SOLUTION OF SYMMETRIC MATRIX EQUATIONS USING STAR TO COMPLETE POLYGON CIRCUIT TRANSFORMATIONS

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Abstract: In this paper, a complete polygon electric circuit is used to model general symmetric linear algebraic systems of equations and an optimum technique is employed to solve the circuit problem based on successive star to complete polygon transformations. The values of the unknowns of a given system of equations are identical with the nodal potential values of its model circuit. The number of arithmetic operations required is about half of that for the Gaussian elimination.

Circuit Model

Consider an arbitrary symmetric system of linear equations with real or complex entries

\[
\begin{aligned}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    \vdots \\
    a_{nn}x_1 + a_{nn}x_2 + \cdots + a_{nn}x_n &= b_n \\
\end{aligned}
\]

(1)

This system can be modeled by employing an electric circuit in the form of a complete polygon (with no magnetic coupling between its branches) and a node-voltage representation of the circuit equations [1], [2]. To construct a passive two-port circuit model with an applied voltage equal to unity, we write (1) in the equivalent form [2]

\[
\begin{aligned}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + a_{1n+1}x_{n+1} &= 0 \\
    a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + a_{2n+1}x_{n+1} &= 0 \\
    \vdots \\
    a_{nn}x_1 + a_{zn}x_2 + \cdots + a_{nn}x_n + a_{nn+1}x_{n+1} &= 0 \\
\end{aligned}
\]

(2)

with \(x_{n+1} = 1\) and \(a_{i,n+1} = -b_i, \ i = 1, 2, \ldots, n\). The unknowns \(x_1, x_2, \ldots, x_n\) in (2) give, respectively, the potentials of the nodes 1, 2, \ldots, \(n\) with respect to the node of reference \(n + 2\) in the circuit in Fig. 1, when a unit voltage \(x_{n+1} = 1\) is applied to the node \(n + 1\). In this model,
The admittance of the branch connected directly between the nodes $i$ and $k$, such that the coefficient $a_{ii}$ in (2) represents the negative sum of the admittances of all the branches connected at the node $i$,

$$a_{ii} = -\sum_{k=1}^{n+2} a_{ik}, \quad a_{ik} = a_{ki}.$$  \hspace{1cm} (3)

This relationship gives the values of the admittances $a_{i,n+2}$, $i = 1, 2, \ldots, n$, of the branches connected at the node of reference $n+2$ in the circuit model.

Even though some of the conductances in the branches of the circuit in Fig. 1 could be negative, the usual techniques of circuit analysis can be applied as long as the circuit is stable.

Method of Circuit Solution

The "stars" at the nodes of the circuit in Fig. 1 are successively transformed in equivalent complete polygons. In a first step, the star of $n+1$ branches connected at the node 1 is transformed into an equivalent complete polygon connected between the nodes 2, 3, $\ldots$, $n+2$, whose branch admittances, represented by the broken line segments in Fig. 2, are given by [1].
\[ a^{(i)}_{ik} = -\frac{a_i a_{ik}}{a_{ii}}, \quad i < k, \quad i = 2, 3, \ldots, n+1, \]
\[ k = 3, 4, \ldots, n+2. \]  

Assume for the time being that \( a_{11} \neq 0 \). The parallel connections of these admittances with the original ones yield the equivalent branch admittances for the new circuit which has only the nodes 2, 3, \ldots, \( n+1, n+2 \),

\[ a^{(l)}_a = a_{ik} + a^{(l)}_{ik}, \quad i < k, \quad i = 2, 3, \ldots, n+1, \]
\[ k = 3, 4, \ldots, n+2. \]  

The transformation procedure continues in the same way for the nodes 2, 3, \ldots and at each step \( \ell, \ell = 1, 2, \ldots, n-1 \), we transform the star at the node \( \ell \), with \( n - \ell + 2 \) branches between the node \( \ell \) and the nodes \( \ell + 1, \ell + 2, \ldots, n+2 \), into a complete polygon whose branches have the admittances

\[ a^{(t-1)}_{ik} = -a^{(t-1)}_{ii} a^{(t-1)}_{ik}/a^{(t-1)}_{ii}, \quad i < k, \quad i = \ell + 1, \ell + 2, \ldots, n+1, \]
\[ k = \ell + 2, \ell + 3, \ldots, n+2, \]  

where

\[ a^{(t-1)}_{ii} = -(a^{(t-1)}_{i,i+1} + a^{(t-1)}_{i,i+2} + \cdots + a^{(t-1)}_{i,n+2}) \]
and is assumed to be different from zero. The admittances in (6) are connected in parallel with the corresponding ones from the previous step. Thus, after the step \( l, l = 1, 2, ..., n - 1 \), we remain with a circuit having only the nodes \( l + 1, l + 2, ..., n + 1, n + 2 \), whose branch admittances are

\[
a_{ik}^{(l') \cdot} = a_{ik}^{(l-1)} + a_{ik}^{(l')}, \quad i < k, \quad i = l + 1, l + 2, ..., n + 1, \\
k = l + 2, l + 3, ..., n + 2.
\]  

\[ (8) \]

Fig. 3. Circuit model after \( n - 1 \) transformations.

After the step \( n - 1 \) the transformed circuit is as shown in Fig. 3. At this stage, the potential \( x_n \) of the node \( n \) can be determined from the potentials \( x_{n+2} = 0, x_{n+1} = 1 \) and the admittances \( a_{n,n+1}^{(n-1)}, a_{n,n+2}^{(n-1)} \) as

\[
x_n = -a_{n,n+1}^{(n-1)} / a_{nn}^{(n-1)}
\]

with

\[
a_{nn}^{(n-1)} = -(a_{n,n+1}^{(n-1)} + a_{n,n+2}^{(n-1)}).
\]

\[ (9) \]

\[ (10) \]

To obtain the potentials of the other nodes, i.e. the other unknowns in the system (1), we use a backward procedure and calculate successively \( x_{n-1}, x_{n-2}, ..., x_2, x_1 \) by applying the Kirchhoff current theorem at the nodes \( n - 1, n - 2, ..., 2, 1 \), respectively. After the step \( n - 2 \) in the forward procedure the transformed equivalent circuit is that in Fig. 3 completed with the node \( n - 1 \) and with the corresponding three admittances connected to it, namely \( a_{n-1,n}^{(n-2)}, a_{n-1,n+1}^{(n-2)} \) and \( a_{n-1,n+2}^{(n-2)} \). Imposing the condition that the total current leaving this node be zero yields

\[
x_{n-1} = -(a_{n-1,n+1}^{(n-2)} + a_{n-1,n+2}^{(n-2)}) / a_{n-1,n-1}^{(n-2)},
\]

\[ (11) \]

where \( a_{n-1,n-1}^{(n-2)} \) is given by (7). Similarly, we apply Kirchhoff's current equation to the node \( l \) of the reduced equivalent circuit obtained after the step \( l - 1 \) in the forward procedure and determine the potential \( x_l \) in terms of the potentials \( x_{l+1}, x_{l+2}, ..., x_n \) already calculated,
\[
x_t = -\left( a_{t,n+1}^{(t-1)} + a_{t,n}^{(t-1)} x_n + a_{t,n-1}^{(t-1)} x_{n-1} + \cdots + a_{t,1}^{(t-1)} x_1 \right) / a_{t+1}^{(t-1)},
\]
\[\ell = n-1, \quad n-2, \ldots, 2, 1.\]

Notice that in this method there is no need to calculate the admittances \( a_{n+1,n+2}^{(t)} \) in (4), (5), (6), (8) connected between the nodes \( n+1 \) and \( n+2 \) in the successive reduced equivalent circuits.

**Complexity of the Method**

At the step \( \ell, \ell = 2, 3, \ldots, n-1 \) of the forward procedure we perform \( (n^2 + 5n + 2)/2 - (2n + 5)\ell/2 + \ell^2/2 \) additions, \( (n^2 + 3n)/2 - (2n + 3)\ell/2 + \ell^2/2 \) multiplications and \( n - \ell \) divisions, while at the first step, \( \ell = 1 \), we have \( n^2 - n \) additions, \( (n^2 + n - 2)/2 \) multiplications and \( n - 1 \) divisions. For the backward calculation of the potentials we perform \( n - \ell \) additions, \( n - \ell \) multiplications and 1 division for each \( \ell, \ell = n-1, n-2, \ldots, 2, 1 \), as well as 1 addition and 1 division to calculate \( x_n \). This gives a total of \( n^3/6 + 2n^2 - 19n/6 + 1 \) additions, \( n^3/6 + n^2 - 7n/6 \) multiplications and \( n^2/2 + n/2 \) divisions. The total number of arithmetic operations involved is about half of that in Gaussian elimination algorithm.

**Conclusion**

A passive electric circuit model is constructed for general symmetric systems of linear equations and its solution is obtained by performing generalized star to complete polygon transformations in the circuit model. The total number of arithmetic operations required is of the order of \( n^3/6 \) additions and \( n^3/6 \) multiplications, that is about half of that involved in the classical Gaussian elimination method. This reduction in the computation time comes naturally with the circuit modeling presented and corresponds to a special variant of LU factorization, namely the LDL^T factorization [3], which is in fact a "symmetric" elimination method. If necessary, a symmetric pivoting technique can be employed in the method presented. When the system matrix is symmetric but not positive definite, then special pivoting techniques may be required to stabilize the solution process [3].

**References**

