GREEN FUNCTIONS OF THE LAPLACE OPERATOR FOR INTERNAL NEUMANN PROBLEMS

BY

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In order to construct the Green function — defined either under the form of a single expression in the considered domain or under the form of two expressions valid each in one of the two subdomains in which this domain is divided — we look firstly for a particular function which should ensure the satisfying of the inhomogeneous conditions at least on some of the portions of the boundary; the difference between the Green function and this particular function will satisfy homogeneous boundary conditions, which permit the use of the method of eigenfunctions. The technique is illustrated by a particular case of the plane (two-dimensional) domain under rectangular form.

1. INTRODUCTION

A general method for the integration of linear partial differential equations with constant coefficients is that which uses Green functions [1].

The internal Neumann problem which refers to the Poisson equation consists in the determination of the point function \( \varphi \) (of the position vector \( r \)) which satisfies the equation

\[
\Delta \varphi (r) = -k_0 \varphi (r) \tag{1}
\]

in the domain \( D \) bounded by the closed surface \( \Sigma \), \( k_0 \) being a given positive constant and \( \varphi \) a given point function, with the inhomogeneous boundary condition

\[
\frac{\partial \varphi}{\partial n} \bigg|_{\Sigma} = \chi (r), \tag{2}
\]

where \( \chi (r) \) is a function given on \( \Sigma \) and \( \frac{\partial}{\partial n} \bigg|_{\Sigma} \) stands for the derivative upon the normal exterior to the domain \( D \), performed on its boundary.

In order that this problem should admit a solution under the assumption that the surface $\Sigma$ is a sufficiently smooth surface [2], the function $\chi(r)$ should satisfy the condition

$$\int_{\Sigma} \chi(r) \delta A = -k_0 \int_D \rho(r) \delta v,$$

(3)

where $\delta A$ and $\delta v$ are the surface and volume elements, respectively.

The formulation of the internal Neumann problem pertaining to the Laplace equation is obtained from the formulation presented above, by setting $\rho \equiv 0$ in relations (1) and (3).

The Green function $G$ for equation (1) under conditions (2) is by definition [3] the solution of the equation

$$\Delta G(r, r_0) = -k_0 \delta(r - r_0),$$

(4)

referring to the same domain $D$, with the inhomogeneous condition on its boundary $\Sigma$

$$\frac{\partial G}{\partial n} \mid_{\Sigma} = -\frac{k_0}{A_\Sigma},$$

(5)

where $r_0$ is the position vector which represents the points at which the source function $\rho$ is distributed, $\delta(r - r_0)$ is Dirac's "delta function" (Dirac's distribution) with the singularity at the point $r = r_0$ and $A_\Sigma$ is the finite area of the closed surface $\Sigma$.

The problem of the determination of Green's function, defined by relations (4), (5), is a standard internal Neumann problem for the considered domain, in which the source function $\rho$ is just $\delta(r - r_0)$ and the function $\chi(r)$ reduces to the constant value $-\frac{k_0}{A_\Sigma}$, such that condition (3) corresponding to this problem be satisfied.

The expression of the "delta function" in the three-dimensional case, in orthogonal curvilinear coordinates, is [3]

$$\delta(r - r_0) = \frac{1}{h_1 h_2 h_3} \delta(x_1 - x_{10}) \delta(x_2 - x_{20}) \delta(x_3 - x_{30}),$$

(6)

where the factors of the form $\delta(x_k - x_{k0})$ ($k = 1, 2, 3$) are the one-dimensional "delta functions", $h_1, h_2, h_3$ are Lamé's coefficients corresponding to the system of coordinates adopted, and $x_k$ and $x_{k0}$ ($k = 1, 2, 3$) are the orthogonal curvilinear coordinates corresponding to the position vectors $r(x_1, x_2, x_3)$ and $r_0(x_{10}, x_{20}, x_{30})$.

If the domain $D$ is divided into two subdomains $D_I$ and $D_{II}$, by a surface $S$ of coordinates drawn through the point $r = r_0$, e.g. by the surface $x_3 = x_{30}$, such that the subdomain $D_I$ should correspond to $x_3 < x_{30}$ and the subdomain $D_{II}$ to $x_3 > x_{30}$, then Green's function for
equation (1) under conditions (2) may be also defined under the form of two expressions as follows [4]

\[ G = \begin{cases} G_I \text{ in the subdomain } D_I, \\ G_{II} \text{ in the subdomain } D_{II}. \end{cases} \]  

(7)

This time \( G_I \) and \( G_{II} \) are the solutions of Laplace’s equations

\[ \Delta G_I = 0 \text{ referring to the subdomain } D_I, \]  

(8)

\[ \Delta G_{II} = 0 \text{ referring to the subdomain } D_{II}, \]  

(9)

with the *inhomogeneous* boundary conditions

\[ \frac{\partial G_I}{\partial n} \bigg|_{S_I} = -\frac{k_0}{\mathcal{A}_\Sigma}, \]  

(10)

and

\[ \frac{\partial G_{II}}{\partial n} \bigg|_{S_{II}} = -\frac{k_0}{\mathcal{A}_\Sigma}, \]  

(11)

if \( S_I \) and \( S_{II} \) are the open surfaces which partially bound the subdomains \( D_I \) and \( D_{II} \), respectively, obtained from the closed surface \( \Sigma \) by its intersection with the separation surface \( S \), if we set the following passing conditions for the separation surface \( S \) between the two subdomains

\[ G_I = G_{II}, \]  

for \( x_3 = x_{30} \) and

\[ \frac{1}{h_3} \left( \frac{\partial G_I}{\partial x_3} - \frac{\partial G_{II}}{\partial x_3} \right) = k_0 \frac{\delta(x_1 - x_{10}) \delta(x_2 - x_{20})}{h_1 h_2}, \]  

for all \( x_1, x_2 \in D. \)  

(13)

This last definition of Green’s function is equivalent to that given by relations (4), (5). In a similar manner one may define Green’s function when as separation surface \( S \) between the two subdomains we choose the surface \( x_1 = x_{10} \) or the surface \( x_3 = x_{30} \). Accordingly in the three-dimensional case there exist three possibilities for defining Green’s function in the manner indicated by relations (7)–(13). By choosing in a convenient manner the subdomains \( D_I \) and \( D_{II} \), Green’s function defined by relations (7)–(13) may present important advantages in many problems intervening in physics and engineering.

In the case of two-dimensional problems of the plane-parallel fields, Green’s functions are defined in a similar manner [1], [4], the domain \( D \) being this time a plane (two-dimensional) domain, whose boundary is the closed curve \( \Gamma \), Dirac’s three-dimensional “delta function” being replaced by a corresponding two-dimensional “delta function”, the area
\( \mathcal{A}_S \) (from (5), (10), (11)) by the length \( l_1 \) of the curve \( \Gamma \), the separation surface \( S \) by the coordinate curve \( C \), and the surfaces \( S_1 \) and \( S_{II} \) (from (10), (11)) by the corresponding curves \( C_1 \) and \( C_{II} \).

Once Green’s function is determined for the considered domain (defined either by relations (4), (5) or by relations (7)—(13)) and the looked for function \( \varphi \) is obtained by simple integrations [4].

2. METHOD FOR CONSTRUCTING GREEN’S FUNCTIONS

In principle, the construction of Green’s function of the Laplace operator for a given domain (see [1], [2], [3]) can be made by adding to Green’s function for the infinite domain, which satisfies the inhomogeneous equation (4), a harmonic function in the considered domain such that to satisfy the boundary conditions corresponding to the type of problem (Dirichlet, Neumann, mixed problem) to which the looked for Green function refers. This technique cannot be practically employed in the general case of some domains which do not present symmetry properties with respect to certain coordinates.

On the other hand, the calculation of the harmonic functions necessary for constructing Green’s functions, defined either under the form of a single expression or under the form of two expressions [4], imposes the determination of the eigenvalues and of the integration constants from the general solutions of Laplace’s equation, obtained for instance through the method of the separation of variables in the system of coordinates adopted; in general this operation cannot be made in the case of inhomogeneous boundary conditions, such as conditions (5) or conditions (10), (11) which intervene in the internal Neumann problem.

In the present paper, the construction of Green’s function defined by relations (4), (5), referring to domains bounded by surfaces (or curves) of coordinates, is made by adding to a particular function, with the aid of which the inhomogeneous boundary condition (5) is satisfied, a function subjected to some homogeneous boundary conditions, such that equation (4) be satisfied. For constructing Green’s function defined under the form of two expressions by relations (7)—(13), we separate in each of the two subdomains a particular harmonic function which should satisfy partially the inhomogeneous boundary conditions (10) and (14), to which one should add one harmonic function in the respective subdomain which satisfies hence homogeneous conditions on some portions of the subdomain boundary, such that the eigenvalues and then the corresponding integration constants could be determined.

The particular functions employed for constructing Green’s functions should satisfy — exactly as the last one — the condition to be symmetrical with respect to their arguments [1].

3. GREEN’S FUNCTION FOR THE INTERIOR OF THE RECTANGLE

In the following we illustrate the manner in which one may construct Green’s function of Laplace’s operator for internal Neumann problems, by considering the plane (two-dimensional) domain bounded by the contour of the rectangle with sides \( a \) and \( b \) (see Figs. 1 and 2).
Green's function can be obtained either under the form of a single expression for the whole considered domain with the aid of a double Fourier series, or under the form of two expressions valid each in one of the two subdomains into which the rectangle is divided, case in which the expressing is made with the aid of some single Fourier series.

a. Expressing of Green's function by a double Fourier series. We use the Cartesian axes of coordinates shown in Fig. 1. The looked for Green function is by definition (see (4), (5) and (6)) the solution of the equation

\[ \Delta G(x, y; x_0, y_0) = -k_0 \delta(x - x_0) \delta(y - y_0), \]  

(14)

referring to the domain \( D \) considered (interior of the rectangle), with the boundary condition

\[ \frac{\partial G}{\partial n} \bigg|_\Gamma = -\frac{k_0}{l_\Gamma}, \]  

(15)

in which the length of the boundary \( \Gamma \) is

\[ l_\Gamma = 2(a + b). \]  

(16)

The boundary condition (15) may be developed as follows

\[ \frac{\partial G}{\partial x} \bigg|_{x=0}^{x=a} = \frac{\partial G}{\partial y} \bigg|_{y=0}^{y=b} = \frac{\partial G}{\partial y} \bigg|_{y=0}^{y=a} - \frac{k_0}{2(a + b)}. \]  

(17)

Green's function is looked for under the form

\[ G = G_p + G_\delta, \]  

(18)

in which the particular function \( G_p \) is a symmetrical function with respect to the arguments \((x, y)\) and \((x_0, y_0)\) of the form

\[ G_p = -\frac{k_0}{2(a + b)} \left[ \left( \frac{x^2 + x_0^2}{a} + \frac{y^2 + y_0^2}{b} \right) - (x + x_0 + y + y_0) \right], \]  

(19)

such that with its aid the boundary condition (17) be satisfied. The Laplacean of the function \( G_p \) is

\[ \Delta_{xy} G_p = -\frac{k_0}{ab}. \]  

(20)

With (20) and (14) there results that the function \( G_\delta \) from (18) satisfies the equation

\[ \Delta_{xy} G_\delta = -k_0 \delta(x - x_0) \delta(y - y_0) + \frac{k_0}{ab}, \]  

(21)
with homogeneous conditions on the boundary of the domain

\[ \frac{\partial G_\delta}{\partial x} \bigg|_{x=0} = \frac{\partial G_\delta}{\partial x} \bigg|_{x=a} = \frac{\partial G_\delta}{\partial y} \bigg|_{y=0} = \frac{\partial G_\delta}{\partial y} \bigg|_{y=b} = 0. \] (22)

The general form of the solution of equation (21) is

\[ G_\delta = \sum_{m} \sum_{n} \left( A_{mn} \cos \frac{m \pi x}{a} + B_{mn} \sin \frac{m \pi x}{a} \right) \left( C_{mn} \cos \frac{n \pi y}{b} + D_{mn} \sin \frac{n \pi y}{b} \right) + \]

\[ + \sum_{m} \left( A_{m0} \cos \frac{m \pi x}{a} + B_{m0} \sin \frac{m \pi x}{a} \right) \left( C_{m0} y + D_{m0} \right) + \]

\[ + \sum_{n} \left( A_{0n} x + B_{0n} \right) \left( C_{0n} \cos \frac{n \pi y}{b} + D_{0n} \sin \frac{n \pi y}{b} \right) + \]

\[ + (A_{00} x + B_{00}) (C_{00} y + D_{00}). \] (23)

On imposing conditions (22) there results that \( B_{mn} = B_{m0} = A_{0n} = A_{00} = 0, \) \( D_{mn} = D_{0n} = C_{m0} = C_{00} = 0 \) and that \( m, n \) are positive integer numbers, \( m, n = 1, 2, 3, \ldots \) Denoting \( A_{mn} C_{mn} \equiv G_{mn} \), \( A_{m0} D_{m0} \equiv G_{m0} \), \( B_{0n} C_{0n} \equiv G_{0n} \) and \( B_{00} D_{00} \equiv G_{00} \), we obtain

\[ G_\delta = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} G_{mn} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} + \]

\[ + \sum_{m=1}^{\infty} G_{m0} \cos \frac{m \pi x}{a} + \sum_{n=1}^{\infty} G_{0n} \cos \frac{n \pi y}{b} + G_{00}. \] (24)

By inserting this expression into equation (21) we have

\[ - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} G_{mn} \left[ \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 \right] \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} = \]

\[ - \sum_{m=1}^{\infty} G_{m0} \left( \frac{m \pi}{a} \right)^2 \cos \frac{m \pi x}{a} - \sum_{n=1}^{\infty} G_{0n} \left( \frac{n \pi}{b} \right)^2 \cos \frac{n \pi y}{b} = \]

\[ = - k_0 \delta (x - x_0) \delta (y - y_0) + \frac{k_0}{ab}. \] (25)
Using the development of Dirac's "delta function" into double Fourier series of cosines (see the Appendix, (A.5))

\[
\delta(x - x_0) \delta(y - y_0) = \frac{4}{ab} \left[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \cos \frac{m \pi x_0}{a} \cos \frac{n \pi y_0}{b} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} + \right.
\]
\[
+ \frac{1}{2} \left( \sum_{m=1}^{\infty} \cos \frac{m \pi x_0}{a} \cos \frac{m \pi x}{a} + \sum_{n=1}^{\infty} \cos \frac{n \pi y_0}{b} \cos \frac{n \pi y}{b} \right) + \frac{1}{4} \right],
\]

we observe that we may identify the coefficients of the series in the two sides of the identity (25). We get

\[
G_{m0} \left[ \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 \right] = k_0 \frac{4}{ab} \cos \frac{m \pi x_0}{a} \cos \frac{n \pi y_0}{b},
\]
\[
G_{m0} \left( \frac{m \pi}{a} \right)^2 = k_0 \frac{2}{ab} \cos \frac{m \pi x_0}{a},
\]
\[
G_{0n} \left( \frac{n \pi}{b} \right)^2 = k_0 \frac{2}{ab} \cos \frac{n \pi y_0}{b}.
\]

Taking account of expressions (27), (24) and (19), Green's function (18) may be written under the form

\[
G(x, y; x_0, y_0) = \text{const} - \frac{k_0}{2(a + b)} \left[ \left( \frac{x^2 + x_0^2}{a} + \frac{y^2 + y_0^2}{b} \right) - (x + x_0 + y + y_0) \right] +
\]
\[
+ k_0 \frac{4}{\pi^2 ab} \left[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \cos \frac{m \pi x_0}{a} \cos \frac{n \pi y_0}{b} \right.
\]
\[
+ \frac{1}{2} \left( a^2 \sum_{m=1}^{\infty} \frac{1}{m^2} \cos \frac{m \pi x}{a} \cos \frac{m \pi x_0}{a} + b^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{n \pi y}{b} \cos \frac{n \pi y_0}{b} \right) \right],
\]

in which we have denoted \( G_{00} = \text{const} \); Green's function is determined up to an arbitrary additive constant.

b. Expressing of Green's function by single Fourier series. If we divide the domain \( D \) into two subdomains \( D_1 \) and \( D_2 \) by the straight
line \( x = x_0 \), as shown in Fig. 2, the looked for Green function can be defined under the form of two expressions as follows (see relations (7)–(13))

\[
G = \begin{cases}
G_I \text{ for } x \in [0, x_0), \ y \in [0, b], \\
G_{II} \text{ for } x \in (x_0, a], \ y \in [0, b],
\end{cases}
\tag{29}
\]

where \( G_I \) and \( G_{II} \) are the solutions of Laplace’s equation

\[
\Delta G_I (x, y; x_0, y_0) = 0, \quad (30)
\]

\[
\Delta G_{II} (x, y; x_0, y_0) = 0, \quad (31)
\]

with the boundary conditions

\[
- \frac{\partial G_I}{\partial x} \bigg|_{y=0} = - \frac{\partial G_I}{\partial y} \bigg|_{x=x_0} = - \frac{\partial G_{II}}{\partial y} \bigg|_{y=b} = \frac{k_0}{2(a+b)}, \quad (32)
\]

and

\[
- \frac{\partial G_{II}}{\partial x} \bigg|_{y=0} = - \frac{\partial G_{II}}{\partial y} \bigg|_{x=a} = - \frac{\partial G_{II}}{\partial y} \bigg|_{y=b} = \frac{k_0}{2(a+b)}, \quad (33)
\]

and with the passing conditions

\[
G_I = G_{II}, \quad (34)
\]

\[
\frac{\partial G_I}{\partial x} - \frac{\partial G_{II}}{\partial x} = k_0 \delta (y-y_0), \quad (35)
\]

The functions \( G_I \) and \( G_{II} \) are looked for under the form

\[
G_I = G_{Ia} + G_{Ib}, \quad (36)
\]

\[
G_{II} = G_{IIa} + G_{IIb}, \quad (37)
\]

in which the particular harmonic functions \( G_{Ia} \) and \( G_{IIa} \) are chosen such that to satisfy conditions (32) and (33) respectively, on the boundaries \( y = 0, \ y = b \) and be symmetrical with respect to the arguments \( (x, y) \) and \( (x_0, y_0) \). We may take

\[
G_{Ia} = \frac{k_0}{2(a+b) b} \left[ (x^2 - y^2) + (x_0^2 - y_0^2) + b(y + y_0) \right], \quad (38)
\]

\[
G_{IIa} = \frac{k_0}{2(a+b) b} \left[ (x^2 - y^2) + (x_0^2 - y_0^2) + b(y + y_0) \right]. \quad (39)
\]
Hence, the functions \( G_{18} \) and \( G_{118} \) satisfy Laplace's equations

\[
\Delta_x G_{18} = 0, \tag{40}
\]
\[
\Delta_y G_{118} = 0, \tag{41}
\]

with homogeneous conditions on the boundaries \( y = 0 \) and \( y = b \). With (38)–(39), (36)–(37) and (32)–(33) there result the boundary conditions

\[
- \frac{\partial G_{18}}{\partial x} \bigg|_{x=0} = -\frac{k_0}{2(a + b)}, \tag{42}
\]
\[
\frac{\partial G_{118}}{\partial y} \bigg|_{y=0} = \frac{\partial G_{118}}{\partial y} \bigg|_{y=b} = 0, \tag{43}
\]

respectively

\[
- \frac{\partial G_{118}}{\partial x} \bigg|_{x=a} = -\frac{k_0}{2(a + b)} \left(1 + \frac{2a}{b}\right), \tag{44}
\]
\[
\frac{\partial G_{118}}{\partial y} \bigg|_{y=0} = \frac{\partial G_{118}}{\partial y} \bigg|_{y=b} = 0, \tag{45}
\]

and with (34)–(35) we get the passing conditions

\[
G_{18} = G_{118}, \tag{46}
\]
\[
\frac{\partial G_{18}}{\partial x} - \frac{\partial G_{118}}{\partial x} = k_0 \delta(y - y_0), \quad \text{for } x = x_0, \quad y \in [0, b]. \tag{47}
\]

The general solution of the two-dimensional Laplace equation ((40) or (41)) is obtained through the method of the separation of variables, having in Cartesian coordinates \( x, y \), the expression

\[
\sum_\lambda (A_\lambda \cosh \lambda x + B_\lambda \sinh \lambda x) \left( C_\lambda \cos \lambda y + D_\lambda \sin \lambda y \right) + (A_0 x + B_0) (C_0 y + D_0). \tag{48}
\]

On imposing the homogeneous conditions (43) or (45), there result \( D_\lambda = 0, \quad C_0 = 0 \) and the eigenvalues \( \lambda = \frac{n\pi}{b} \), \( n \) being a positive integer number, \( n = 1, 2, 3, \ldots \)
Hence, the expressions of $G_{I\delta}$ and $G_{II\delta}$ may be written as follows

$$G_{I\delta} = \sum_{n=1}^{\infty} \left( A_{n1} \cosh \frac{n\pi x}{b} + B_{n1} \sinh \frac{n\pi x}{b} \right) \cos \frac{n\pi y}{b} + A_{01} x + A_{01} x_0 + B_{01}, \quad (49)$$

$$G_{II\delta} = \sum_{n=1}^{\infty} \left( A_{nII} \cosh \frac{n\pi x}{b} + B_{nII} \sinh \frac{n\pi x}{b} \right) \cos \frac{n\pi y}{b} + A_{0II} x + A_{0II} x_0 + B_{0II}. \quad (50)$$

By imposing the conditions (42), (44) and (46) we get

$$B_{n1} = 0, \quad A_{01} = \frac{k_0}{2(a + b)}, \quad (51)$$

$$A_{nII} \sinh \frac{n\pi a}{b} + B_{nII} \cosh \frac{n\pi a}{b} = 0, \quad A_{0II} = -\frac{k_0}{2(a + b)} \left(1 + \frac{2a}{b}\right), \quad (52)$$

$$A_{n1} \cosh \frac{n\pi x_0}{b} = A_{nII} \cosh \frac{n\pi x_0}{b} + B_{nII} \sinh \frac{n\pi x_0}{b}, \quad B_{01} = B_{0II}. \quad (53)$$

Likewise, imposing the condition (47) we obtain

$$\sum_{n=1}^{\infty} \left( \frac{n\pi}{b} \right) \left( (A_{n1} - A_{nII}) \sinh \frac{n\pi x_0}{b} - B_{nII} \cosh \frac{n\pi x_0}{b} \right) \cos \frac{n\pi y}{b} +$$

$$+ A_{01} - A_{0II} = k_0 \delta(y - y_0). \quad (54)$$

On using the development of Dirac’s delta function into single Fourier series of cosines (see the Appendix, (A.4))

$$\delta(y - y_0) = \frac{2}{b} \left[ \sum_{n=1}^{\infty} \cos \frac{n\pi y_0}{b} \cos \frac{n\pi y}{b} + \frac{1}{2} \right], \quad (55)$$

and taking account of the fact that $A_{01} - A_{0II} = \frac{k_0}{b}$ (from (51) and (52)), one may observe that the identification of the coefficients of the series in the two sides of the identity (54) can be made; thus

$$(A_{n1} - A_{nII}) \sinh \frac{n\pi x_0}{b} - B_{nII} \cosh \frac{n\pi x_0}{b} = k_0 \frac{2}{\pi n} \cos \frac{n\pi y_0}{b}. \quad (56)$$

Relations (51), (52), (53) and (56) determine all the constants of integration from expressions (49) and (50). Taking also account of
expressions (36)—(39), Green’s function (29) may be finally written under the form

\[ G_1(x, y; x_0, y_0) = \]

\[ = \frac{k_0}{2(a + b)} [(x^2 - y^2) + (x_0^2 - y_0^2)] + \frac{k_0}{2(a + b)} [(x + y) + (x_0 + y_0)] - k_0 \frac{x_0}{b} + \]

\[ + k_0 \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\text{ch} \frac{n\pi x}{b} \cos \frac{n\pi y}{b} \text{ch} \frac{n\pi x_0}{b} \cos \frac{n\pi y_0}{b}}{\text{sh} \frac{n\pi a}{b}} + G_0 \]  \hspace{1cm} (57)

for \( x \in [0, x_0), y \in [0, b] \) and

\[ G_{11}(x, y; x_0, y_0) = \]

\[ = \frac{k_0}{2(a + b)} [(x^2 - y^2) + (x_0^2 - y_0^2)] + \frac{k_0}{2(a + b)} [(x + y) + (x_0 + y_0)] - k_0 \frac{x}{b} + \]

\[ + k_0 \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\text{ch} \frac{n\pi (a - x)}{b} \cos \frac{n\pi y}{b} \text{ch} \frac{n\pi x}{b} \cos \frac{n\pi y_0}{b}}{\text{sh} \frac{n\pi a}{b}} + G_0 \]  \hspace{1cm} (58)

for \( x \in (x_0, a], y \in [0, b] \), where the additive arbitrary constant \( (B_{01} = B_{011}) \) has been denoted by \( G_0 \).

Expressions (57) and (58) may be condensed in a single one

\[ G(x, y; x_0, y_0) = \]

\[ = \frac{k_0}{2(a + b)} [(x^2 - y^2) + (x_0^2 - y_0^2)] + \frac{k_0}{2(a + b)} [(x + y) + (x_0 + y_0)] - k_0 \frac{x_0}{b} + \]

\[ + k_0 \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\text{ch} \frac{n\pi x}{b} \cos \frac{n\pi (a - x)}{b} \text{ch} \frac{n\pi y}{b} \cos \frac{n\pi y_0}{b}}{\text{sh} \frac{n\pi a}{b}} + G_0, \]  \hspace{1cm} (59)

where \( x_< \) and \( x_> \) represent the smallest and the greatest of the magnitudes \( x \) and \( x_0 \), respectively.
If we divide the domain $D$ into two subdomains $D_{\text{III}}$ and $D_{\text{IV}}$ by the straight line $y = y_0$ (see Fig. 2), the Green function defined under the form of two expressions is obtained from relations (57)–(59) by replacing $x$ by $y$, $x_0$ by $y_0$, and $a$ by $b$ and conversely:

$$ G_{\text{III}}(x, y; x_0, y_0) = - $$

$$ - \frac{k_0}{2(a + b)a} [(x^2 - y^2) + (x_0^2 - y_0^2)] + \frac{k_0}{2(a + b)} [(x + y) + (x_0 + y_0)] - k_0 \frac{y_0}{a} + $$

$$ + k_0 \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \frac{\cos m\pi x}{\sinh m\pi b} \frac{\cos m\pi y}{\sinh m\pi b} + G_0, \quad (60) $$

for $x \in [0, a]$, $y \in [0, y_0)$ and

$$ G_{\text{IV}}(x, y; x_0, y_0) = - $$

$$ - \frac{k_0}{2(a + b)a} [(x^2 - y^2) + (x_0^2 - y_0^2)] + \frac{k_0}{2(a + b)} [(x + y) + (x_0 + y_0)] - k_0 \frac{y_0}{a} + $$

$$ + k_0 \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \frac{\cos m\pi x}{\sinh m\pi b} \frac{\cos m\pi y}{\sinh m\pi b} + G_0, \quad (61) $$

for $x \in [0, a]$, $y \in (y_0, b]$. Likewise, expressions (60) and (61) may be condensed in a single one

$$ G(x, y; x_0, y_0) = - $$

$$ - \frac{k_0}{2(a + b)a} [(x^2 - y^2) + (x_0^2 - y_0^2)] + \frac{k_0}{2(a + b)} [(x + y) + (x_0 + y_0)] - k_0 \frac{y_0}{a} + $$

$$ + k_0 \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \frac{\cos m\pi x}{\sinh m\pi b} \frac{\cos m\pi y}{\sinh m\pi b} + G_0, \quad (62) $$

where $y_<$ and $y_>$ represent the smallest and the greatest of the magnitudes $y$ and $y_0$, respectively.
4. OBSERVATIONS AND CONCLUSIONS

The standard internal Neumann problem (4), (5), whose solution is the Green function \( G \), has been decomposed into other two standard internal Neumann problems referring to the same domain. In the first problem we have determined the function \( G_p \) corresponding to a source function \( p \) uniformly distributed in the whole domain, of value equal to the average value of Dirac's "delta function" on the respective domain, with inhomogeneous boundary conditions (5). In the second problem we have determined the function \( G_s \) corresponding to a source function \( p \) equal to the difference between the Dirac's "delta function" and its average value on the considered domain, with homogeneous boundary conditions.

The problem of the determination of Green's function by a single expression in the whole domain corresponds to the integration of an inhomogeneous (Poisson) equation, while the problem of the determination of Green's function under the form of two expressions valid each in one of the two subdomains into which the considered domain is divided, corresponds to the integration of two homogeneous (Laplace) equations and to the setting of a passing condition at the separation surface between the two subdomains.

From the example treated in paragraph 3 one may observe that the identification of the coefficients of the corresponding series in the two sides of relations (25) and (54) is possible — and accordingly the problem of the construction of Green's function admits a solution — only if the boundary conditions are formulated in a manner corresponding to relations (17) and (32)—(33) in agreement with the general condition (3).

One may observe the symmetry of the calculated Green functions: by changing \( x \) by \( x_0 \) and \( y \) by \( y_0 \), and conversely, expression (28) remains unchanged; by changing \( y \) by \( y_0 \) and conversely, expressions (57) and (58) remain unchanged but changing \( x \) by \( x_0 \) and conversely, expression (57) transforms into expression (58) and conversely; in a similar manner by interchanging the places of the variables \( x \) and \( x_0 \), expressions (60) and (61) remain unchanged. If, however, we interchange the places of the variables \( y \) and \( y_0 \) expression (60) transforms into expression (61) and conversely.

The Green functions for the interior of the rectangle in the case in which the coordinate axes are chosen in a manner different from that shown in Figs. 1 and 2, are obtained from expressions (28) and (57)—(62) by performing the corresponding translations of the coordinate axes.

The Green functions of type (28) have the advantage of the determination of the looked for function \( \varphi \) (solution of the problem (1)—(3)) by a single expression valid throughout the whole domain. With the aid of the Green functions of type (57)—(62), the function \( \varphi \) is determined by several expressions valid in various subdomains of the considered domain; however, these last expressions are more advantageous as con-
cerns the numerical calculations since single series which intervene are more rapidly convergent than the double series which intervene in the expressions obtained by using Green functions of type (28). In addition, expressions of the form (57)–(62) permit the obtaining of some summation formulae for the corresponding series from expressions of the form (28); by particularizing the values of the magnitudes \( x, y, x_0, y_0 \) and \( a, b \), these summation formulae may assume forms useful for the performing of some independent numerical calculations.

By using the method presented in this paper, one may construct Green functions of the Laplace operator for internal Neumann problems referring to domains bounded by surfaces of coordinates in various systems of coordinates; proceeding from such Green functions one may obtain directly solutions for a great number of problems which appear in various branches of physics and engineering.

**APPENDIX**

**DEVELOPMENT OF DIRAC'S "DELTA FUNCTION" INTO EVEN FOURIER SERIES**

The development of the one-dimensional "delta function" \( \delta(x-x_0) \), defined in the domain \( x \in [0, a] \), into Fourier series of cosines is made by prolonging this "function" symmetrically with respect to the ordinate axis in the range \([-a, 0]\). There results the series

\[
\delta(x-x_0) = \delta_0 + \sum_{m=1}^{\infty} \delta_m \cos \frac{m \pi x}{a}, \quad (A.1)
\]

whose coefficients are obtained by using the properties of the "delta function" [3]

\[
\delta_0 = \frac{1}{a} \int_{-a}^{0} \delta(x-x_0) \, dx = \frac{1}{a}, \quad (A.2)
\]

\[
\delta_m = \frac{2}{a} \int_{-a}^{0} \delta(x-x_0) \cos \frac{m \pi x}{a} \, dx = \frac{2}{a} \cos \frac{m \pi x_0}{a}, \quad (A.3)
\]

Accordingly, we have the development

\[
\delta(x-x_0) = \frac{2}{a} \left[ \frac{1}{2} + \sum_{m=1}^{\infty} \cos \frac{m \pi x}{a} \cos \frac{m \pi x_0}{a} \right]. \quad (A.4)
\]
In a similar manner one obtains the development of the two-dimensional "delta function" \( \delta(x-x_0) \delta(y-y_0) \) defined in the domain \( x \in [0, a] \), \( y \in [0, b] \), into double Fourier series of cosines

\[
\delta(x-x_0) \delta(y-y_0) = \frac{4}{ab} \left[ \frac{1}{4} + \frac{1}{2} \left( \sum_{m=1}^{\infty} \cos \frac{m\pi x}{a} \cos \frac{m\pi x_0}{a} + \sum_{n=1}^{\infty} \cos \frac{n\pi y}{b} \cos \frac{n\pi y_0}{b} \right) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \cos \frac{m\pi x_0}{a} \cos \frac{n\pi y_0}{b} \right].
\]

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