SINGLE BOUNDARY INTEGRAL EQUATION FOR STATIC FIELDS
IN THE PRESENCE OF PENETRABLE BODIES

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Abstract. Laplacian electrostatic fields in the presence of dielectric bodies are usually analyzed using an integral equation for the density of the polarization charge over their surfaces. The advantage of this equation consists in its computational efficiency for systems of homogeneous bodies and in that the potential and the field intensity are calculated everywhere by employing elementary Coulombian formulass. In this paper, however, a single-source boundary integral equation is derived using the Green representation of the potential outside the dielectric bodies, a single surface source representation of the potential inside the bodies, and applying the boundary conditions. The formulation presented is important since it can easily be extended to structures containing layered bodies to derive a boundary integral equation substantially more efficient than the existent equations. Based on field analogies, the results obtained for electrostatic fields can directly be translated for other static fields in the presence of penetrable bodies.

Introduction

Consider a homogeneous dielectric body of permittivity \( \varepsilon \) immersed in an electrostatic field of potential \( \Phi_{0s} \) produced by given sources in an unbounded homogeneous three-dimensional space of permittivity \( \varepsilon_0 \).

![Figure 1. Dielectric body in electrostatic field.](image)

The resultant potential \( \Phi_0 \) in the region \( V_0 \) outside the dielectric body can be written as

\[
\Phi_0(r) = \Phi_{0s}(r) + \Phi'_0(r), \quad r \in V_0
\] (1)

where \( \Phi_0 \) is the potential induced by the presence of the body. The potential \( \Phi \) in the region \( V \) inside the body and \( \Phi'_0 \) in \( V_0 \) satisfy the Laplace equation

\[
\nabla^2 \Phi(r) = 0, \quad r \in V
\] (2)

and

\[
\nabla^2 \Phi'_0(r) = 0, \quad r \in V_0
\] (3)

with the boundary conditions at the surface \( S \) of the body

\[
\Phi(r) = \Phi'_0(r), \quad r \in S
\] (4)

\[
\frac{\varepsilon}{\varepsilon_0} \frac{\partial \Phi(r)}{\partial n} = \frac{\varepsilon_0}{\varepsilon} \frac{\partial \Phi'_0(r)}{\partial n}, \quad r \in S
\] (5)
and with
\[ \Phi'_0(r) \to 0 \quad \text{as} \quad r \to \infty \] (6)

where \( r \) is the position vector of the observation point and \( \partial / \partial n \) denotes the normal derivative.

An integral equation formulation of this field problem can be attempted using the Green function representation [1] of the potential in \( V \) and in \( V_0 \), separately, and imposing the boundary conditions at \( S \). Following such an approach would yield a pair of coupled surface integral equations in \( \Phi \) and \( \partial \Phi / \partial n \) over \( S \), similar to the coupled integral equations in the theory of electromagnetic wave scattering. A boundary integral equation for a single unknown function was developed by employing the Coulombian model of the polarized dielectric bodies [2] and the Ampère model of the magnetized bodies [3]. Single surface integral equations were also obtained for the scattering of electromagnetic and acoustic waves (a short presentation of some references is made in [4]), and the computational efficiency of various two-dimensional formulations was analyzed in [5].

In this paper, a single integral equation is formulated for three- and two-dimensional Laplacian fields in the presence of penetrable bodies by using the classical Green representation of the potential outside the body, a single surface source representation of the potential inside the body, and by directly imposing the interface conditions.

Formulation

Let us use for the three-dimensional potential \( \Phi_0 \) in \( V_0 \) the Green representation [1]
\[ \Phi_0(r) = \Phi_0 s(r) + \frac{1}{4 \pi \varepsilon_0} \left[ \int_S \frac{\Phi(s(r))}{R} dS' + \int_S \rho^d(r) \frac{\partial}{\partial n} \left( \frac{1}{R} \right) dS' \right], \quad r \in V_0 \] (7)

where \( R = |r - r'| \) \( r' \) is the position vector of the integration point, and
\[ \rho^s(r) = -\varepsilon_0 \frac{\partial \Phi_0(r)}{\partial n}, \quad \rho^d(r) = \varepsilon_0 \Phi_0(r), \quad r \in S \] (8)

with the derivatives taken with respect to the direction of the normal shown in Fig. 1; \( \rho^s \) and \( \rho^d \) represent the densities of a single layer and of a double layer of electric charge over \( S \).

The potential \( \Phi \) in \( V \) can be represented in terms of a single unknown function \( \rho \) over \( S \), as a single-layer potential or a double-layer potential, or as a linear combination of the potentials of a single and a double layer of same density. Using, for instance, a single-layer potential of density \( \rho \) gives
\[ \Phi(r) = \frac{1}{4 \pi} \int_S \frac{\rho(s(r))}{R} dS', \quad r \in V \] (9)

To shorten the writing, the integral operators \( \mathcal{G}^s \) and \( \mathcal{G}^d \) are defined by
\[ \mathcal{G}^s x = \frac{1}{4 \pi} \int_S \frac{x(r')}{R} dS', \quad r \in S \] (10)
\[ \mathcal{G}^d x = \frac{1}{4 \pi} \int_S x(r') \frac{\partial}{\partial n} \left( \frac{1}{R} \right) dS', \quad r \in S \] (11)

with the integral in (11) taken in principal value. Then, the potentials on the boundary can be written in the form
\[ \Phi_0(r) = \Phi_0 s(r) + \frac{1}{\varepsilon_0} \left[ \mathcal{G}^s \rho^s + \frac{1}{2} I + \mathcal{G}^d \rho^d \right], \quad r \in S \] (12)
\[ \Phi (r) = \frac{1}{\varepsilon} \mathcal{G}^s \rho, \quad \quad \quad r \in S \] (13)

where \( I \) is the identity operator and \( S \) is assumed to be a smooth surface, with \( r \to S \) in (12) from \( V_0 \). The normal derivative of \( \Phi \) on \( S \) is obtained from (9) as
\[
\frac{\partial \Phi(r)}{\partial n} = \frac{1}{\varepsilon} \left( \frac{1}{2} I + Q^s \right) \rho, \quad r \in S
\]  

where \( r \rightarrow S \) from \( V \) and the operator \( Q^s \) is defined by

\[
Q^s \rho = \frac{1}{4\pi} \int \frac{\rho(r')}{\varepsilon} \frac{1}{\varepsilon_0} \overline{\vec{F}'} \cdot \overline{\vec{F}}' \, dS', \quad r \in S
\]  

with the integral taken in principal value. The discontinuities in the double-layer potential and in the normal derivative of the single-layer potential are taken into account by adding \( I/2 \) to \( \mathcal{G}^d \) in (12) and to \( Q^s \) in (14), respectively.

To satisfy the boundary conditions (5) and (4) we obtain from (8), (14) and (13)

\[
\rho^s(r) = -\left( \frac{1}{2} I + Q^s \right) \rho, \quad \rho^d(r) = \frac{\varepsilon_0}{\varepsilon} \mathcal{G}^s \rho, \quad r \in S
\]  

Substituting (16) in (12) and, then, (13) and (12) in (4) yields the surface integral equation for \( \rho \)

\[
\left[ \frac{1}{\varepsilon_0} \mathcal{G}^d + \frac{1}{\varepsilon} \mathcal{G}^s \left( \frac{1}{2} I + Q^s \right) \right] \rho = \Phi_{0s}(r), \quad r \in S
\]  

Remark. Using the classical Green representation of the potential \( \Phi \) in \( V \), the potential on the boundary \( S \) is expressed as

\[
\Phi(r) = \mathcal{G}^s \frac{\partial \Phi}{\partial n} + \left( \frac{1}{2} I - \mathcal{G}^d \right) \Phi, \quad r \in S
\]  

where \( r \rightarrow S \) from \( V \). Thus,

\[
\frac{\partial \Phi(r)}{\partial n} = \left( \mathcal{G}^s \right)^{-1} \left( \frac{1}{2} I + \mathcal{G}^d \right) \Phi, \quad r \in S
\]  

On the other hand, from (14) and (13)

\[
\frac{\partial \Phi(r)}{\partial n} = \left( \frac{1}{2} I + Q^s \right) \left( \mathcal{G}^s \right)^{-1} \Phi, \quad r \in S
\]  

Equations (19) and (20) yield the commutative property

\[
\mathcal{G}^s Q^s = \mathcal{G}^d \mathcal{G}^s
\]  

and the single integral equation (17) is, thus, simplified to

\[
\left[ \frac{1}{\varepsilon_0} \left( \frac{1}{2} + \frac{1}{\varepsilon} \right) I + \left( \frac{1}{\varepsilon} - \frac{1}{\varepsilon_0} \right) \mathcal{G}^d \right] \mathcal{G}^s \rho = \Phi_{0s}(r), \quad r \in S
\]  

with only one operator-operator multiplication.

The solution of this integral equation gives the single function \( \rho \) over \( S \). Once \( \rho \) is obtained, \( \rho^s \) and \( \rho^d \) are determined from (16), and the potential everywhere in \( V_0 \) and \( V \) is calculated with (7) and (9), respectively. Since the potential satisfies the equations (2) and (3), and the boundary conditions (4) - (6), according to the uniqueness theorem for Laplacian fields [1], the formulation presented gives the sought solution to the potential problem considered, if the equation (22) has a unique solution.
Example

Consider an infinitely extended plane interface between two semispaces of permittivities $\varepsilon_1$ and $\varepsilon_2$, and a point charge $Q$ located as shown in Fig. 2.

In this special configuration, the operators $\mathcal{G}^d$ and $\mathcal{Q}^s$ are null operators and the single surface integral equation reduces to

$$\frac{1}{2} \left( \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} \right) \mathcal{G}^s \rho = \frac{Q}{4\pi\varepsilon_1 R_S}, \quad r \in S \quad (23)$$

Thus,

$$\mathcal{G}^s \rho = \frac{\varepsilon_2 Q}{2\pi(\varepsilon_1 + \varepsilon_2) R_S}, \quad \rho^s = -\frac{\rho}{2}, \quad \rho^d = \frac{\varepsilon_1}{\varepsilon_2} \mathcal{G}^s \rho, \quad r \in S \quad (24)$$

and the potentials on the boundary $S$ are (see (12) and (13))

$$\Phi_1(r) = \frac{Q}{4\pi\varepsilon_1 R_S} + \frac{1}{2\varepsilon_1} \left( -1 + \frac{\varepsilon_1}{\varepsilon_2} \right) \mathcal{G}^s \rho$$

$$= \frac{Q}{4\pi\varepsilon_1 R_S} + \frac{Q}{4\pi\varepsilon_1 R_S} \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}, \quad r \in S \quad (25)$$

$$\Phi_2(r) = \frac{Q}{2\pi(\varepsilon_1 + \varepsilon_2) R_S}, \quad r \in S \quad (26)$$

The potential $\Phi_1$ in $V_1$ is as if produced in an unbounded space of permittivity $\varepsilon_1$ by the original point charge $Q$ and by its "image" $Q'$ located in $V_2$, as shown in Fig. 2,

$$\Phi_1(r) = \frac{Q}{4\pi\varepsilon_1 R} + \frac{Q'}{4\pi\varepsilon_1 R'}, \quad Q' = Q \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}, \quad r \in V_1 \quad (27)$$

The potential $\Phi_2$ in $V_2$ is as if produced in an unbounded space of arbitrary permittivity $\varepsilon$ by a single point charge $Q'$ replacing the original charge in $V_1$,

$$\Phi_2(r) = \frac{Q''}{4\pi\varepsilon R'}, \quad Q' = Q \frac{2\varepsilon}{\varepsilon_1 + \varepsilon_2}, \quad r \in V_2 \quad (28)$$

Fig. 2. Point charge in the presence of a plane dielectric interface.
Concluding Remarks

Another formulation can be developed by using the Green representation of the potential inside the body and a single surface source representation for the potential outside the body. As well, instead of being the density of a single layer, the single unknown function on the boundary of the body can be chosen to be the density of a double layer. Thus, four different single integral equations can be constructed for the field problem considered [4]. Also, the potential in one of the two regions can be represented as a linear combination of the potential of a single layer and a double layer of same density, which allows the construction of two more general integral equations for an unknown function over the boundary of the body.

The expressions presented are valid for smooth surfaces. When the body boundary is not smooth, then the potential and its normal derivative at a point on an edge or at a vertex are evaluated by using the actual solid single under which a vanishing neighborhood on the boundary is seen from the point considered.

The formulations for two-dimensional field problems are obtained replacing $1/(4\pi R)$ by the two-dimensional Green function $1/(2\pi \ln(1/R))$.

Single integral equations for Laplacian fields in the presence of layered material body structures derived by a recursive application of the procedure presented in this paper are published elsewhere.

References


