Scalar Potential Formulations for Magnetic Fields Produced by Arbitrary Electric Current Distributions in the Presence of Ferromagnetic Bodies

Ioan R. Ciric, Fellow, IEEE

Department of Electrical and Computer Engineering, University of Manitoba, Winnipeg, MB R3T 5V6, Canada

In this paper it is shown how to analyze stationary or quasistationary magnetic fields due to arbitrary distributions of electric current in terms of single-valued scalar potentials. The total scalar potential in a specified region outside the magnetic material bodies, not containing electric currents, is obtained by superposing two single-valued Laplacian scalar potentials. One is produced by the given current distribution in an unbounded space and is determined from the free-space values of the normal component of the magnetic field intensity over the region boundary. The other one is defined in the entire region outside the magnetic bodies and is employed in order to satisfy the boundary conditions. When the magnetic bodies can be considered to be linear and homogeneous or piecewise homogeneous, with no electric current within them, the field inside the bodies can also be expressed in terms of Laplacian scalar potentials. If the material of the bodies can be approximated to be ideal ferromagnetic, then the exterior field problem is a Dirichlet problem and is solved independently of the field inside the bodies. The Neumann boundary condition for the interior field problem is provided from the solution of the exterior problem. Differential equation and integral equation formulations, as well as an illustrative application example, are presented. Since the resultant field is computed in terms of only single-valued scalar potentials, the solution techniques based on the formulations presented in this paper are considerably more efficient than those in usual solution methods.

Index Terms—Electromagnetics, magnetic field modeling, magnetic scalar potential.

I. INTRODUCTION

Consider a simply connected magnetic body of permeability $\mu$, occupying a region $V_B$ bounded by a closed surface $S$, placed in a linear and homogeneous unbounded region of permeability $\mu_0$, as shown in Fig. 1. There is no conduction electric current within the body. Given stationary or quasistationary electric currents are located outside the body and produce in an unbounded homogeneous space, in the absence of the body, a magnetic field of intensity $\mathbf{H}_0$ which can be determined by applying appropriate Biot-Savart formulas. In the presence of the body, the resultant magnetic field outside the body can be analyzed by superposing $\mathbf{H}_0$ and a field derived from a Laplacian scalar potential, while the resultant field inside the body can be expressed in terms of only a scalar potential [1]. These potentials are obtained by using various partial differential equation methods or integral equation methods [1]–[3] and by coupling the exterior and the interior field problems through the conditions to be satisfied across the boundary $S$. The resultant field outside the body remains expressed in a vector form and the boundary conditions at the surface of the body have to be applied to the normal and the tangential components of the vector fields outside and inside the body. This complicates the field solution and requires a substantial amount of computation. In the particular case when the external field $\mathbf{H}_0$ can be approximated to be uniform its magnetic scalar potential is a known linear function of the Cartesian coordinates and the exterior field problem can be treated in terms of only scalar potentials, as in the case of electrostatic field problems. A similar treatment can be applied in special situations where a single-valued scalar potential can be found for $\mathbf{H}_0$, for example for fields produced by current-carrying turns or by various two-dimensional or axisymmetric distributions of surface and volume currents, based on models that contain fictitious distributions of magnetic charge and of magnetization [4]–[7]. But, in general, for magnetic fields produced by arbitrarily distributed currents, this modeling technique cannot be applied and the usage of single-valued scalar potentials would require either the introduction of “cuts” in the field region [1], [8]–[12] or special numerical algorithms [13]–[15], which increases the difficulty of the field analysis. In a previous paper [16], we introduced a single-valued magnetic scalar potential for fields produced by arbitrary distributions of current in free-space regions where the field analysis in the presence of magnetic bodies is to be performed, such that the resultant field problem can be solved employing only scalar potentials, thus eliminating complicated “cut” procedures, otherwise required to be implemented. Using this potential allows for the magnetic fields in the regions where the potential is defined to be formulated in a manner...
similar to the classical formulations for scalar potential fields, for instance the formulations for the electrostatic potential [3].

In this paper, we present new single-valued scalar potential formulations for fields from arbitrary current distributions in the presence of ferromagnetic bodies ($\mu \gg \mu_0$) which are approximated to be ideal ferromagnetic bodies, i.e., with $\mu \rightarrow \infty$. The exterior field problem becomes now a Dirichlet problem for the scalar potential, independent of the field inside the ferromagnetic bodies, and the interior field problem is formulated as a Neumann problem, with the boundary condition provided from the solution of the exterior problem.

II. MAGNETIC FIELD PROBLEM FORMULATION

For an ideal ferromagnetic body ($\mu \rightarrow \infty$), the magnetic field lines of the total field outside the body are normal to its surface and the field intensity is equal to zero inside it. The magnetic flux density inside the body is finite due to the continuity of its normal component across the surface. Let us first define and show how to construct a single-valued scalar potential $\Phi_0$ for the field $H_0$ produced by arbitrarily distributed given currents in a free space region [1 6].

A. Scalar Potential for $H_0$

A closed surface $S_0$ is conveniently chosen to separate the electric current region from the region $V_{S_0}$ in which the ferromagnetic body is located and where the magnetic field is to be analyzed (see Fig. 1). The scalar potential $\Phi_0$ in $V_{S_0}$ is defined with

$$H_0(r) = -\nabla \Phi_0(r), \quad r \in V_{S_0}$$

(1)

where $H_0$ is the magnetic field intensity produced in an unbounded space by the given current distribution and the position vector $r$ represents the point of observation. Obviously, $\Phi_0$ satisfies the Laplace equation

$$\nabla^2 \Phi_0(r) = 0, \quad r \in V_{S_0}$$

(2)

under the Neumann boundary condition

$$\frac{\partial \Phi_0(r)}{\partial n_0} = -H_{0n}(r), \quad r \in S_0$$

(3)

where $H_{0n}$ is the component of $H_0$ normal to $S_0$ along the direction of the unit vector $n_0$ shown in Fig. 1 and is obtained, for arbitrary current distributions, by applying the Biot-Savart formulas. On the other hand, $\Phi_0$ in $V_{S_0}$ can be expressed in integral form as given in (13), with $\Phi_0$ on $S_0$ obtained by solving the integral equation (14) (see Section III).

Once this scalar potential is determined for a given distribution of current, it can be used for any field problem relative to various configurations of magnetic bodies located inside the region $V_{S_0}$ considered.

B. Exterior Problem

The resultant magnetic field intensity $H$ outside the ferromagnetic body in the region bounded by $S \cup S_0$ and the surface at infinity, when the region $V_{S_0}$ is extended to infinity (see Fig. 1), is now derived from a single-valued scalar potential $\Phi$ as

$$H(r) = -\nabla \Phi(r), \quad r \in V'_S \cap V_{S_0}$$

(4)

with

$$\Phi(r) = \Phi_0(r) + \Phi'(r), \quad r \in V'_S \cap V_{S_0}$$

(5)

where $\Phi'$ is due to the presence of the body and is defined and satisfies the Laplace equation in the entire region $V'_S$ outside $S$

$$\nabla^2 \Phi'(r) = 0, \quad r \in V'_S.$$  

(6)

For the special case of an ideal ferromagnetic body, since the field lines outside the body are perpendicular to its boundary, the surface $S$ of the body is now equipotential, i.e., $\Phi$ is constant over $S$

$$\Phi(r) = K, \quad r \in S.$$  

(7)

On the other hand, according to the law of magnetic flux,

$$\oint_S \mu_0 H : dS = 0$$

(8)

i.e.,

$$\oint_S \frac{\partial \Phi}{\partial n} dS = 0$$

(9)

where $\oint_S$ denotes a closed surface integral and $\frac{\partial \Phi}{\partial n}$ the normal derivative. Thus, the solution of the exterior problem is obtained by imposing the Dirichlet condition in (7), the condition in (9) and, for an unbounded outside region, the conditions at infinity, with $\Phi_0$ within $V_{S_0}$ determined as shown in Section II-A. It gives the field everywhere in $V'_S \cap V_{S_0}$ and also the value of the constant $K$ in (7) with respect to a reference potential. The exterior field problem is solved independently of the field inside the body and its solution also provides the normal component of the flux density $\mu_0 \frac{\partial \Phi}{\partial n}$ on $S$, i.e., the boundary condition for the interior problem.

C. Interior Problem

Once the exterior problem is solved, the interior field problem consists in the solution of a scalar potential equation under Neumann boundary conditions. For a linear and homogeneous ferromagnetic body, the magnetic flux density $B_0$ inside the body is curlless [1] and is derived from a scalar potential $\Psi$ as

$$B_0(r) = -\nabla \Psi(r), \quad r \in V_S.$$  

(10)

$\Psi$ satisfies the equation

$$\nabla^2 \Psi(r) = 0, \quad r \in V_S$$

(11)

under the condition that the normal component of the magnetic flux density is continuous across the boundary $S$, i.e.,

$$\frac{\partial \Psi(r)}{\partial n} = \mu_0 \frac{\partial \Phi(r)}{\partial n}, \quad r \in S$$

(12)
with the right-hand side known from the solution of the exterior problem.

III. SURFACE INTEGRAL EQUATION FORMULATIONS

The magnetic field outside and inside the ferromagnetic bodies can also be determined using boundary integral equations for the scalar potentials in Section II, which are, in general, more convenient for real world body shapes.

For the exterior problem, as mentioned before, the total field can be obtained, in a straightforward procedure, by superposing the field produced in an unbounded space by the given distribution of electric current, computed with Biot-Savart formulas, and the field derived from a Laplacian scalar potential to be determined by imposing the condition that the tangential component of the resultant field is equal to zero at the boundary \( S \) of each ideal ferromagnetic body [1] (see Fig. 1). Employing a Green function representation of this potential yields a boundary integral equation which, unfortunately, does not allow an efficient numerical computation. Obviously, more efficient integral equations can be constructed whenever the total field can be expressed in terms of only scalar potentials.

In this paper, we formulate surface integral equations for a field analysis using scalar potentials based on the single-valued potential \( \Phi_0 \) presented in Section II-A. First, the potential \( \Phi_0 \) in (1)–(3) produced in the region \( V_{S_0} \) by the given current distribution (see Fig. 1) is expressed using its Green function representation, equivalent to (2)–(3), i.e.,

\[
\Phi_0(r) = -\frac{1}{4\pi} \int_{S_0} \left[ \frac{\mathbf{R} \cdot \mathbf{n}_0(r')}{R^3} - \Phi_0(r') \right] dS', \quad r \in V_{S_0}
\]  

(13)

where \( \mathbf{R} = r - r' \), \( R = |\mathbf{R}| \), \( r \) and \( r' \) are the position vectors of the observation point and the source point, respectively. Bringing the observation point on \( S_0 \) yields the surface integral equation for \( \Phi_0 \) [16]

\[
\alpha_0(r)\Phi_0(r) + \int_{S_0} \frac{\mathbf{R} \cdot \mathbf{n}_0(r')}{R^3} \Phi_0(r') dS' = -\int_{S_0} \frac{1}{R} H_{0n}(r') dS', \quad r \in S_0
\]  

(14)

where \( \int \) denotes the integral in principal value and \( \alpha_0 \) is the solid angle under which a small neighborhood of \( V_{S_0} \) is seen from the observation point on \( S_0 \) (for smooth sections of \( S_0 \), i.e., with the exception of points on the edges or at vertices, \( \alpha_0 = 2\pi \)). Once \( \Phi_0 \) on \( S_0 \) is obtained, \( \Phi_0 \) at any point inside \( V_{S_0} \) is calculated with (13). It should be noted that the boundary \( S_0 \) of \( V_{S_0} \) can be chosen such that the numerical computations in (13) and (14) are much alleviated. For instance, if \( S_0 \) is constructed from plane sections parallel to the Cartesian coordinate planes, then, for each such section, \( \mathbf{R} \cdot \mathbf{n}_0 \) and \( H_{0n} \) in (13), (14) are just one Cartesian component of the position vector \( \mathbf{R} \) and of the Biot-Savart vector integral used to compute \( \mathbf{H}_0 \), respectively.

Second, using again the Green function representation, the potential \( \Phi' \) in (6), due to the presence of the body, is expressed in the form

\[
\Phi'(r) = \frac{1}{4\pi} \int_{S} \left[ \frac{\mathbf{R} \cdot \mathbf{n}'}{R^3} \Phi'(r') - \frac{1}{R} \frac{\partial \Phi'(r')}{\partial n'} \right] dS', \quad r \in V'_S
\]  

(15)

or, with (5)

\[
\Phi'(r) = \frac{1}{4\pi} \int_{S} \frac{\mathbf{R} \cdot \mathbf{n}'(r')}{R^3} \Phi'(r') dS' - \frac{1}{4\pi} \int_{S} \frac{1}{R} \frac{\partial \Phi'(r')}{\partial n'} dS' - \frac{1}{4\pi} \int_{S} \left[ \frac{\mathbf{R} \cdot \mathbf{n}'}{R^3} \Phi_0(r') - \frac{1}{R} \frac{\partial \Phi_0(r')}{\partial n'} \right] dS', \quad r \in V'_S.
\]  

(16)

The last integral in (16) is equal to zero (see (13) for \( S_0 \rightarrow S \) and with the observation point being outside the body) and the potential \( \Phi \) in the first integral in (16) is constant on \( S \) (see (7)). Since the solid angle under which the whole body is seen from outside is equal to zero, i.e.,

\[
\int_{S} \frac{\mathbf{R} \cdot \mathbf{n}}{R^3} dS' = 0, \quad r \in V'_S
\]  

(17)

we get

\[
\Phi'(r) = -\frac{1}{4\pi} \int_{S} \frac{1}{R} \frac{\partial \Phi'(r')}{\partial n'} dS', \quad r \in V'_S
\]  

(18)

and

\[
\Phi(r) = \Phi_0(r) - \frac{1}{4\pi} \int_{S} \frac{1}{R} \frac{\partial \Phi'(r')}{\partial n'} dS', \quad r \in V'_S \cap V_{S_0}.
\]  

(19)

This expression is used to calculate \( \Phi \) at any point in \( V'_S \cap V_{S_0} \) when \( \Phi_0 \) on \( S_0 \) is known. Equation (19) is also used to derive a first boundary integral equation. Namely, bringing the observation point on \( S \), with \( \Phi = K \) on \( S \) (see (7)), (19) yields

\[
\int_{S} \frac{1}{R} \frac{\partial \Phi'(r')}{\partial n'} dS' = 4\pi \Phi_0(r) - 4\pi K, \quad r \in S.
\]  

(20)

This surface integral equation in \( \frac{\partial \Phi}{\partial n} \), together with the condition in (9), gives the normal derivative of \( \Phi \) as a function of position over \( S \) and the value \( K \) of the constant potential on \( S \) for a chosen reference of the potential.

Another integral equation can be derived from (19) by taking the gradient on both sides and, then, bringing the observation point on \( S \) and projecting along the normal direction. This yields

\[
\frac{\partial \Phi(r)}{\partial n} = -H_{0n}(r) + \frac{1}{4\pi} \int_{S} \frac{\mathbf{R} \cdot \mathbf{n} \cdot \partial \Phi'(r')}{R^3} dS', \quad r \in S
\]  

(21)

where \( \mathbf{\hat{n}} \) is the unit normal at the observation point on \( S \) (see Fig. 1) and \( H_{0n} \) is now the component of \( \mathbf{H}_0 \) along \( \mathbf{\hat{n}} \). Since \( \frac{\partial \Phi}{\partial n} \) is not defined at the points on \( S \) which are at vertices or on edges, for the purpose of numerical computation this equation
is only written for points that are on the smooth sections of $S$. 
At these points, (21) becomes

$$\frac{\partial \Phi(r')}{\partial n} - \frac{1}{2\pi} \int_{S_c} \frac{\mathbf{R} \cdot \mathbf{n'}}{R^3} \frac{\partial \Phi(r'')}{\partial n''} dS'' = -2H_0n(r), \quad r \in S. \quad (22)$$

Again, the solution of (22) must satisfy the condition in (9) to give the unique and correct normal derivative $\frac{\partial \Phi}{\partial n}$ over the boundary $S$ of the ferromagnetic body that is needed for the interior field problem.

For the interior field problem, the Laplacian scalar potential inside each linear and homogeneous body (see (10) and (11)) can be represented as

$$\Psi(r) = -\frac{1}{4\pi} \int_{S} \left[ \frac{\mathbf{R} \cdot \mathbf{n'}}{R^3} \Psi(r') - \frac{1}{R} \frac{\partial \Psi(r')}{\partial n'} \right] dS', \quad r \in V_S. \quad (23)$$

Bringing the point of observation on $S$ yields the surface integral equation in

$$\alpha_i(r')\Psi(r') + \int_{S} \frac{\mathbf{R} \cdot \mathbf{n'}}{R^3} \Psi(r')dS' = \frac{1}{R} \int_{S} \frac{\partial \Psi(r')}{\partial n'} dS', \quad r \in S \quad (24)$$

where $\frac{\partial \Phi}{\partial n}$ on $S$ is equal to $\mu_0\frac{\partial \Phi}{\partial n}$ on $S$ (see (12)) which is known from the solution of (20) or (22) in the exterior field problem and $\alpha_i$ is the solid angle under which a small neighborhood of $V_S$ is seen from a point of observation on $S$. Once $\Psi$ on $S$ is determined, the potential at any point inside the ferromagnetic body is obtained from (23) and the magnetic flux density $\mathbf{B}$, with (10). Evidently, in the particular case of an ideal ferromagnetic body considered in this paper, the integral equation for the exterior problem is uncoupled from that for the interior problem.

It should be noted that (22) allows the computation of $\frac{\partial \Phi}{\partial n}$ over $S$ without being necessary to previously determine the scalar potential $\Phi_0$ as a function of position on $S$. In (22), one only has to have the normal component $H_0n$ on $S$ which is computed by applying Biot-Savart formulas to the given distribution of electric current. Therefore, when the field analysis is required only for the inside of the body or when the first objective is to calculate the forces exerted on the body, then, the usage of (22) would be the best choice in order to obtain $\frac{\partial \Phi}{\partial n}$ needed in (24), (23) or in Maxwell’s stress tensor, respectively. To compute the scalar potential outside the ferromagnetic body, at any point in the region $V_S^c \cap V_{S_0}$, one uses (19) with the potential $\phi_0$ calculated from (13), with only scalar surface integrals to be evaluated. On the other hand, the total magnetic field intensity in (4) can also be computed by taking the gradient of both sides in (19) but, then, one has to evaluate a vector surface integral and, also, the vector volume integrals in the Biot-Savart formulas to get $\mathbf{H}_0$ at all the observation points.

The resultant field intensity inside the current region (i.e., inside $S_0$ in Fig. 1) is obtained, when needed, as $\mathbf{H}_0 + \mathbf{H}'$, where $\mathbf{H}_0$ is due to the current distribution and is computed with Biot-Savart formulas, and $\mathbf{H}' = -\nabla \Phi'$ is due to the presence of the ferromagnetic body, with $\Phi'$ already determined as shown in (18).

IV. ILLUSTRATIVE APPLICATION EXAMPLE

A simple example for which an elementary solution is available in the literature is given in this section to show the application of the proposed method based on the usage of the single-valued scalar potential $\Phi_0$ defined in Section II-A or, in integral form, in (13)–(14). Consider a ferromagnetic sphere of radius $r_s$, placed in the magnetic field produced in an unbounded region of permeability $\mu_0$ by a coaxial circular turn of radius $a$ and carrying an electric current $I$, as shown in Fig. 2. The scalar potential in (1) due to the current-carrying turn in the absence of the sphere is obtained from (2)–(3). For the region $r \leq r_0 < c$, the normal component of the magnetic field intensity in (3), only due to the current turn, is known for this simple geometry [2], i.e.,

$$H_0n(r, \theta) = H_0s(r, \theta) = \frac{Ia}{2c} \sum_{n=1}^{\infty} \frac{(r)}{n} P_n^{1} \left( \frac{b}{c} \right) \times P_n(\cos \theta), \quad r < c \quad (25)$$

where $\theta$ are spherical coordinates, $P_n$ are Legendre polynomials and $P_n^1$ are associated Legendre functions of the first kind.

Applying the method of separation of variables yields [16]

$$\Phi_0(r, \theta) = -\frac{Ia}{2c} \sum_{n=1}^{\infty} \frac{1}{n} \frac{(r)}{c} P_n^{1} \left( \frac{b}{c} \right) \times P_n(\cos \theta) + K_0, \quad r < c \quad (26)$$

and the resultant potential in (5) is

$$\Phi(r, \theta) = \sum_{n=0}^{\infty} C_n^{r' - (n+1)} P_n^{1} \left( \frac{b}{c} \right) \times P_n(\cos \theta) + \frac{C_0^r}{r} + K_0, \quad r_s \leq r < c. \quad (27)$$

The potential $\Phi'$ in the region $r \geq r_s$ due to the presence of the sphere has the general form

$$\Phi'(r, \theta) = \sum_{n=0}^{\infty} C_n^{r' - n} P_n(r') \times P_n(\cos \theta), \quad r \geq r_s \quad (28)$$

and the resultant field intensity in (5) is

$$\Phi(r, \theta) = \sum_{n=1}^{\infty} \left[ -\frac{Ia}{2c} \frac{1}{n} \frac{(r)}{c} P_n^{1} \left( \frac{b}{c} \right) + C_n^{r' - n+1} \right] \times P_n(\cos \theta) + \frac{C_0^r}{r} + K_0, \quad r_s \leq r < c \quad (29)$$

The normal derivative of $\Phi$ at the sphere surface is

$$\frac{\partial \Phi}{\partial n} \bigg|_{r=r_s} = \frac{\partial \Phi}{\partial r} \bigg|_{r=r_s} \times \frac{1}{r_s} \left[ \frac{Ia}{2c} \frac{(r_s)}{c} P_n^{1} \left( \frac{b}{c} \right) \right] \times P_n(\cos \theta) - \frac{C_0^r}{r_s}, \quad (30)$$
Taking into account that \( \int_0^\pi P_n(\cos \theta') \sin \theta' d\theta' = 0, \quad n = 1, 2, \ldots \), the condition (9), with (30), gives \( C'_n = 0 \). Assuming the sphere to be of ideal ferromagnetic material, the condition (7) that the resultant potential is constant over the surface of the sphere, \( \Phi(r_s, \theta) = K \), yields the coefficients \( C'_n, \quad n = 1, 2, \ldots \), and the constant \( K \),

\[
C'_n = \frac{La}{2c} \left( \frac{r_s}{c} \right)^n P_n \left( \frac{b}{c} \right) r_s^{n+1},
\]

\[
K = K_s. \tag{31}
\]

Thus, the resultant potential for the exterior problem is

\[
\Phi(r, \theta) = -\frac{La}{2c} \sum_{n=1}^{\infty} \frac{2n + 1}{n} \left( \frac{r_s}{c} \right)^{n-1} P_n \left( \frac{b}{c} \right) P_n(\cos \theta) + K_s, \quad r_s \leq r < c
\]

and its normal derivative needed for the interior field problem is

\[
\frac{\partial \Phi}{\partial r} \bigg|_{r=r_s} = -\frac{La}{2c^2} \sum_{n=1}^{\infty} \frac{2n + 1}{n} \left( \frac{r_s}{c} \right)^{n-1} P_n \left( \frac{b}{c} \right) P_n(\cos \theta). \tag{33}
\]

The potential \( \Psi \) in (10) inside the sphere, assumed to be of linear, isotropic and homogeneous material, satisfies the Laplace equation (11) under the Neumann boundary condition (12). The general form of \( \Psi \) is

\[
\Psi(r, \theta) = \sum_{n=1}^{\infty} D_n r^n P_n(\cos \theta) + K_s, \quad r \leq r_s. \tag{34}
\]

Imposing the boundary condition (12), with (33), gives the coefficients \( D_n \)

\[
D_n = -\mu_0 \frac{La}{2c} \frac{2n + 1}{n^2} e^{-n} P_n \left( \frac{b}{c} \right), \quad n = 1, 2, \ldots \tag{35}
\]

and, thus

\[
\Psi(r, \theta) = -\mu_0 \frac{La}{2c} \sum_{n=1}^{\infty} \frac{2n + 1}{n^2} \left( \frac{r_s}{c} \right)^{n-1} P_n \left( \frac{b}{c} \right)
\]

\[
\times P_n(\cos \theta) + K_s, \quad r \leq r_s \tag{36}
\]

with the additive constant \( K_s \) used to choose the reference potential. The magnetic flux density \( B_i \) in \( r \leq r_s \) is calculated with (10).

The scalar potential \( \Phi \) in the region \( r > c \) is determined by the superposition of the scalar potential \( \Phi_0 \) for \( r > a \) (27) produced by the circular turn and the potential \( \Phi' \) in (28), (31) due to the presence of the sphere.

When the radius of the current turn \( a \to \infty \), with the ratio \( \frac{L}{2a} \) maintained constant, one gets the special case of the ferromagnetic sphere in a uniform magnetic field of intensity \( H_0 - H_0 \vec{x} \), with \( H_0 = I/(2a) \).

It can easily be checked that the results in (33) and (36) satisfy the integral equations (20), (22) and (24), respectively.

V. CONCLUSION AND REMARKS

As shown in Sections II-A and III ((13), (14)), a single-valued scalar potential for the magnetic field due to arbitrary distributions of electric current can be determined in a simple manner and, thus, the formulations for and the analysis of fields in the presence of ferromagnetic bodies is performed in terms of only scalar potentials. Obviously, this reduces substantially the computational effort required to determine the magnetic field quantities and the forces in real world, three-dimensional, structures.

It can be noticed that the integral equations (20) and (22) with the condition in (9) correspond formally to the potential problem of an uncharged conductor at a potential \( K \) in an external electrostatic field of potential \( \Phi_0 \), where \( \frac{\partial \Phi}{\partial n} \) on \( S \) is proportional to the surface density of electric charge (see, for instance, the Coulomb integral equation and Robin’s integral equation in [1]). But, while in the electrostatic case \( \Phi_0 \) is known in the whole space for given distributions of electric charge and \( \frac{\partial \Phi}{\partial n} \) is zero inside the conductor, the magnetic scalar potential \( \Phi_0 \) due to given distributions of electric currents is known (after solving (1)–(3) or (14)) only in the region \( V_{r_s} \) (see (1)–(2) or (13)) and \( \mu_0 \frac{\partial \Phi}{\partial n} = \Phi_{enc} \) when crossing the surface of the ferromagnetic body due to the continuity of the normal component of the magnetic flux density. As well, the expression of the total magnetic scalar potential in (19) is valid outside the ferromagnetic body only where the potential \( \Phi_0 \) is defined.

The scalar potential method presented and the equations derived in this paper can be applied in the same way to problems with multiple ferromagnetic bodies and even when the bodies have holes but with zero total electric current passing through each hole and linking ferromagnetic material. The surface \( S \) becomes now the union of the surfaces of the bodies, including the hole surfaces, but (7) and (20) must be written separately for observation points on each body, with different constant potentials \( K \). As well, the condition in (9), associated with (20) or (22), is written separately for each body. Solution of these systems of equations yields \( \frac{\partial \Phi}{\partial n} \) at the observation points on the surfaces of all the bodies and, in the case of (20), also the values of the constant potentials for each of them. The functions \( \frac{\partial \Phi}{\partial n} \) over the ferromagnetic body surfaces provide the boundary...
condition for the potential problem inside individual bodies. As well, in the case the given electric currents are distributed in different regions, the surface $S_0$ is constituted from the union of the surfaces separating various current regions from the region of interest in which the ferromagnetic bodies are located. For a three-dimensional study of the magnetic fields in various electric machines with open stator and rotor slots, for example, the surface $S$ is constituted from the slotted stator and rotor surfaces, while the closed surface $S_0$ is constructed using the two slotted armature surfaces facing the air gap and two other sections around the end connections of the coil sides, such that the entire winding system is inside $S_0$.

The specific differential and integral equation formulations given in this paper refer to ideal ferromagnetic bodies, when the exterior and interior field problems are uncoupled, but the scalar potential methodology presented can easily be applied to the case with bodies of finite permeability.

When the electromagnetic system contains not only ferromagnetic bodies but also simply connected bodies which can be approximated to be ideal diamagnetic (for example superconductors or metallic conductors at sufficiently high frequencies), the single-valued potential outside the bodies is expressed in the same way as shown in this paper, but with appropriate boundary conditions to be imposed at the surfaces of different bodies. The single-valued scalar potential due to arbitrary current distributions used in this paper can also be employed for the computation of the field outside induced solid conductors at low frequencies.

REFERENCES


Ioan R. Ciric (SM’84–F’97) is a Professor of Electrical Engineering at the University of Manitoba, Winnipeg, MB, Canada. His major research interests are in the mathematical modeling of stationary and quasistationary fields, electromagnetic fields in the presence of moving solid conductors and levitation, field theory of special electrical machines, dc corona ionized fields, analytical and numerical methods for wave scattering and diffraction problems, propagation along waveguides with discontinuities, transients, and inverse problems. Dr. Ciric presented for the first time a generalization of basic operators and theorems in vector analysis and of Maxwell’s equations for regions with material and source discontinuities. He initiated new directions of research regarding the field analysis based on new rotational-translational addition theorems and the formulation and application of single-source reduced surface integral equations for wave scattering, for eddy-current problems and for stationary fields. Dr. Ciric has published archival scientific papers and has made presentations at international conferences and symposia. He also published electrical engineering textbooks and book chapters on scalar potential magnetic field modeling and on electromagnetic scattering by complex objects. Dr. Ciric is a Fellow of the Electromagnetics Academy. He is listed in Who’s Who in Electromagnetics.