Convergence acceleration in the polarization method for nonlinear periodic fields

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Abstract

Purpose - The purpose of this paper is to present three novel techniques aimed at increasing the efficiency of the polarization fixed point method for the solution of nonlinear periodic field problems.

Design/methodology/approach - Firstly, the characteristic $B-M$ resulting from the constitutive relation $B-H$ is replaced by a relation between the components of the harmonics of the vectors $B$ and $M$. Secondly, a dynamic overrelaxation method is implemented for the convergence acceleration of the iterative process involved. Thirdly, a modified dynamic overrelaxation method is proposed, where only the relation $B-M$ between the magnitudes of the field vectors is used.

Findings - By approximating the actual characteristic $B-M$ by the relation between the components of the harmonics of the vectors $B$ and $M$, the amount of computation required for the field analysis is substantially reduced. The rate of convergence of the iterative process is increased by implementing the proposed dynamic overrelaxation technique, with the convergence being further accelerated by applying the modified dynamic overrelaxation presented. The memory space is also well reduced with respect to existent methods and accurate results for nonlinear fields in a real world structure are obtained utilizing a normal size processor notebook in a time of about one-half of one minute when no induced currents are considered and of about one minute when eddy currents induced in solid ferromagnetic parts are also fully analyzed.

Originality/value - The originality of the novel techniques presented in the paper consists in the drastic approximations proposed for the material characteristics of the nonlinear ferromagnetic media in the analysis of periodic electromagnetic fields. These techniques are highly efficient and yield accurate numerical results.

Keywords Convergence acceleration, Periodic nonlinear fields, Polarization fixed point method, Electromagnetic fields, Vectors

Paper type Research paper

I. Introduction

The polarization fixed-point method (PFPM) has clear advantages with respect to the Newton-Raphson method for the solution of electromagnetic field problems in nonlinear media (Hantila, 1975; Hantila et al., 2000). It employs a magnetic permeability that is maintained constant during the iterations, the nonlinearity of the relationship $B-H$ being

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada and the Romanian National Council of Scientific Research, CNCSIS-UEFISCSU, under project PNII-IDEI 2197/2008.
transferred to the nonlinear dependence $B \cdot M$ and, as a consequence, the system matrices remain unchanged, with only a free term being modified. As well, the method can efficiently be applied to media with hysteresis, in both stationary and variable with time regimes (Dlala et al., 2007; Bottauscio et al., 2003). When a fictitious free space permeability is adopted, one can apply simple integral methods for the solution of the electromagnetic field problem at each iteration (Hantila et al., 2007; Albanese et al., 1998). One of the most important advantages of the PFPM consists in the fact that it allows a separate solution of each harmonic for nonlinear field problems in a periodic regime (Ciric and Hantila, 2007; Ausserhofer et al., 2007). Using the eddy-current integral equation for the solution of the field problem at each iteration, this method has allowed the development of an efficient procedure for the analysis of the solidification of ferromagnetic materials in motion (Ciric et al., 2009). PFPM uses a Picard-Banach iterative process whose convergence is insured for a correct choice of the fictitious permeability employed in the computation (Hantila, 1974). Unfortunately, the rate of convergence of the PFPM is smaller and smaller as one approaches the exact solution.

One strategy for improving the rate of convergence of the iterative process involved consists in choosing an optimal fictitious permeability (Hantila, 1974; Hantila et al., 2000) that yields an as small as possible value of the contraction factor. This technique can be utilized when the finite element method is employed to determine the electromagnetic field at each iteration (Dlala et al., 2007; Ausserhofer et al., 2008). It cannot be implemented when integral methods are employed, where the choice of the free space permeability is necessary and the polarization correction is performed, as in the case of eddy-current integral equation solutions in a quasistationary regime (Albanese et al., 1998) or in the case of a Green function method solution in a static regime (Hantila et al., 2007). Another strategy is to first employ the PFPM to obtain iteration solutions which are close to the final solution and, then, to apply the stable Newton-Raphson method which has a superior convergence (Yuan et al., 2005; Kuczmann, 2008). The drawback of this approach is that, whenever involving the Newton-Raphson method, the system matrix changes and the periodic regime can only be analyzed by using successive time steps. The convergence can also be improved by applying at each iteration a dynamic overrelaxation (Hantila and Grama, 1982; Hantila et al., 2000), which can be implemented independently of the solution method. This procedure is extremely efficient for the analysis of the static fields (Chiampi et al., 1996), but for the periodic regime it is inconvenient, since it requires a tremendous computation and memory effort.

In this paper, an extension of the dynamic overrelaxation technique to the computation of periodic fields is proposed, where the magnetization correction at each iteration is performed based on the dependence between the Fourier components of the magnetization and the magnetic induction, instead of that between their instantaneous values. The proposed procedure for computing the fundamental harmonic is much more efficient than the method of static permeability, which is used in commercial software.

II. Nonlinearity treatment in PFPM

The nonlinear relationship $H = F(B)$ is rewritten in the form:

$$B = \mu(H + M)$$  \hspace{1cm} (1)

where $\mu$ is a constant and $M$ takes into account the nonlinearity (Hantila, 1975; Hantila et al., 2000). We rearrange equation (1) as:
and choose $v = 1/\mu$ such that the function $G$ is a contraction, i.e.:

$$\|G(B_1) - G(B_2)\|_\mu \leq \lambda \|B_1 - B_2\|_v$$  \hspace{1cm} (3)

where $\lambda < 1$ and the norm is defined by:

$$\|U\|_v = \left( \frac{1}{T} \int_0^T \int_{\Omega} U \cdot (vU) d\Omega dt \right)^{1/2}$$  \hspace{1cm} (4)

with $T$ being the period and $\Omega$ the space region. For isotropic media, the contraction is insured if, at each space point, one chooses $\mu \in (0, 2\mu_{\min})$ (Hantila, 1974; Hantila et al., 2000), where $\mu_{\min}$ is the minimum value of the differential permeability in the region considered. In particular, for any medium, $\mu$ can be chosen to be the free space permeability, when the contraction factor is:

$$\lambda = 1 - \frac{\mu_0}{\mu_{\max}}$$  \hspace{1cm} (5)

where $\mu_{\max}$ is the maximum value of the differential permeability. $\lambda$ has usually a value close to unity, which results in a small convergence rate. If one chooses:

$$\frac{1}{\mu} = v = \frac{v_{\max} + v_{\min}}{2},$$

the smallest value of the contraction factor is attained, namely:

$$\lambda_{\text{opt}} = \frac{\mu_{\max} - \mu_{\min}}{\mu_{\max} + \mu_{\min}},$$

which can also be close to unity for big differences between $\mu_{\max}$ and $\mu_{\min}$. In evolutionary processes (e.g. in the case of hysteretic media), it is possible to determine the interval of variation of the magnetic induction and to retain the corresponding section of the magnetization characteristic, thus making smaller the difference between $\mu_{\max}$ and $\mu_{\min}$ (Dlala et al., 2007). In a periodic regime, this variation interval can be corrected in terms of the magnetic induction values within a period (Yuan et al., 2005). But these corrections yield modifications in the system matrix. It should be noted that, when the magnetization correction is conducted through the field intensity $H$, then, in the above equations $B$ is replaced by $H$ and $\mu$ by $v$, and the convergence is only insured if $\mu > \mu_{\max}/2$.

## III. PFPM contraction

### A. Periodic regime without eddy currents

At any time $t$, the magnetic field quantities satisfy the equations:

$$\nabla \cdot B = 0, \quad \nabla \times H = J_0, \quad B = \mu(H + M)$$  \hspace{1cm} (6)

where $J_0$ is the electric current density due to the given sources. All the vectors in equation (6) are time periodic of period $T$. To given $\mu$, $J_0$ and for any boundary conditions, each distribution of $M$ yields a unique field solution, with the magnetic
induction obtained in the form $B = T(M) = L(M) + B_{j_0}$, where $L(M)$ is due to $M$ in an unbounded space, while $B_{j_0}$ corresponds to $J_0$ and satisfies the boundary conditions. It can be shown that the function $T$ is nonexpansive. The periodic field problem solution can be obtained by choosing a sufficiently big number of time steps within a period and by solving iteratively the nonlinear stationary field problem at each time step. Even though the final field values at a time step can be used to initialize the iterative process at the next time step, the necessary computational effort is exceedingly high. In what follows, the magnetization $M$ is approximated by a truncated Fourier series $M = M_a = S(M)$, where the function $S$ is nonexpansive, and $B$ is determined for each harmonic separately.

**B. Periodic regime with eddy currents**

The equations for the quasistationary electromagnetic field in the conductive region $\Omega$ are:

$$\nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \times H = \frac{1}{\rho} E + J_0, \quad B = \mu(H + M)$$  \hspace{1cm} (7)

where $E$ is the intensity of the induced electric field and $\rho$ is the resistivity. As in the previous case, for given $\mu$, $\rho$, $J_0$ and for any boundary conditions, the magnetic induction is uniquely determined in terms of magnetization, $B = T(M)$, with $T$ being nonexpansive. As shown in Ciric and Hantila (2007) and Ausserhofer et al. (2007), the magnetic induction can be determined very efficiently by solving the field problem separately, for each harmonic involved.

**C. Picard-Banach sequence**

Starting with an arbitrary initial distribution of magnetization $M^{(0)}$, the solution of the periodic field problem is obtained through the Picard-Banach scheme:

$$\cdots M^{(n-1)} \overset{S}{\longrightarrow} M_a^{(n-1)} \overset{T}{\longrightarrow} B^{(n)} \overset{G}{\longrightarrow} M^{(n)} \cdots$$  \hspace{1cm} (8)

By a direct check, it can be seen that the composition $W = G \circ T \circ S$ is a contraction.

**IV. Dynamic overrelaxation**

After $n$ iterations, the error with respect to the limit $M_a^*$ of the sequence in equation (8) satisfies the relation:

$$\|M_a^{(n)} - M_a^*\| \leq \frac{\|M_a^{(n+1)} - M_a^{(n)}\|}{1 - \lambda}. \hspace{1cm} (9)$$

Owing to the contraction $W$ in equation (8), the error in the field problem solution becomes smaller and smaller with each subsequent iteration.

Suppose that $M_a^{(n-1)}$ is determined following equation (8), i.e.:

$$B^{(n)} = T(M_a^{(n-1)}), \quad M_a^{(n)} = \Psi(B^{(n)})$$  \hspace{1cm} (10)

with $\Psi = S \circ G$. The overrelaxation is performed by employing a new value $M_a^{(n)}$ instead of $M_a^{(n)}$, namely:
\[ M^{(n)}_a = M^{(n-1)}_a + \theta \Delta M^{(n)}_a \]  
where \( \Delta M^{(n)}_a = M^{(n)}_a - M^{(n-1)}_a \) and the overrelaxation factor \( \theta \) is determined to obtain the smallest value of:

\[ f(\theta) = \|M^{(n+1)}_a - M^{(n)}_a\|_\mu^2 = \|\psi \circ T(M^{(n)}_a) - M^{(n)}_a\|_\mu^2. \]  

Since \( T \) is a linear operator, we have:

\[ B^{(n+1)} = T(M^{(n)}_a) = B^{(n)} + \theta \Delta B^{(n+1)} \]  
with \( \Delta B^{(n+1)} = L(\Delta M_a^{(n)}) \), and equation (12) becomes:

\[ f(\theta) = \|\psi(B^{(n+1)}) - M^{(n)}_a\|_\mu^2. \]

\( \theta \) is determined by applying the secant method to:

\[ f'(\theta) = 2 \left( \frac{d\psi}{dB}|_{B^{(n+1)}} \Delta B^{(n+1)} - \Delta M_a^{(n)} \right) \]  

\[ = 0 \]  

where \( \langle \rangle \) indicates the time average of the inner product over \( \Omega \) (equation (4)) and:

\[ \frac{d\psi}{dB}|_{B^{(n+1)}} \]

is the Frechet derivative of the operator \( \psi \) at \( B^{(n+1)} \). When retaining \( N \) harmonics in the Fourier series expansion of the field quantities, we have:

\[ M_a(t) = \sum_{k=1,3,\ldots,2N-1} \sqrt{2}(M^{'(n+1)}_k \sin(k\omega t) + M^{''(n+1)}_k \cos(k\omega t)) \]  

and, thus:

\[ M_a^{(n+1)} = \psi(B_1^{(n+1)}) = \sum_{k=1,3,\ldots,2N-1} \sqrt{2}(M_a^{(n+1)}M^{'(n+1)}_k \sin(k\omega t) + M^{''(n+1)}_k \cos(k\omega t)) \]  

with each Fourier component of \( M_a^{(n+1)} \) depending nonlinearly on all the components of \( B^{(n+1)} \), e.g.:

\[ M^{(n+1)}_k = \psi_k(B_1^{(n+1)}, B_2^{(n+1)}, B_3^{(n+1)}, B_4^{(n+1)}, \ldots). \]  

The Frechet derivative of \( \psi \) becomes the Jacobian of \( \psi \) with respect to these components. Since we approximate the components of \( \psi \) by piecewise linear functions with respect to each variable, the Jacobian is evaluated very easily.

Consider in what follows 2D periodic field problems and assume one retains the first three harmonics, where each harmonic has four components (equations (18) and (20)). Each component of the harmonics \( B_k, k = 1, 3, 5 \), is defined, respectively, through \( N_1, N_3, N_5 \) values, which yields \( N_1^4 \times N_3^4 \times N_5^4 \) values for the 12 components of the harmonics of \( \psi \). As a function of \( \bar{B} \), \( \psi \) is constructed by linear interpolations.
It has been shown in Ciric and Hantila (2007) that the most efficient strategy consists first in retaining only the fundamental harmonic and, then, refining the solution by introducing successively the higher order harmonics. Since the weight of the fundamental is the greatest, its determination requires most of the computation time. This is why, we start by applying the above procedure to the fundamental harmonic. Equation (15) is now simplified to:

\[
\int \Omega \sum_{i=1}^{4} \left( \sum_{j=1}^{4} \frac{d\Psi_j}{dB_j} \right) \cdot (\Psi_i(B_j^{(n+1)}) - M_i^{(n)}) d\Omega = 0
\]  

(19)

where the following notation is used for the subscripts of the components of \( B \) (or \( M \)):

\[
B_{1x}^{(n+1)} = B_{1x}^{(n+1)} \quad B_{1y}^{(n+1)} = B_{1y}^{(n+1)} \quad B_{3x}^{(n+1)} = B_{3x}^{(n+1)} \quad B_{4x}^{(n+1)} = B_{4x}^{(n+1)}.
\]  

(20)

Usually, one iteration is sufficient to obtain a satisfactory value for \( \theta \), rarely being necessary two or at most three iterations. To calculate the components of \( \Psi \) at a point \((B_1, B_2, B_3, B_4)\) we determine the numerical values of:

\[
B_x(t_i) = \sqrt{2}(B_1 \sin(\omega t_i) + B_2 \cos(\omega t_i))
\]

\[
B_y(t_i) = \sqrt{2}(B_3 \sin(\omega t_i) + B_4 \cos(\omega t_i))
\]

(21)

for a chosen number of time steps, \( t_i \in [0, T] \). At each time step, the magnetization is corrected through the function \( G \) (equation (2)). In the case of an isotropic medium, when \( G \) is a scalar function, we get:

\[
M_x(t_i) = \frac{G(B(t_i))B_x(t_i)}{B(t_i)} \quad M_y(t_i) = \frac{G(B(t_i))B_y(t_i)}{B(t_i)} \quad B(t_i) = \left(B_x^2(t_i) + B_y^2(t_i)\right)^{1/2}
\]

(22)

Next, the components of the operator \( \Psi = S \cdot G \) are to be determined through Fourier analysis. The above procedure yields \( \Psi \) in a numerical form only for the fundamental harmonic. Once these numerical values associated with the fundamental are tabulated, we replace the tedious calculations required for the Fourier analysis in the iterative process with simple interpolations. The important advantage of the proposed technique consists in the fact that these numerical values for the fundamental are determined only once for a given nonlinear material and, then, they can be used for the solution of various related periodic field problems.

V. Modified technique for determining \( \Psi \)

The huge amount of numerical computation necessary to determine \( \Psi \) for the \( N_1^4 \) arguments \((N_1^3 \text{ in 3D structures})\), defined by the \( N_1 \) values of the components of the fundamental harmonic, can further be drastically reduced by a modification which simplifies the proposed dynamic overrelaxation. This consists in applying the procedure described in the previous section in order to obtain a single table of numerical values of \( \Psi \) calculated for a number of values of \( B \), thus constructing a piecewise linear scalar function \( \Psi = h(B) \). The components of the actual vector function \( \Psi \) for the fundamental harmonic are obtained approximately with:

\[
\Psi(B) = \frac{h(B)B}{B} \quad B = \left(B_1^2 + B_2^2 + B_3^2 + B_4^2\right)^{1/2}
\]

(23)
and the Jacobian with:

\[
\frac{d\Psi}{dB} = \left( \frac{h'(B) - h(B)}{B^3} \right) (BB) + \frac{h(B)}{B} \mathbf{I}
\]  

(24)

where \((BB)\) is a dyad and \(\mathbf{I}\) is the identity dyadic. The approximation of \(\Psi\) in equation (23) satisfies the property that the components of \(\Psi\) for \(\mathbf{B} = (1, 0, 1, 0)\) (vector field aligned to a fixed direction) are equal, in a different order, to the components of \(\Psi\) for \(\mathbf{B} = (1, 0, 0, 1)\) (rotating field). Generated numerical results show that this approximation is very good. Deviations from the exact solution are subsequently corrected, when introducing higher harmonics in the iterative procedure.

VI. Illustrative examples

Computation examples are given below to show the efficiency of the proposed overrelaxation and modified overrelaxation techniques as applied to the solution of periodic nonlinear field problems, without and with eddy currents induced. A field solution at each iteration is derived by using the integral method based on the Green function for an unbounded space (Hantila et al., 2007), where the permeability is taken to be everywhere the permeability of free space \(\mu_0\). In the region with nonlinear materials \(\Omega_f\), a discretization grid with \(n_f\) elements \(\omega_k\) is used, within each element the magnetization \(\mathbf{M}\) being considered to be constant.

A. Periodic regime without eddy currents

The magnetic induction at any point is expressed as:

\[
\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{2\pi} \sum_{k=1}^{n_f} \int_{\partial \omega_k} \frac{k \times R}{R^2} (\mathbf{M}_k \cdot d\mathbf{l}') + \mathbf{B}_{j_0}
\]  

(25)

where \(\partial \omega_k\) is the boundary of \(\omega_k\), \(d\mathbf{l}'\) is the vector length element of \(\partial \omega_k\), \(\mathbf{R} = \mathbf{r} - \mathbf{r}'\), \(R = |\mathbf{R}|\), \(\mathbf{r}\) and \(\mathbf{r}'\) are the position vectors of the observation point and integration point, respectively, \(\mathbf{k}\) is the longitudinal unit vector, and \(\mathbf{B}_{j_0}\) is the magnetic induction due to the given distribution of current density \(\mathbf{j}_0\). The average value of the magnetic induction over the element \(\omega_k\) is:

\[
\bar{B}_i = \frac{1}{\sigma_i} \int_{\omega_k} \mathbf{B}(\mathbf{r}) dS = -\frac{\mu_0}{\sigma_i} \sum_{k=1}^{n_f} \bar{\alpha}_{ik} \mathbf{M}_k + \bar{B}_{j_0}
\]  

(26)

where:

\[
\bar{\alpha}_{ik} = \frac{1}{2\pi} \int_{\partial \omega_k} \int_{\partial \omega_k} \ln R(d\mathbf{l}_i d\mathbf{l}'_k),
\]

\(\sigma_i\) is the area of the element \(\omega_i\) and \((d\mathbf{l}_i d\mathbf{l}'_k)\) is the dyad of the vector line elements \(d\mathbf{l}_i\) and \(d\mathbf{l}'_k\). The averaging operator in equation (26) is nonexpansive and, thus, the convergence of the PPFM is insured. The tensor \(\bar{\alpha}_{ik}\) and its Cartesian components satisfy the properties \(\bar{\alpha}_{ik} = \bar{\alpha}_{ki}\), \(\alpha_{ik_{xy}} = \alpha_{ik_{yx}}\) and \(\alpha_{ik_{xy}} + \alpha_{ik_{yx}} = 0\), for \(i \neq k\), and \(\alpha_{ik_{xy}} + \alpha_{ik_{yx}} = \sigma_i\), for \(i = k\), which are used to reduce the memory effort. \(\omega_k\) are chosen to be quadrilateral elements (Figure 1) and now \(\bar{\alpha}_{ik}\) can be calculated through exact analytic formulas.
B. Periodic regime with eddy currents

The integral equation employed to obtain the eddy currents at each odd harmonic \( n \) of angular frequency \( n \omega \) is (Ciric and Hantila, 2007):

\[
pJ_n(r) + \frac{\mu_0}{2\pi} \text{Im} \left( \int_{\Omega} J_n(r') \ln \frac{1}{R} dS' \right) = -\frac{\mu_0}{2\pi} \text{Im} \left( \int_{\Omega_0} J_0(r') \ln \frac{1}{R} dS' \right)
\]

\[+ \int_{\Omega_0} k \cdot (\nabla \times M_n(r')) \ln \frac{1}{R} dS' + \oint_{\partial \Omega_0} \ln \frac{1}{R} (M_n(r') \cdot dl') + C_l\]

(27)

where \( J_n \) is the current density in the conducting regions of cross-section \( \Omega \), \( J_0 \) is the given current density in the nonferromagnetic coil regions of cross-section \( \Omega_0 \), and \( C_l \) is a constant for each disjoint conducting region \( l \) which is determined by specifying its total current. For a given harmonic, the matrix associated with equation (27) remains the same for all the necessary iterations. From each harmonic \( n \) of the magnetization, we obtain the \( n \)th harmonic of the induced current density by solving equation (27) and, then, the \( n \)th harmonic of the magnetic induction is calculated from:

Figure 1. Field domain with its ferromagnetic region discretized in 820 elements
Consider the cross-section of an electromagnetic structure as shown in Figure 1, where the linear dimensions are given in millimeters. The ferromagnetic region is divided into 820 quadrilateral elements. The ferromagnetic material has everywhere the same characteristic $B-H$, as shown in Figure 2, for which the contraction factor is very close to unity, $\lambda = 0.99991$. For the periodic regime with eddy currents, a resistivity $\rho = 10^{-7} \text{Dm}$ is taken for the outer cylindrical shell. To test the performance of the modified overrelaxation procedure, the total currents in the four coil sections are (in complex) $I_0$, $-jI_0$, $-I_0$, $jI_0$, as shown in Figure 1, with the current density constant over each section, such that a rotating magnetic field is produced. Two values for the electric current $I_0$ are chosen, 600 and 1,500 A, to correspond, respectively, to a weak and to a pronounced saturation. We consider therefore four cases, namely $ST_1$, $ED_1$, $ST_2$ and $ED_2$, as shown in Figure 3.

To construct the function $\Psi(B)$, necessary in the dynamic overrelaxation technique, we divided uniformly the interval between $-2T$ and $2T$ of the fundamental of $B$ into 40 segments and used the components of $B$ at the 41 points within this interval. Thus, for the fundamental harmonic, this function is defined numerically using a table of $4 \times 41^2$ values. In the case of the modified dynamic overrelaxation only half of these values are retained, namely, those corresponding to the 21 points with positive values of magnetic induction. For the Fourier analysis employed when determining $\Psi$ and for the iterative PFPM procedure, the functions of time have been approximated by using 320 time steps per period, with a linear variation within each step. The relative error for the fundamental is evaluated as:

$$\frac{||\Delta M^{(n)}||_p}{||M^{(n)}||_p}$$

and it is shown in Figure 3 versus the computation time for the four cases mentioned above. In each of them, the rate of convergence is presented for the old PFPM procedure (Ciric and Hantila, 2007) with overrelaxation and for the proposed modified...
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Figure 3.
Rate of convergence for the fundamental:
1 – modified technique with dynamic overrelaxation;
2 – modified technique with constant overrelaxation ($\theta = 1.95$);
3 – modified technique without overrelaxation;
4 – old PFPM procedure (Ciric and Hantila, 2007) with overrelaxation
technique without and with overrelaxation, using a constant overrelaxation factor of $\theta = 1.95$, as well as for the modified technique with dynamic overrelaxation. In the case of a strong saturation, the old iterative technique cannot be used to decrease the relative error below certain values (for example, below $1.4 \times 10^{-4}$ for a current $I_0 = 1,500$ A (Figure 3(c) and (d))). Instead, the modified technique with dynamic overrelaxation yields rapidly very small relative errors, e.g. $10^{-12}$ in 240.4 s for the case $ED_2$. This shows the excellent performance of this technique for the solution of the periodic regime, even though such a small error is not needed in practice. The harmonic content of the magnetic induction at the point $P$ in Figure 1 is shown in Table I for the four cases considered in Figure 3, with the magnetic induction components given in rms values. To illustrate the overall computation times required for the case of a strong saturation ($I_0 = 1,500$ A), in Table II, beside the time required for the fundamental harmonic shown in column “1”, obtained by the modified dynamic overrelaxation technique, are given the additional times required when introducing the third harmonic, in column “1 + 3”, and the fifth harmonic, in column “1 + 3 + 5”, the latter two being obtained by the previously proposed procedure (Ciric and Hantila, 2007) with overrelaxation. The numerical results in all the cases considered were generated using a program developed in Visual Fortran 6.6 and employing a 2.5 GHz processor.

<table>
<thead>
<tr>
<th>Case</th>
<th>$k$</th>
<th>$B_{x1}$ (T)</th>
<th>$B_{x2}$ (T)</th>
<th>$B_{y1}$ (T)</th>
<th>$B_{y2}$ (T)</th>
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<td>$ST_1$</td>
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<td>$ED_2$</td>
<td>1</td>
<td>$-1.17$</td>
<td>$-3.51 \times 10^{-1}$</td>
<td>$1.09$</td>
<td>$3.20 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>$-1.77 \times 10^{-1}$</td>
<td>$-2.17 \times 10^{-1}$</td>
<td>$1.67 \times 10^{-1}$</td>
<td>$2.04 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>$-1.05 \times 10^{-2}$</td>
<td>$-9.54 \times 10^{-2}$</td>
<td>$9.43 \times 10^{-3}$</td>
<td>$8.95 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

**Table I.** Harmonic content of the magnetic induction at the point $P$ in Figure 1, for the four cases in Figure 3

*Note: $k$ indicates the harmonic order*

<table>
<thead>
<tr>
<th>Case</th>
<th>Harm.</th>
<th>1</th>
<th>1 + 3</th>
<th>1 + 3 + 5</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ST_2$</td>
<td>1 Error</td>
<td>$10^{-5}$</td>
<td>$10^{-4}$</td>
<td>$10^{-4}$</td>
<td>35.54</td>
</tr>
<tr>
<td></td>
<td>4 $t$ (s)</td>
<td>4.42</td>
<td>12.22</td>
<td>18.9</td>
<td></td>
</tr>
<tr>
<td>$ED_2$</td>
<td>1 Error</td>
<td>$2 \times 10^{-4}$</td>
<td>$10^{-4}$</td>
<td>$10^{-4}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4 $t$ (s)</td>
<td>230</td>
<td>11.2</td>
<td>19.2</td>
<td>260.4</td>
</tr>
<tr>
<td></td>
<td>4 Error</td>
<td>$10^{-4}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4 $t$ (s)</td>
<td>25</td>
<td>17</td>
<td>16</td>
<td>58</td>
</tr>
</tbody>
</table>

**Table II.** Computation times for cases $ST_2$ and $ED_2$ in Figure 3

*Notes: 1 - Modified technique with dynamic overrelaxation; 4 - old PFPM procedure (Ciric and Hantila, 2007) with overrelaxation*
notebook. All the computations were also performed using a refined discretization grid, with 2,100 quadrilateral elements. Only very slight modifications of the results were observed, with the rates of convergence practically in the same relation with respect to each other as in the case of the previous discretization grid.

VII. Conclusions
Application of the PFPM to the analysis of periodic regimes (Ciric and Hantila, 2007; Ausserhofer et al., 2007) allows for the electromagnetic field solution to be obtained for each harmonic separately, which constitutes a particularly important advantage with respect to other methods available in the literature. Replacing the B-M characteristic with the relation between the fundamental components of the magnetic induction and magnetization yields a spectacular reduction of the amount of computation necessary to obtain the fundamental harmonics, by eliminating the necessity to perform the Fourier analysis at each iteration and in each subregion. This, obviously, results in a substantial increase in the rate of convergence of the iterative process. The dynamic overrelaxation procedure further accelerates the convergence. These two procedures are efficiently applied to the iterative process associated with the determination of the fundamental harmonics which, due to the smaller contributions of the higher harmonics, brings the field quantities values closer to the actual periodic values. This initial approximation is then corrected by adding higher harmonics and employing the previously developed iterative technique. The modified dynamic overrelaxation, also proposed and illustrated in this paper for 2D structures, is expected to be highly efficient for the field solution in 3D structures as well. This modified technique is applied to determine very efficiently a good approximation of the fundamental harmonic and, then, the addition of successive higher harmonics is dealt with by employing the previously developed iterative technique (Ciric and Hantila, 2007). For strongly nonlinear media, where the contributions of the fundamental and the third harmonics are comparable, the dynamic overrelaxation procedure can be initiated for both these harmonics, but the number of equal segments within the interval considered for the fundamental of the magnetic induction should be reduced to allow for a number of segments within the interval for the third harmonic. For example, for a similar computational effort as in the cases presented in the previous section, one can choose about eight equal segments between $-2T$ and $2T$ for the fundamental, and four equal segments between $-0.8T$ and $0.8T$ for the third harmonic. Owing to the high content of third harmonic, the approximation in equation (23) is not acceptable any more and, thus, the modified dynamic overrelaxation cannot be applied in this case.

References


About the authors

Ioan R. Ciric is a Professor of Electrical Engineering at the University of Manitoba, Winnipeg, Canada. His major research interests are in the mathematical modelling of electromagnetic fields, field theory of special electrical machines, dc corona ionized fields, methods for wave scattering and diffraction problems, transients, and inverse problems.

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