Iterative FEM-BEM Technique for an Efficient Computation of Magnetic Fields in Regions with Ferromagnetic Bodies

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Abstract — Inside the ferromagnetic bodies the magnetic field is determined by applying the Finite Element Method (FEM) with a Dirichlet boundary condition for the magnetic potential. The resulting tangential component of the field intensity is used as boundary condition for the exterior field problem whose solution is obtained by the Boundary Element Method (BEM) which provides the new boundary condition for the interior problem. The nonlinearity of the highly permeable ferromagnetic material is treated by implementing the Polarization Fixed Point Method (PFPM), where the magnetic polarization is corrected in terms of the field intensity. While preserving the intrinsic advantages of the FEM and the BEM, separately, the proposed technique also allows to easily take into consideration the terminal voltage of the coils as input data and a simple field solution for multiply connected structures, such as transformers, electrical machines, and a multitude of electromagnetic devices.

Index Terms—boundary integral equations, hybrid FEM-BEM techniques, iterative methods, multiply connected regions, nonlinear media.

I. INTRODUCTION

DIFFiculties encountered when employing the FEM for the analysis of fields in unbounded regions or in structures with moving bodies, as well as those related to parasitic sources artificially generated at the interfaces between the discretization elements, with undesirable implications in force calculations, have determined the development of various integral and hybrid methods [1]-[7]. The application of the PFPM [8] has allowed the extension of integral formulations to nonlinear media with periodic eddy currents and moving bodies [9]-[10].

In this work, a novel iterative FEM-BEM procedure is proposed for an efficient solution of magnetic field problems in unbounded regions in the presence of ferromagnetic bodies. A vector potential formulation is illustrated for two-dimensional fields in simply and multiply connected structures. The field problems inside and outside the ferromagnetic regions are solved separately and repeatedly with the interior field determined by applying the FEM under Dirichlet boundary conditions and the exterior one by the BEM under Neumann boundary conditions, the solution in each region providing successively the boundary condition for the problem in the other region. A highly rate of convergence is insured by treating the material nonlinearity with the PFPM, where the magnetic polarization is corrected in terms of the field intensity.

II. FIELD PROBLEM IN LINEAR MAGNETIC BODIES

Assuming a null current density, the magnetic vector potential \( A' = kA' \) for a two-dimensional structure satisfies the following equation inside the region \( \Omega \) occupied by linearized ferromagnetics:

\[
-\nabla \cdot \left( \frac{1}{\mu} \nabla A' \right) = k \left( \nabla \times \left( \frac{1}{\mu} I \right) \right)
\]

where \( k \) is the unit vector in the longitudinal direction, \( \mu \) is the magnetic permeability, and \( I \) represents a given distribution of magnetic polarization. The potential \( A' \) is known on the boundary \( \partial \Omega \) of \( \Omega \). We represent \( A' \) in \( \Omega \) as

\[
A' = \sum_{i \in \{ n \}} A'_i \varphi_i + \sum_{i \in \{ n_B \}} A'_i \varphi_i
\]

where \( \{ n \} \) and \( \{ n_B \} \) represent the sets of nodes of the discretization mesh inside \( \Omega \) and on \( \partial \Omega \), respectively, and \( \varphi_i \) and \( A'_i \) are the linear nodal element function and the potential values at the node \( i \). Applying the FEM yields

\[
\sum_{i \in \{ n \}} a_{ji} A'_i = -\sum_{i \in \{ n_B \}} a_{ji} A'_i + t_j, \quad j \in \{ n \}
\]

with

\[
a_{ji} = \frac{\int_{\Omega} \nabla \varphi_j \cdot \left( \frac{1}{\mu} \nabla \varphi_i \right) dS}{\Omega}
\]

\[
t_j = \frac{\int_{\Omega} \left( \nabla \varphi_j \times \kappa \right) \left( \frac{1}{\mu} I \right) dS}{\Omega}
\]

The values \( A'_i \) are known for \( i \in \{ n_B \} \) and the matrix of \( a_{ji} \)'s in (3) is sparse, symmetric and positive definite. The memory space for the matrix entries is practically proportional to \( n \) and the solution of (3) is performed very rapidly using various sparse matrix techniques. From the
solution of (3) one first obtains $A'$ at the interior nodes and then the tangential field intensity on $\partial \Omega$.

III. FIELD PROBLEM IN FREE SPACE

First, assume the ferromagnetic region to be simply connected. The potential integral equation on the boundary $\partial \Omega$ of the free space unbounded region $\Omega_0$ is obtained from

$$\alpha A(r) = \oint_{\partial \Omega} \frac{(R \cdot n)}{R^2} A(r') dl' - \oint_{\partial \Omega} \frac{1}{R} \frac{\partial A(r')}{\partial n} dl' + A_0(r) \quad (6)$$

where $\alpha$ is the solid angle under which a small neighborhood of $n_0$ is seen from the observation point, $r$ and $r'$ are the position vectors of the observation and the source points, respectively, $R = r - r'$, $R = |R|$, $n$ is the unit vector in the direction normal to the boundary (see Fig. 1), and $A_0$ is the vector potential due to the given current distribution,

$$A_0(r) = \mu_0 \frac{\theta}{S} \left( \int_{S_+} \frac{1}{R} ds' - \int_{S_-} \frac{1}{R} ds' \right) = \theta (Q_+(r) - Q_-(r)) \quad (7)$$

for a total $\theta$ of Ampère-turns of the coils, considered to be uniformly distributed over the cross sections of areas $S_+$ and $S_-$ of the coil sides carrying current in the direction of $k$ and $-k$, respectively, with $S_+ = S_- = S$ (see Fig. 2). For coupling the BEM applied to (6) with the FEM in Section II, the vector potential is taken to have a linear variation on each boundary element with a constant normal derivative. Locating the observation point at the boundary nodes, (6) is transformed into

$$\Lambda A = MA + N \frac{\partial A}{\partial n} + \theta C \quad (8)$$

where the symbols $A$, $\frac{\partial A}{\partial n}$, and $C$ are now used to denote the matrices $n_B \times 1$ of the boundary values of $A$ and $\frac{\partial A}{\partial n}$ in (6) and of $Q_+ - Q_-$ in (7), $\Lambda$ is the diagonal matrix of the values of $\alpha$ at the boundary nodes, and $M$ and $N$ are some $n_B \times n_B$ matrices. Equation (8) gives $A$ when $\frac{\partial A}{\partial n}$ is known,

$$A = Z \frac{\partial A}{\partial n} + \theta T \quad (9)$$

with $Z = (\Lambda - M)^{-1} N$ and $T = (\Lambda - M)^{-1} C$. At any observation point inside $\Omega_0$ the vector potential is calculated from

$$2\pi A(r) = M'(r)A + N'(r) \frac{\partial A}{\partial n} + \theta (Q_+(r) - Q_-(r)) \quad (10)$$

where $M'$ and $N'$ are $n \times n_B$ matrices.

IV. ITERATIVE PROCEDURE

At the boundary $\partial \Omega$ between $\Omega_0$ and $\Omega$ we connect the vector potential and its normal derivative in the two regions by imposing the continuity condition for the vector potential and for the tangential component of field intensity, i.e.

$$\frac{\partial A}{\partial n} = \frac{1}{\mu_r} (\frac{\partial A'}{\partial n} - I_t) \quad (11)$$

where $I_t$ is the tangential component of the polarization. The bigger the value of relative permeability $\mu_r$, the stronger a contraction will result from (11) for the normal derivatives of the potential. Thus, the following iterative process is proposed:

a) having known $\frac{\partial A}{\partial n}$ on the elements of $\partial \Omega$, $A' = A$ at the nodes of $\partial \Omega$ are obtained from (9), i.e. by BEM;

b) from $A'$ on the boundary, one determines $A'$ in $\Omega$ by the FEM and, then, $\frac{\partial A'}{\partial n}$ on $\partial \Omega$;

c) $\frac{\partial A}{\partial n}$ in a) is corrected with (11).

This cycle is repeated until the difference between $\frac{\partial A}{\partial n}$ for two successive iterations satisfies an error condition, for instance

$$\int_{\partial \Omega} \left( \frac{\partial A}{\partial n}_{\text{old}} - \frac{\partial A}{\partial n}_{\text{new}} \right)^2 dl < \epsilon^2 \int_{\partial \Omega} \left( \frac{\partial A}{\partial n}_{\text{new}} \right)^2 dl \quad (12)$$

with $\epsilon$ conveniently small. The matrices $Z$ and $T$ are computed only once, before the start of the iterative process.

When the terminal voltages of the coils are given, instead of the coil currents, the magnetic fluxes of the coils are known. For illustrating the computation procedure in this case, consider a system with only one coil. The average magnetic flux per coil turn is obtained from (10) as

$$\Phi = \int_{\partial \Omega} \frac{\partial A}{\partial n} dl + C \quad (13)$$

with the entries of the $1 \times n_B$ matrices $M_\Phi$ and $N_\Phi$ obtained from
\[ M_{\Phi} = \frac{1}{2\pi S} \left( \int_{S_+} M'(r) dS - \int_{S_-} M'(r) dS \right), \]  
(14)

\[ N_{\Phi} = \frac{1}{2\pi S} \left( \int_{S_+} N'(r) dS - \int_{S_-} N'(r) dS \right), \]  
(15)

and

\[ Q_{\Phi} = \frac{1}{2\pi S} \left( \int_{S_+} (Q_+(r) - Q_-(r)) dS - \int_{S_-} (Q_+(r) - Q_-(r)) dS \right) \]  
(16)

Taking into account (9), we get

\[ \Phi = W \frac{\partial A}{\partial n} + \theta \gamma \]  
(17)

where \( W = M_{\Phi} Z + N_{\Phi} \) and \( \gamma = M_{\Phi} T + Q_{\Phi} \). We remark that if two different values of \( \theta \), say \( \theta' \) and \( \theta'' \), correspond (according to (17)) to \( \Phi^i \) and \( \Phi^u \), respectively, then the differences \( \Delta \theta = \theta' - \theta'' \) and \( \Delta \Phi = \Phi^i - \Phi^u \) are simply related as

\[ \Delta \Phi = \gamma \Delta \theta \]  
(18)

As a consequence, the step a) in the above iterative procedure is modified as follows. For an arbitrary \( \theta_0 \), \( \Phi_0 \) is determined with (17) and, then, to the imposed value of \( \Phi \) corresponds a \( \theta = \theta_0 + \Delta \theta \), with \( \Delta \theta \) determined from (18), i.e. \( \Delta \Phi = \Phi \gamma \). This value of \( \theta \) is used in (9) and we continue with step b). \( \Phi_0 \) is a \( 1 \times n_B \) matrix and \( \gamma \) a scalar which are, again, computed only once, before starting the iterative process.

V. TREATMENT OF MULTIPLY CONNECTED REGION

To explain the technique proposed for the field analysis in multiply connected regions, consider the magnetic circuit of a single-phase transformer, sketched with only one of its coils in Fig. 2. On the boundaries \( \partial \Omega^e \) and \( \partial \Omega^i \) of the two nonmagnetic regions \( \Omega^e \) and \( \Omega^i \) we consider, respectively, \( n_B^e \) and \( n_B^i \) nodes. As in (9), for \( \Omega_0^e \) we have

\[ A^e = Z^e \frac{\partial A^e}{\partial n} + \theta T^e \]  
(19)

but only with the coil side of cross-sectional area \( S_+ \). \( Z^e \) is a \( n_B^e \times n_B^e \) matrix and \( T^e \) a \( n_B^e \times 1 \) matrix. For the inner subregion \( \Omega_0^i \), the potential is determined up to an additive constant and the matrix \( A^i - M^i \) is now singular. Let's, therefore, fix a zero value of reference for the potential at a chosen node \( P^i \) and determine the potentials at the other \( n_i^i = n_i^e - 1 \) nodes from

\[ A_i^i = Z_i^i \frac{\partial A^i}{\partial n} + \theta T^i + c_0 \]  
(20)

where \( Z^i \) and \( T^i \) are \( n_i^i \times n_B^i \) and \( n_i^i \times 1 \) matrices, respectively, \( Z^i = \left( A_i^i - M_i^i \right)^{-1} N_i^i \), \( T^i = \left( A_i^i - M_i^i \right)^{-1} C_1^i \) with \( A_i^i \), \( M_i^i \) and \( N_i^i \) being the matrices in (8) but for only \( n_i^i \) nodes, and \( C_1^i \) the \( n_i^i \times 1 \) matrix of the boundary values \( Q_i^i \) in (7).

In the case when the coil current is given, for known values of the normal derivative, the potential on \( \partial \Omega^e \) is obtained with (19) and that on \( \partial \Omega^i \) with (20), the latter up to an unknown additive constant \( c_0 \). The potential in \( \Omega \) satisfies the equation (see (3))

\[ \sum_{i \in \{n\}} a_{ji} A_{i} = - \sum_{i \in \{n_B^e\}} a_{ji} A_{i} - c_0 \sum_{i \in \{n_B^i\}} a_{ji} + \theta j, \quad j \in \{n\} \]  
(21)

where \( n_B = n_B^e + n_B^i \). \( A^i \) in \( \Omega \) can be determined as follows. First, without any additive constant, we apply the FEM in \( \Omega \) and obtain the potential \( A^e \) from (3), with the boundary conditions given by \( A^e = A^e \) on \( \partial \Omega^e \) and \( A^i = A^i \) on \( \partial \Omega^i \). \( A^e \) and \( A^i \) being those computed from (19) and (20), respectively.

In general, this solution corresponds to an incorrect total \( \theta'' \) of Ampère-turns, i.e. \( \theta'' = \theta' \),

\[ \frac{1}{\mu_0} \int_{\partial \Omega^i} \frac{\partial A^i}{\partial n} dl = \theta'' \]  
(22)
Then, we obtain the potential \( A^c \) in \( \Omega \) corresponding to a zero potential on \( \partial \Omega^e \), a unity value of it on \( \partial \Omega^l \) and to a null polarization, from

\[
\sum_{i \in \mathbb{N}} a_{ji} A^c = - \sum_{i \in \mathbb{N}} a_{ji} A^l, \quad j \in \mathbb{N} \tag{23}
\]

i.e. (21) with \( c_0 = 1 \) and \( t_j = 0 \). This \( A^c \) corresponds to a \( \theta^c \) Ampère-turns of the coil. The correct potential \( A' \) in \( \Omega \) is given by

\[
A' = A'' + A^c \frac{\Delta \theta}{\theta^c} \tag{24}
\]

where \( \Delta \theta = \theta - \theta^c \). \( A^c \) is calculated only once before starting iterating.

For the case when the coil terminal voltage is given, we notice that, while the potential at the point \( p_i \) on \( \partial \Omega^l \) is equal to zero for the problem in \( \Omega^l \), the actual potential at \( p_i \) has now a value \( c_0 \in \mathbb{R} \) to be determined. The average magnetic flux per turn is evaluated by applying (17) to the interior and exterior subregions, and Ampère's theorem to the boundaries \( \partial \Omega^l \) and \( \partial \Omega^e \),

\[
\Phi = W I \frac{\partial A^l}{\partial n} - \frac{1}{\mu_0} \oint_{\partial \Omega^l} \frac{\partial A^l}{\partial n} dl + c_0
\]

\[
- W I \frac{\partial A^e}{\partial n} + \frac{1}{\mu_0} \oint_{\partial \Omega^e} \frac{\partial A^e}{\partial n} dl
\]

This yields the constant \( c_0 \) which has to be added to the potential in \( \Omega^l \) computed from (20).

**VI. FIELD ANALYSIS IN NONLINEAR MEDIA**

The field problem in nonlinear ferromagnetic regions is solved iteratively by employing the PFPM \([8]\). The nonlinear constitutive relation \( B = \mathbf{F}(H) \) is replaced by

\[
B = \mu H + I \tag{26}
\]

where \( \mu \) is a constant and the additive term \( I \), playing the role of a magnetic polarization, is corrected iteratively in terms of \( H \) from

\[
I = \mathbf{F}(H) - \mu H = \mathbf{G}(H) \tag{27}
\]

Starting with some distribution of \( I \), a magnetic field \( (B, H) \) is determined using (26) and then, \( I \) is corrected using (27). These two steps are repeated until the norm of the difference between the polarizations in two successive iterations is sufficiently small.

**VII. ILLUSTRATIVE EXAMPLES**

The magnetic field of the electromagnet in Fig. 3 is produced by a coil with 5,000 turns carrying a direct current of 49 mA. The thickness of the ferromagnetic region is 19.5 mm and the stacking factor of the laminations is 0.92. For the \( B-H \) characteristic shown in Fig. 4, the minimum and the maximum values of the differential permeability are
The magnetic flux of the winding when the voltage drop corresponding to its resistance is negligible. The region occupied by the ferromagnetic material is doubly connected. The transformer is operated on no-load with the same \( B-H \) relationship with the PFPM, whose convergence is insured when the correction of the magnetic polarization is made in terms of the magnetic field intensity and the constant \( \mu \) in (26) is chosen to be greater than half of the maximum value of the differential permeability in the operating zone. The field problem inside the ferromagnetic bodies is solved by implementing the FEM under Dirichlet boundary conditions, the system matrix being sparse, symmetric and positive definite. The required memory space and the computation time are very small. The boundary conditions are corrected by solving the exterior field problem applying the BEM where the stiffness matrix is dense, but is calculated only once, before the start of the iterations.

As well, an extension of the proposed procedure to field problems for multiply connected regions is presented, where the BEM applied to bounded nonmagnetic regions allows the determination of the vector potential up to an additive constant. It should be remarked that the technique developed in this paper deals easily with situations when either the coil currents or their terminal voltages are given, thus the flux-current relationships necessary for the modeling of field structures by lumped parameter circuits being efficiently established.

In the proposed procedure, all the advantages of the FEM are kept regarding the system matrix structure and, at the same time, all the advantages of the BEM are exploited. It is also advantageous to apply the procedure to structures with moving bodies, since only the entries of the BEM matrices corresponding to the coupling between the bodies in relative motion are to be recalculated for various specific positions. In the air regions the magnetic field equations are exactly satisfied, thus avoiding the fictitious generation of the parasitic current sheets specific to the FEM. As a consequence, when calculating forces, the flux of the magnetic stress tensor is rigorously independent of the integration surfaces.

**VIII. CONCLUSIONS**

The main contribution to the computational electromagnetics of the work presented consists in the efficient iterative procedure for the solution of the equations in the hybrid FEM-BEM for structures containing nonlinear ferromagnetic bodies. The rapid convergence of the procedure is determined by the big difference between the permeabilities of the ferromagnetic bodies and that of the air. This difference is preserved by treating the nonlinearity of \( B-H \) relationship with the PFPM, whose convergence is insured when the correction of the magnetic polarization is made in terms of the magnetic field intensity and the constant \( \mu \) in (26) is chosen to be greater than half of the maximum value of the differential permeability in the operating zone.
IX. REFERENCES


X. BIOGRAPHIES

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