STABILIZING A SINGLE INTEGRAL EQUATION FOR TRANSVERSE ELECTRIC WAVE SCATTERING BY DIELECTRIC CYLINDERS

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Introduction

The problem of wave scattering by homogeneous dielectric obstacles can be formulated by using a single integral equation (SIE) in terms of only one unknown surface density [1]. Such a formulation assumes half the number of unknown entities required by the more common coupled surface integral equations and thus yields increased computational efficiency as demonstrated in [2]. However, SIEs formulated exclusively in terms of either electric or magnetic surface current suffer from non-uniqueness at eigen frequencies of the associated interior problem. A method of stabilizing the SIE, by employing a combination of surface electric and magnetic current defined in terms of a single unknown density, has been suggested in [3]-[5]. The first numerical implementation of the combined source SIE formulation was presented in [6] for the problem of transverse magnetic (TM) waves scattered by a homogeneous dielectric circular cylinder.

In this paper, a combined source formulation of the SIE is derived for the problem of transverse electric (TE) waves scattered by a system of homogeneous multiply-connected dielectric cylinders. The effectiveness of the method is demonstrated by numerical example. Particular attention is paid to the selection of the assumed ratio of electric current to magnetic current.

Formulation

Consider a system of \( n \) multiply-connected homogeneous dielectric cylinders of arbitrary cross section which are arranged parallel to the \( z \)-axis and illuminated by a TE plane wave (\( E_z = 0 \)). A necessary feature of this system is that each homogeneous dielectric subdomain is completely surrounded by free-space. The magnetic field only has a \( H_z \) component which satisfies a homogeneous Helmholtz equation in each dielectric region

\[
(V^2 + k_i^2)H_z(r) = 0, \quad r \in V_i, \quad i = 1, 2, \ldots, n, \tag{2}
\]

where the subdomain \( V_0 \) is the union of the surrounding free-space and all the free-space voids within the multiply-connected dielectric cylinders \( V_i \), 

\[
k_0 = \sqrt{\varepsilon_0 \mu_0} \quad \text{and} \quad k_i = \sqrt{\varepsilon_i \mu_i}
\]

are the wave numbers of the free-space and of the dielectric media, respectively, and \( r \) is the vector indicating the field point. A harmonic time dependence, \( \exp(j\omega t) \), is assumed and suppressed; \( j = \sqrt{-1} \). The tangential components of the magnetic and electric fields are continuous across each interface

\[
\Delta H_z(r) = 0, \quad r \in S_i \tag{3}
\]

\[
\Delta E_z(r) = 0, \quad r \in S_i \tag{4}
\]
where \( E_i \) is related to the normal derivative of \( H_i \), \( E_i = (j/\omega) \partial H_i / \partial n \). The total magnetic field in the collective free-space region \( V_0 \) is the sum of the known incident field \( H_i^{inc} \) and the unknown scattered field \( H_i^s \).

Construction of a SIE begins with the definition of a particular field \( H_0 \) which is identical to the true scattered field in the collective free-space region \( V_0 \) but is unrestrained elsewhere. We represent \( H_0 \) by using a layer of combined fictitious electric and magnetic current distributed on the collective interface \( S_0 = \bigcup_i S_i \) and radiating in an unbounded free-space. The combined electric and magnetic current distributions share a common unknown density \( J_0 \); however, the ratio of electric current to magnetic current may be arbitrarily manipulated. The particular field \( H_0 \) and its limit are expressed in a convenient operator notation as

\[
H_0(r) = (a H_0^n + b H_0^m) J_0, \quad r \notin S_0
\]

(5)

\[
(H_0(r))_+ = (a H_0^n + b(1/2 + H_0^m)) J_0, \quad (r \to S_0) \in V_0,
\]

(6)

respectively, where the linear operators \( H_0^n \) and \( H_0^m \) provide the electric current and magnetic current contributions to the free-space field \( H_0 \), respectively, I is the identity operator and \( a \) and \( b \) are constants to be optimized for stability. The weakly-singular operator \( H_0^m \) is defined as

\[
H_0^m J_0 = -\frac{j \omega \varepsilon_0}{4} \int_{S_0} J_0(r') H_0^{(2)}(k_0 |r - r'|) dl',
\]

(7)

while the Cauchy-singular operator \( H_0^s \) is defined as the principal value of the integral

\[
H_0^s J_0 = \frac{j}{4} \int_{S_0} J_0(r') \frac{\partial}{\partial n} H_0^{(2)}(k_0 |r - r'|) dl',
\]

(8)

where \( r \) and \( r' \) are the position vectors of the field point and source point, respectively, and \( H_0^{(2)} \) is the zero order Hankel function of the second kind.

Next, we exploit the boundary conditions (3)-(4) in order to constrain our scattered field representation by the Kirchhoff integral representation of the internal magnetic field in each dielectric region \( V_i \) and, in doing so, arrive at the SIE for TE wave scattering

\[
\left[ a \left( -\frac{1}{2} I + H_0' \right) H_0^n + H_0^m \left( -\frac{1}{2} I + E_0^n \right) \right] + b \left[ \frac{1}{2} (1 + \varepsilon_i) I - \frac{1}{2} (H_0' + H_i') + H_i' \left( H_0^m - \varepsilon_i H_i' \right) + H_i^m \left( E_0^m - \varepsilon_i E_i' \right) \right] J_0
\]

\[
= \left( \frac{1}{2} I - H_i' \right) H_i^{inc} - H_i^m E_i^{inc}, \quad i = 1, 2, \ldots, n,
\]

(9)

where the Cauchy-singular operator \( E_0^n \) and the hypersingular operator \( E_0^m \) are given by

\[
E_0^n J_0 = -\frac{j}{4} \int_{S_0} J_0(r') \frac{\partial}{\partial n} H_0^{(2)}(k_0 |r - r'|) dl'
\]

(10)

\[
E_0^m J_0 = \frac{-1}{4 \omega \varepsilon_0} \int_{S_0} J_0(r') \frac{\partial^2}{\partial n \partial n} H_0^{(2)}(k_0 |r - r'|) dl', \quad \alpha = 0, i,
\]

(11)
respectively, and the operators \( H_i^m \) and \( H_i^r \) are given by (7) and (8), respectively, upon replacing the medium subscript '0' by 'i'. The commutative relationship \( H_i^m E_i^r = \frac{1}{2} \left( H_i^r \right)^2 \) is invoked so that the hypersingular operator \( E_0^c \) only appears in the weakly-singular difference \( E_0^c - E_i^c \), where \( E_i^c = E_i / \varepsilon_0 \) is the relative permittivity of the region \( V_i \).

Upon solving (9) for \( J_0 \), the true scattered field in \( V_0 \) is given by (5), while the solution in each homogeneous dielectric region \( V_i \) is

\[
H_i(r) = H_i^m \left\{ \left( a \left( \frac{1}{2} J_0 + E_0^c \right) + b E_i^c \right) J_0 + E_i^c \right\} + H_i^r \left\{ H_i^m J_0 + H_i^m E_0^c \right\}, \quad r \in V_i. \tag{12}
\]

**Stability of the SIE**

We now examine the role played by the constants \( a \) and \( b \) in the stability of the SIE. Recognizing that (9) is a Fredholm equation of the second kind, it has a unique solution if and only if the associated homogeneous equation has only the trivial solution. Since fields constructed from the solution of (9) via (5) and (12) satisfy the TE scattering problem, we expect \( H_i^m = H_0 \) to vanish in \( V_0 \) in the absence of an incident field. However, since no constraint on \( H_0 \) in \( V_i \) has been placed, we have left open the possibility that \( H_0 \) may be non-zero in \( V_i \), while vanishing in \( V_0 \). Such a non-vanishing \( H_0 \) must satisfy the associated interior problem

\[
\left( \nabla^2 + k_0^2 \right) H_0(r) = 0, \quad r \in V_i \tag{13}
\]

\[
a H_0(r) + \frac{b}{j \omega \varepsilon_i \partial n} H_0(r) = 0, \quad r \in S_i. \tag{14}
\]

Thus, (9) has a unique solution if and only if \( k_0 \) is not an eigenvalue of (13)-(14). Standard energy considerations [3] suggest that Re\( (a/b) > 0 \) is a sufficient condition for uniqueness.

**Numerical Results and Discussion**

As a simple numerical example, the TE monostatic radar cross section (RCS) of a single homogeneous dielectric circular cylinder (relative permittivity \( \varepsilon_r = 4 \) and radius \( r_c \)) is calculated in the vicinity of \( k_0 r_c = 6.38 \) which corresponds to an eigen frequency of the unstable TE SIE (i.e. \( b = 0 \) in (9)). The resonant behavior of the unstable SIE is plotted in Fig. 1, together with the results of three combined source formulations. These three traces were selected to show a transition in the perturbation of the combined source solution — from the expected slowly varying RCS — which was found by sweeping the argument of the complex ratio \( h/a \), while fixing the magnitude \( |b/a| = 4.5 \times 10^{-6} \). The \( 139^\circ \) argument provided the best detuning of the interior resonance at this arbitrary magnitude. It is interesting to note that the singular frequency \( k_0 r_c = 6.38 \) yields a stationary point of the various combined source results. A similar stationary point is found at this frequency in the remaining three quadrants of the complex plane (not shown here).
Conclusions

A single integral equation for the scattering of transverse electric waves by a system of multiply-connected dielectric cylinders has been presented. The SIE overcomes the stability problem associated with the "magnetic current only" SIE formulation by employing a layer of combined electric and magnetic current having a single unknown density. Corresponding numerical results presented for the case of a single homogeneous circular dielectric cylinder are considerably improved. However, the optimum ratio of electric current to magnetic current is not predicted by the theory; rather, it must be determined experimentally.

References