1.2 SEPARATION OF VARIABLES FOR ELECTROMAGNETIC SCATTERING BY SPHEROIDAL PARTICLES

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1. INTRODUCTION

Exact analytic solutions to physical problems allow, in general, a more direct interpretation of the results and a better understanding of the physical mechanism than numerical method solutions. The method of separation of variables can be used to obtain exact solutions for the scattering of electromagnetic waves by objects whose surfaces coincide with coordinate surfaces in certain curvilinear orthogonal systems. In this paper, we present exact analytic solutions to problems of scattering of electromagnetic waves by single spheroids and also by systems of spheroids in arbitrary orientation, using the method of separation of variables. The vector spheroidal wave functions involved can be calculated numerically with a specified accuracy. These solutions can be used as benchmarks for evaluating the efficiency of various simpler, approximate methods of solution. Numerical results obtained are presented in the form of normalized bistatic and backscattering cross sections for single spheroids as well as for configurations of arbitrarily oriented spheroids.

2. SCATTERING BY A COATED SPHEROID

The application of the method of separation of variables is illustrated in this Section by deriving an exact solution to the problem of scattering of electromagnetic waves by a homogeneous dielectric spheroid coated with a confocal lossy dielectric material of arbitrary thickness. The electric and magnetic fields inside the spheroid, within the coating, and outside the coating, are expressed in terms of a set of vector spheroidal wave functions, and the solution is obtained by imposing the appropriate boundary conditions at each spheroidal surface.

2.1 Formulation

Consider a linearly polarized monochromatic plane electromagnetic wave with an electric field intensity of unit amplitude, incident on a dielectric spheroid with a confocal lossy dielectric coating of arbitrary thickness. The material of the spheroid and of the coating layer are assumed to be linear, homogeneous, isotropic, and lossy, with permitivities \( \varepsilon_1 \) and \( \varepsilon_2 \), and permeabilities \( \mu_1 \) and \( \mu_2 \), respectively. The semi-axial lengths of the spheroidal core are denoted by \( a_2 \) and \( b_2 \), and those of the outer surface of the confocal layer by \( a_1 \) and \( b_1 \). The inner and outer spheroidal surfaces are defined by \( \xi = \xi_2 \) and \( \xi = \xi_1 \), respectively, with \( \xi \) being the radial coordinate of a spheroidal coordinate system whose origin is at the center \( O \) of the spheroid. The major axis of the spheroid is along the \( z \)-axis of the Cartesian system \( Oxyz \).

Without any loss of generality, the incident plane can be considered to be the \( xy \)-plane \( (\phi_i = 0) \). The incident propagation vector makes an angle \( \theta_i \) with the \( z \)-axis. A linearly polarized incident wave can in general be resolved into transverse electric (TE) and transverse magnetic (TM) components. These components are usually defined in terms of the polarization angle \( \zeta \), which is the angle between the direction of the incident electric field intensity vector and the direction of the normal to the plane of incidence. For TE polarization \( \zeta = 0 \) and for TM polarization \( \zeta = \pi/2 \). It should be noted that a time dependence \( \exp(j\omega t) \) is assumed in all of the derivations and the formulae given in this paper.

The incident electric field \( E_i \) and magnetic field \( H_i \) can be expressed in terms of vector spheroidal wave functions \( M \) and \( N \) as (Sinha and MacPhie, 1983; Cooray and Ciric, 1989; Cooray and Ciric, 1991)

\[
E_i = \tilde{M}_i^{(1)T} \tilde{I} , \quad H_i = j(\varepsilon_0/\mu_0)^{1/2} \tilde{N}_i^{(1)T} \tilde{I} \tag{1}
\]

with the overbar denoting a column matrix and \( T \) denoting the transpose of a matrix. The elements of \( \tilde{M}_i^{(1)T} \) and \( \tilde{N}_i^{(1)T} \) are the spheroidal vector wave functions \( M_{m,n}^{\pm(1)} \) and \( N_{m,n}^{\pm(1)} \), respectively, and the elements of \( \tilde{I} \) are the known incident field expansion coefficients.

The scattered fields \( E_s \) and \( H_s \) for \( \xi > \xi_1 \), can be expanded in terms of vector spheroidal wave functions of the fourth kind as (Cooray and Ciric, 1989)

\[
E_s = \tilde{M}_s^{(4)T} \tilde{\alpha} , \quad H_s = j(\varepsilon_0/\mu_0)^{1/2} \tilde{N}_s^{(4)T} \tilde{\alpha} \tag{2}
\]

where the elements of \( \tilde{X}_X^{(4)T} \) \( (X = M, N) \) are the vector wave functions \( X_{m,n}^{\pm(4)} \) and \( X_{m,n}^{\mp(4)} \), and those of \( \tilde{\alpha} \) are the unknown expansion coefficients.

The field \((1)E_t \) transmitted in the region \( \xi_2 < \xi < \xi_1 \) contains both the first and the second kinds of vector spheroidal wave functions, whereas the field \((2)E_t \) transmitted in the region \( \xi < \xi_2 \) only contains the vector wave functions of the first kind. Thus, their expansions in terms of vector spheroidal wave functions can be written in the matrix forms

\[
(1)E_t = (1)\tilde{M}_t^{(1)T} \tilde{\beta} + (1)\tilde{N}_t^{(2)T} \tilde{\gamma} , \tag{3}
\]

\[
(2)E_t = (2)\tilde{M}_t^{(1)T} \tilde{\delta} . \tag{4}
\]

where the elements of matrices \((1)\tilde{M}_t^{(1)T} \) and \((1)\tilde{N}_t^{(2)T} \) are obtained from the corresponding elements of the
matrix $\tilde{M}_s^T$ by replacing the spheroidal vector wave functions of the fourth kind by those of the first kind and the second kind, respectively, taking into account the permittivity and the permeability of the coating material. The elements of the matrix $(\tilde{M}_s^T)^T$ are obtained from the corresponding elements of the matrix $(M_s^T)^T$, taking into account the permittivity and the permeability of the material inside the spheroid.

The expansions of the corresponding magnetic fields can be written as

\[
(1) H_t = j(\epsilon_1/\mu_1)^{1/2} [(1)N_s^T \beta + (1)N_s^T \beta], \\
(2) H_t = j(\epsilon_2/\mu_2)^{1/2} (2)N_s^T \delta,
\]

where the matrices $(1)N_s^T$, $(1)N_s^T$ and $(2)N_s^T$ are obtained from $(1)M_s^T$, $(1)M_s^T$ and $(2)M_s^T$, respectively, by replacing the vector wave functions $M$ by $N$.

### 2.2 Boundary Conditions

The boundary conditions require that the tangential components of the total electric and magnetic fields across each of the spheroidal surfaces $\xi = \xi_1$ and $\xi = \xi_2$ be continuous, i.e.,

\[
(\mathbf{E}_+ + \mathbf{E}_-) \times \hat{\xi}_{\xi = \xi_1} = (\mathbf{E}_+ + \mathbf{E}_-) \times \hat{\xi}_{\xi = \xi_2}, \quad \text{(7)}
\]

\[
(\mathbf{H}_+ + \mathbf{H}_-) \times \hat{\xi}_{\xi = \xi_1} = (\mathbf{H}_+ + \mathbf{H}_-) \times \hat{\xi}_{\xi = \xi_2}, \quad \text{(8)}
\]

and

\[
(1)\mathbf{E}_+ \times \hat{\xi}_{\xi = \xi_1} = (2)\mathbf{E}_+ \times \hat{\xi}_{\xi = \xi_2}, \quad \text{(9)}
\]

\[
(1)\mathbf{H}_+ \times \hat{\xi}_{\xi = \xi_1} = (2)\mathbf{H}_+ \times \hat{\xi}_{\xi = \xi_2}, \quad \text{(10)}
\]

with $\hat{\xi}$ being the unit vector normal to the spheroidal surface. After substituting the different electric and magnetic fields in Eqs. (7)-(10) with the respective expressions in Eqs. (1)-(6), applying the orthogonality properties of the trigonometric functions and the spheroidal angle functions, and integrating correspondingly over each spheroidal surface, a matrix equation is finally obtained, the solution of which is of the form (Cooray et al., 1990)

\[
\tilde{S} = [G] \tilde{T}, \quad \text{(11)}
\]

where $\tilde{S} = [\tilde{\beta} \tilde{\gamma} \tilde{\alpha} \tilde{\delta}]^T$ is the column matrix of the unknown coefficients, and $[G]$ is the system matrix whose elements are independent of the direction and polarization of the incident wave.

### 2.3 Scattered Far Field

The scattered electric field in the far zone at an observation point having spherical coordinates $r, \theta, \phi$ can be written by using the asymptotic forms of the vector spheroidal wave functions $M_{mn}^{\pm4}$, and $M_{mn}^{\pm4}$, in the form

\[
E_e(r, \theta, \phi) = \frac{e^{-jkr}}{k_0} [F_\theta(\theta, \phi) \hat{\theta} + F_\phi(\theta, \phi) \hat{\phi}], \quad \text{(12)}
\]

where $\hat{\theta}$ and $\hat{\phi}$ are unit vectors in the $\theta$- and $\phi$-directions, respectively, and $k_0$ is the wavenumber in free space. Explicit expressions of $F_\theta$ and $F_\phi$ are given in Cooray and Cirić (1989).

The normalized bistatic cross section is given by

\[
F_0 = k_0^2 r^2 |E_e(r, \theta, \phi)|^2 = |F_\theta(\theta, \phi)|^2 + |F_\phi(\theta, \phi)|^2. \quad \text{(13)}
\]

The normalized backscattering cross section is obtained from Eq. (13) with $\theta = 0$ and $\phi = 0$.

### 2.4 Quantitative Results

Numerical results are presented in this Section as plots of the normalized bistatic cross section at axial incidence, for two different prolate spheroids. Since all the matrices in the field expressions have infinite dimensions, numerical results are computed by truncating appropriately these matrices and the infinite series involved. For the illustrative examples given here, considering only the $\phi$-harmonics $e^{j0}$, $e^{j2\phi}$, and $e^{j4\phi}$ in $M_{mn}$ and $N_{mn}$, and $n=|m|, |m|+1, \ldots, |m|+5$ for each value of $m$, is sufficient to obtain a two significant digit accuracy in the computed scattering cross sections. Figure 1 shows plots of normalized bistatic cross section versus the scattering angle, for a dielectric spheroid with $a=\lambda/4$, relative permittivity $3.0$, coated with a material of relative permittivity $2.13-0.055$ and thickness $0.02\lambda$, for two axial ratios.

### 3. SCATTERING BY ARBITRARILY ORIENTED SPHEROIDS

Here we apply the method of separation of variables to obtain an exact analytic solution to the problem of scattering of a monochromatic uniform plane electromagnetic wave, of arbitrary polarization and angle of incidence, by an ensemble of dielectric prolate spheroids in arbitrary orientation. First, the incident, scattered, and transmitted electromagnetic fields are expanded in terms of appropriate vector spheroidal wave functions as in Section 2.1. The boundary conditions at the surface...
of a given spheroid are then imposed by expressing the electromagnetic field scattered by all the other spheroids in terms of the spheroidal coordinates attached to the spheroid under consideration, employing the rotational-translational addition theorems for vector spheroidal wave functions. The solution of the resultant set of algebraic equations yields the unknown scattered and transmitted field expansion coefficients, in terms of which the electromagnetic field can be evaluated at any given point.

3.1 Problem Formulation

Consider \( N \) arbitrarily oriented prolate spheroids, with their centers located at the origins \( O_r \) of the local Cartesian coordinate systems \( O_r x_r y_r z_r , r = 1, 2, \ldots , N \), attached to the \( N \) spheroids. The major axes of these spheroids are along the \( z \)-axes of the respective local Cartesian systems. The position of each of the origins \( O_r \) with respect to the global Cartesian coordinate system \( Oxyz \) is given by the spherical coordinates \( d_r, \theta_r, \phi_r \), and each system \( O_r x_r y_r z_r \) is rotated with respect to \( Oxyz \) through the Euler angles \( \alpha_r, \beta_r, \gamma_r \).

Suppose a linearly polarized uniform plane electromagnetic wave with an electric field intensity of unit amplitude is incident on the system of spheroids at an angle \( \theta _i \) with respect to the \( z \)-axis of the global system \( Oxyz \), the direction of propagation being in the \( xz \)-plane (\( \phi _i = 0 \)). This linearly polarized incident wave is decomposed into its TE and TM components using the polarization angle \( \zeta \), with respect to the \( xz \)-plane, as described in Section 2.1. The medium in which the spheroids are embedded is assumed to be isotropic and nonconducting, of permittivity \( \varepsilon \) and permeability \( \mu \).

The incident electric field \( ^r \mathbf{E}_i \) and the magnetic field \( ^r \mathbf{H}_i \) in the system \( O_r x_r y_r z_r \) can be expanded in terms of prolate spheroidal vector wave functions of the first kind expressed in terms of the coordinates of the \( r \)-th spheroidal system in the matrix form (Cooray and Ciric, 1991)

\[
^r \mathbf{E}_i = ^r \mathbf{M}_i^{(1)T} \mathbf{r}, \quad ^r \mathbf{H}_i = j(\varepsilon/\mu)^{1/2} \mathbf{N}_i^{(1)T} \mathbf{r},
\]

where \( \mathbf{r} \) is a column matrix whose elements are the known incident field expansion coefficients.

The electric field \( ^q \mathbf{E}_s \) and the magnetic field \( ^q \mathbf{H}_s \) scattered by the \( q \)-th spheroid can be expanded in terms of a set of vector spheroidal wave functions of the first kind, expressed in terms of spheroidal coordinates associated with \( O_r x_r y_r z_r \), taking into account the permittivity and the permeability of the medium inside the \( q \)-th spheroid, as (Cooray, 1990)

\[
^q \mathbf{E}_s = ^q \mathbf{M}_s^{(1)T} \mathbf{r}^q , \quad ^q \mathbf{H}_s = j(\varepsilon/\mu)^{1/2} \mathbf{N}_s^{(1)T} \mathbf{r}^q ,
\]

for \( q = 1, 2, \ldots , r-1, r+1, \ldots , N \). These \( N-1 \) secondary incident fields and the primary incident plane wave field determine the electric field \( ^r \mathbf{E}_s \) and the magnetic field \( ^r \mathbf{H}_s \) scattered by the \( r \)-th spheroid, which can be expanded as in Eq. (15) in the form

\[
^r \mathbf{E}_s = ^r \mathbf{M}_s^{(4)T} \mathbf{r}^q , \quad ^r \mathbf{H}_s = j(\varepsilon/\mu)^{1/2} \mathbf{N}_s^{(4)T} \mathbf{r}^q .
\]

The electric field \( ^r \mathbf{E}_s \) and the magnetic field \( ^r \mathbf{H}_s \) transmitted inside the \( r \)-th spheroid can be expanded in terms of a set of vector spheroidal wave functions of the first kind, expressed in terms of spheroidal coordinates associated with \( O_r x_r y_r z_r \), taking into account the permittivity and the permeability of the medium inside the \( r \)-th spheroid, and \( \mathbf{r}^0 \) is the column matrix which contains the unknown coefficients in the series expansion.

3.2 Imposing the Boundary Conditions

On the surface of each dielectric spheroid \( \xi _r = \xi _0 \) \( (r = 1, 2, \ldots , N) \), the tangential components of both electric and magnetic fields must be continuous across the boundary, i.e.,

\[
\left[ ^r \mathbf{E}_i + \sum_{q=1}^N [(1 - \delta_{qr}) ^q \mathbf{E}_s + ^q \mathbf{E}_s] \right] \times ^r \mathbf{\xi} = ^r \mathbf{E}_i \times ^r \mathbf{\xi},
\]

(21)

\[
\left[ ^r \mathbf{H}_i + \sum_{q=1}^N [(1 - \delta_{qr}) ^q \mathbf{H}_s + ^q \mathbf{H}_s] \right] \times ^r \mathbf{\xi} = ^r \mathbf{H}_i \times ^r \mathbf{\xi},
\]

(22)

for \( r = 1, 2, \ldots , N \), where \( \delta_{qr} \) is the Kronecker delta and \( ^r \mathbf{\xi} \) is the unit vector normal to the surface \( \mathbf{\xi} = \xi _0 \) of the \( r \)-th spheroid. After substituting the different electric and magnetic fields in Eqs. (21) and (22) with the respective expressions in Eqs. (14), (17)-(20), applying the orthogonality properties of the trigonometric functions and the spheroidal angle functions, and integrating correspondingly over each spheroidal surface, a matrix equation is finally obtained in the form (Cooray, 1990)

\[
\mathbf{S}_N = [G_N] \mathbf{I}_N ,
\]

(23)

where \([G_N]\) is the spheroid system matrix, which is independent of the direction and polarization of the incident wave, \( \mathbf{I}_N \) is the column matrix of the known incident field expansion coefficients, and \( \mathbf{S}_N \) is the column matrix of the unknown expansion coefficients.
3.3 Scattering Cross Sections

Illustrative numerical results are presented in this paper for a system of two spheroids in an arbitrary configuration. Using the asymptotic expressions for different vector spheroidal wave functions of the fourth kind, the electric field intensity in the far zone can finally be written as in (12). The explicit expressions of \( E_\alpha \) and \( F_\beta \) are given in Cooray (1990). The scattering cross sections are computed from Eq. (13).

![Graph of Scattering Cross Sections](image)

Fig. 2. Normalized scattering cross sections for two identical dielectric prolate spheroids of semi-major axis length \( \lambda/4 \), axial ratio 2, and relative permittivity 2.0, with one spheroid rotated by the Euler angles \( 30^\circ, 45^\circ, 90^\circ \) and displaced by a distance \( 1 \) \( \lambda/2 \) and \( 2 \) \( \lambda \) along the major axis of the other spheroid.

3.4 Numerical Results

Normalized bistatic and backscattering cross sections in the far field are computed for a system of two nonmagnetic prolate spheroids in arbitrary orientation. The bistatic cross sections are calculated for an axial incidence. Numerical results of a given accuracy can be obtained by truncating appropriately the series and the matrices involved. In order to obtain results with an accuracy of two significant digits for the examples considered in this Section, it has been found sufficient to consider only the \( \phi \)-harmonics \( e^{i\phi}, e^{i2\phi}, \) and \( e^{i3\phi} \), so that the index \( m \) in the vector wave functions \( X_{mn} \) (\( X = M, N \)) associated with the matrix elements in Eqs. (21) and (22) is limited to \( m = -2, -1, 0, 1, 2, \) and to associate with \( m \) only \( n = |m|, |m|+1, \ldots, |m|+5 \). Details regarding how this truncation affects the size of the different submatrices in the system matrix are given explicitly in Cooray (1990).

Results are presented in Fig. 2 for normalized bistatic and backscattering cross sections for a system of two identical dielectric prolate spheroids \( A \) and \( B \), of semi-major axis length \( \lambda/4 \), relative permittivity 2.0, and axial ratio 2, with the orientation of \( B \) relative to \( A \) given by the Euler angles \( \alpha = 30^\circ, \beta = 45^\circ, \gamma = 90^\circ \), and the center of \( B \) displaced relative to that of \( A \) along the z-axis of the coordinate system attached to \( A \). As the distance between the centers of the spheroids increases from \( \lambda/2 \) to \( \lambda \), more oscillations appear in both the bistatic and backscattering cross sections due to the phenomenon of multiple scattering in a more resonant system. The maxima and minima of the backscattering cross section patterns corresponding to TE and TM polarizations occur at the same angle of incidence in both cases.

4. CONCLUSIONS

Exact solutions to problems of scattering by spheroidal bodies, obtained by the method of separation of variables, are presented. Analytic expressions have been derived in terms of spheroidal wave functions for a coated dielectric spheroid and for systems of different dielectric spheroids in arbitrary orientation, under a plane wave illumination, with arbitrary polarization and direction of propagation. Numerical results are plotted for normalized bistatic and backscattering cross sections for a single spheroid and for two spheroids in the resonance region, where the major-axis length of the spheroids is comparable with the wavelength of the exciting wave.

REFERENCES


