ANALYSIS OF CIRCULAR MICROSTRIP RESONATORS USING TWO-DIMENSIONAL TELEGRAPHISTS' EQUATIONS

A. Tugulea and I.R. Ciric

Department of Electrical and Computer Engineering
The University of Manitoba
Winnipeg, Manitoba, Canada R3T 5V6

INTRODUCTION

The resonant frequency of a microstrip resonator is usually calculated on the basis of a model where it is assumed that the electromagnetic field exists only in a bounded region between the conducting strip and the ground base. This implies that the scattered field at the side surface of the resonator is neglected. Resonant frequencies for the homogeneous rectangular, circular, and ring microstrip resonators were computed by many authors. In this paper we determine the resonant frequency of an inhomogeneous circular microstrip resonator based on the two-dimensional telegraphists' equations.

For time-harmonic fields, the first order two-dimensional equations are [1]

$$-\nabla U = Z\mathbf{J}_s \qquad (1)$$

$$-\nabla \cdot \mathbf{J}_s = YU \qquad (2)$$

where U is the transverse voltage, between the strip and the conducting base, $\mathbf{J}_s$ is the surface current density on the conducting strip, $Z = R + j\omega L$ is the longitudinal impedance per square, and $Y = G + j\omega C$ is the transverse admittance per unit area. The corresponding second order equations are

$$\nabla^2 U - \gamma^2 U = 0 \qquad (3)$$

$$\nabla^2 \mathbf{J}_s - \gamma^2 \mathbf{J}_s = 0 \qquad (4)$$

where $\gamma = \sqrt{ZY}$ is the complex propagation constant.

APPLICATION TO A HOMOGENEOUS CIRCULAR MICROSTRIP RESONATOR

To check the validity of the proposed solution technique, we first use the two-dimensional equations to determine the resonant frequencies of a homogeneous circular resonator for which numerical results are available in the literature [2]. Consider a resonator as shown in Fig. 1, with a homogeneous and isotropic lossless dielectric, of permittivity ε and permeability μ. In the model employed in this paper the problem region is truncated to $r \in (0,a)$, $z \in (0,h)$ by vertical walls which are either ideal ferromagnetics ($\mu \to \infty$) or perfect conductors ($\sigma \to \infty$) [3].

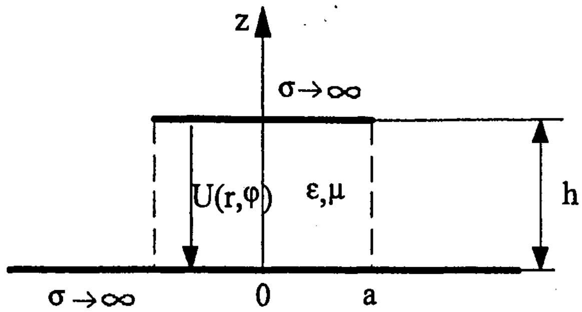

Fig. 1. Circular microstrip resonator.

Equation (3) can be written in circular cylindrical coordinates r, φ, z in the form

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2}+\omega^2\varepsilon\mu U=0 \tag{5}$$

and the associated boundary conditions are

$$\left.\frac{\partial U(r,\varphi)}{\partial r}\right|_{r=a}=0, \qquad \text{if } r=a \text{ is a magnetic wall} \tag{6}$$

$$U(a,\varphi)=0\ , \qquad \text{if } r=a \text{ is an electric wall.} \tag{7}$$

Applying the separation of variables and enforcing the regularity conditions at $r=0$ yields solutions of the form

$$U_n(r,\varphi)=(A_n\cos n\varphi+B_n\sin n\varphi)J_n(\beta r) \tag{8}$$

where n is an integer, $J_n(\beta r)$ is the Bessel function of the first kind and order n, and $\beta^2=\omega^2\varepsilon\mu$. Imposing the boundary conditions gives $J_n'(\beta a)=0$ for the model with a magnetic vertical wall, and $J_n(\beta a)=0$ for the model with an electric vertical wall. For $n=0$, the eigenvalues are obtained from the roots of the equations $J_1(\beta a)=0$ and $J_0(\beta a)=0$, respectively, for the magnetic wall and for the electric wall. The first root of $J_1(\beta a)=0$ is $\beta a\approx 3.831$, which corresponds to the TE_{010} mode [4], and for $a=10\,\text{mm}$, $\varepsilon_r=9.7$, $\mu_r=1$ the resonant frequency is 5.85 GHz. For $n=1$, the equation $J_1'(\beta a)=0$ gives $\beta a\approx 1.841$, which corresponds to the TE_{110} mode and the resonant frequency is now 2.81 GHz. Similarly, for $n=2$ one has $\beta a\approx 3.054$, which corresponds to the TE_{210} mode and the respective frequency is 4.66 GHz, and for $n=3$, $\beta a\approx 4.201$, which corresponds to the TE_{310} mode with the resonant frequency 6.41 GHz. TE_{110} is the fundamental mode and TE_{210}, TE_{010}, and TE_{310} are higher modes of oscillation. The results obtained with the proposed method are similar to those presented in [4] using the classical field theory based on Maxwell's equations.

ANALYSIS OF AN INHOMOGENEOUS CIRCULAR RESONATOR

In Fig. 2 the substrate for $r \in (0,a)$ has the permittivity ε_1 and permeability μ_1, and for $r \in (a,b)$, ε_2 and μ_2, respectively. It is assumed that the dielectric materials are lossless, homogeneous, and isotropic. The equations for the two regions are

$$\frac{\partial^2 U_1}{\partial r^2}+\frac{1}{r}\frac{\partial U_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U_1}{\partial \varphi^2}+\omega^2\varepsilon_1\mu_1 U_1=0, \qquad r \in (0,a) \tag{9}$$

$$\frac{\partial^2 U_2}{\partial r^2}+\frac{1}{r}\frac{\partial U_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U_2}{\partial \varphi^2}+\omega^2\varepsilon_2\mu_2 U_2=0, \quad r \in (a,b). \tag{10}$$

Applying the method of separation of variables and enforcing the regularity conditions at $r=0$ yields the following solutions with no dependence on φ:

$$U_1(r)=A_1 J_0(\beta_1 r), \qquad \beta_1=\omega_1\sqrt{\varepsilon_1\mu_1}, \quad r \in (0,a) \tag{11}$$

$$U_2(r)=A_2 J_0(\beta_2 r)+B_2 Y_0(\beta_2 r), \quad \beta_2=\omega_2\sqrt{\varepsilon_2\mu_2}, \quad r \in (a,b) \tag{12}$$

where Y_0 is the Bessel function of the second kind and order zero.

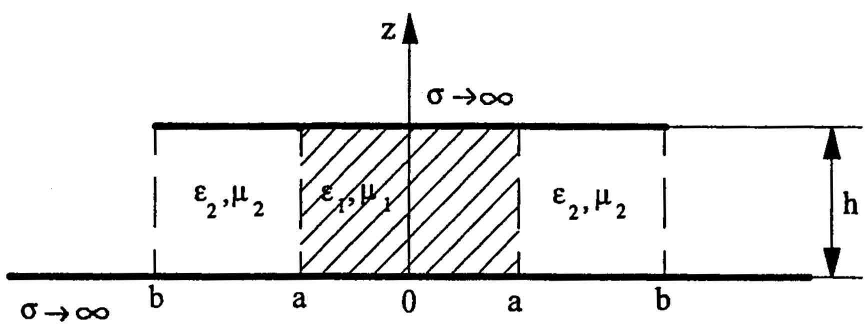

Fig. 2. Circular resonator with inhomogeneous dielectric.

Imposing the boundary conditions

$$U_1(a,\varphi)=U_2(a,\varphi) \tag{13}$$

$$\left.\frac{1}{\mu_1}\frac{\partial U_1(r,\varphi)}{\partial r}\right|_{r=a} = \left.\frac{1}{\mu_2}\frac{\partial U_2(r,\varphi)}{\partial r}\right|_{r=a} \tag{14}$$

$$\left.\frac{\partial U_2(r,\varphi)}{\partial r}\right|_{r=b} = 0, \quad \text{if } r=b \text{ is a magnetic wall} \tag{15}$$

$$U_2(b,\varphi) = 0, \qquad \text{if } r=b \text{ is an electric wall,} \tag{16}$$

the following transcendental equations are derived, corresponding to (15) and (16), respectively:

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\frac{J_1(\beta_1 a)}{J_0(\beta_1 a)} = \sqrt{\frac{\varepsilon_2}{\mu_2}}\frac{J_1(\beta_2 a)Y_1(\beta_2 b) - J_1(\beta_2 b)Y_1(\beta_2 a)}{J_0(\beta_2 a)Y_1(\beta_2 b) - J_1(\beta_2 b)Y_0(\beta_2 a)} \tag{17}$$

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\frac{J_1(\beta_1 a)}{J_0(\beta_1 a)} = \sqrt{\frac{\varepsilon_2}{\mu_2}}\frac{J_1(\beta_2 a)Y_0(\beta_2 b) - J_0(\beta_2 b)Y_1(\beta_2 a)}{J_0(\beta_2 a)Y_0(\beta_2 b) - J_0(\beta_2 b)Y_0(\beta_2 a)}. \tag{18}$$

For instance, if $a = 10\,\text{mm}$, $b = 15\,\text{mm}$, $\varepsilon_{r1} = 9.7$, $\varepsilon_{r2} = 2.32$, and $\mu_{r1} = \mu_{r2} = 1$ the first resonant frequency from (17) is 3.68 GHz and the first resonant frequency from (18) is 2.57 GHz.

CONCLUSION

The two-dimensional telegraphists' equations have been used to determine resonant frequencies of an inhomogeneous microstrip circular resonator. The solution technique is simple and equations like (17) and (18) can be easily solved employing computation tools such as Mathematica or Matlab. Other structures, e.g. microstrip resonators with variable dielectric permittivity can also be treated in the manner presented in this paper.

REFERENCES

[1] R. Radulet and A. Tugulea, "The circuit theory of the microwave two- dimensional structures", *Rev. Roum. Sci. Techn. – Electrotech. et Energ.*, vol. 28, no. 4, pp. 337-348, Dec. 1983.

[2] J. Watkins, "Circular resonant structures in microstrip", *Electronics Letters*, vol. 5, p.524, 1969.

[3] R. Bonetti and P. Tissi, "Analysis of planar disk networks", *IEEE Trans. Microwave Theory Techn.*, vol. MTT-26, pp. 471-477, July 1978.

[4] A.M. Khilla, "Ring and disk resonator CAD model", *Microwave Journal*, pp. 91-105, Nov. 1984.