FIELD THEORY OF RELUCTANCE MOTORS
WITH SEGMENTAL ROTORS

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The paper presents the first example of employing the series matching method in the electrical machine field: the rigorous determination of the magnetic field in the reluctance motor with segmental rotor, taking into account the curvature of its armatures. The longitudinal and transverse reactances, the torques, the input current and the power factor are calculated exactly and computed results are given under the form of characteristic diagrams, useful from the design standpoint.

LIST OF SYMBOLS

- polar coordinates,
- amplitude of current sheet density on the stator surface,
- number of pairs of poles,
- load angle,
- pulsation of current in the stator winding,
- geometrical dimensions in Fig. 1,
- vacuum permeability,
- magnetic induction and magnetic field vectors,
- vector magnetic potential in the regions (i), (g) and (s), respectively,
- constants of integration,
- integers,
- $z$-component of electrical field vector and $\varphi$-component of magnetic field vector,
- rotor length,
- segment-arc/polar pitch ratio,
- relative airgap,
- relative thickness of rotor segments,
- adimensional longitudinal and transverse synchronous reactances,

I. Introduction

The electromagnetic field in the electrical machines has a quite complex structure since their geometry is a very complicated one and even by adopting some simplified models, the classical methods of exact solution of boundary value problems are applicable only in very few cases. The usual method of separation of variables, for instance, cannot be used if there are boundaries which cannot be completely described by constant coordinate surfaces for the entire region under investigation.

A general procedure of exact solution of the complex region boundary value problems is that of the so-called series matching method [1], [2]. In principle this procedure is very simple and follows the underneath steps.

Firstly, the complex boundary region (the whole domain) is divided into a few simpler regions (subdomains) entirely bounded by constant coordinate surfaces, so that the general expression of the solution of the field partial differential equation in each out of these simpler regions can be written by means of a convenient eigenfunction system. The eigenvalues in the partial solutions thus obtained are determined by employing some of the boundary conditions corresponding to the whole domain.

Secondly, in order to determine all the constants of integration in the partial solutions, it is necessary to employ the rest of the boundary conditions relative to the whole domain and to impose the passing conditions at all the separation surfaces between the adjoining subdomains. Since the eigenfunctions corresponding to the different subdomains are of different kinds or, if they are of the same kind, the spectres of eigenvalues relating to them are different, the integration constants result as the unknowns of an infinite set of algebraic simultaneous equations.

Mostly there it is necessary to determine the partial solution in one subdomain only or even just a few of its first constants of integration,
sometimes only one of them. The process of substitution and elimination of the unknowns in the complete set of equations must therefore be led so that the integration constants relating that subdomain or those subdomains in which the solutions are looked for be finally retained.

Thirdly, the final infinite set of equations is to be solved numerically, of course by retaining a finite number of unknowns and an equal number of equations. The accuracy of the computed results depends on how many unknowns and equations are to be considered. Usually these results converge very slowly to the exact ones as the number of unknowns and equations taken into account increases. This is why it is recommended to use some particular techniques such as that employing the Hahn's functions in the Fourier series case, in order to improve the convergence of the intervening series. When possible, it is very useful to get the final infinite set of equations under two different forms, so that the exact numerical value of each integration constant be between the numerical values of that integration constant obtained by retaining the same finite number of unknowns and equations in the two forms of that set of equations. Thus the absolute errors of numerical calculus made when retaining a certain finite number of equations can be evaluated.

This paper presents the first example of employing the above method in the electrical machine field, namely the exact calculation of the electromagnetic field in a reluctance motor with segmental rotor. This kind of motor has been considered in [3], but in that paper the reluctances were calculate approximately, supposing a simplified shape of the magnetic field lines.

Fig. 1.—Geometry for the calculation of field in a reluctance motor with segmental rotor.

To proceed with the analysis, the motor rotor is assumed to have a length much greater than its radius and a plane-parallel model as it is shown in Fig. 1. The rotor segments and the stator magnetic circuit
are from soft magnetic material supposed to be linear, with the permeability \( \mu = \infty \). To take into account the curvature of the machine, a polar coordinate system fixed with the rotor will be utilized. The linear density of the current sheet on the internal surface of the stator is

\[
J_s = J_m \sin p(\varphi - \gamma),
\]

where only the fundamental harmonic of the current distribution along the airgap corresponding to the stator winding has been considered.

In this paper only the synchronous performance of the motor is studied.

2. ANALYSIS

2.1. DETERMINATION OF THE MAGNETIC FIELD VALUES

Taking into account the geometrical periodicity and symmetry of the model adopted for the reluctance motor with segmental rotor, it is sufficient to determine the magnetic field only in the subdomains (i) \( r \in [0, a], \varphi \in \left[ -\frac{\pi}{p}, \frac{\pi}{p} \right] \), (g) \( r \in [b, c], \varphi \in \left[ -\frac{\pi}{p}, \frac{\pi}{p} \right] \) and (s) \( r \in [a, b], \varphi \in [-\pi, \pi] \) (Fig. 1).

The calculation of magnetic field values may be carried out with the aid of the vector potential, defined by the relations

\[
B = \mu_0 H = \text{curl} \ A, \ \text{div} \ A = 0.
\]

For the adopted model the vector potential may be chosen directed along the \( z \)-axis and depending only on \( r \) and \( \varphi \). This vector potential satisfies Laplace's equation

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 A}{\partial \varphi^2} = 0,
\]

which has a general solution of the following form [4]:

\[
A(r, \varphi) = \sum_\lambda (C_\lambda r^\lambda + D_\lambda r^{-\lambda}) (M_\lambda \cos \lambda \varphi + N_\lambda \sin \lambda \varphi) +
\]

\[
+ \left( C_0 + D_0 \ln \frac{1}{r} \right) (M_0 \varphi + N_0).
\]
By making use of the underneath regularity and boundary conditions

\[ A(r, \varphi) = A \left( r, \varphi \pm \frac{2\pi}{p} \right), \]

\[ A(r, \varphi) = -A \left( r, \varphi \pm \frac{\pi}{p} \right), \]

in (i) and (g),

\[ A(0, \varphi) = \text{finite}, \]

\[ \frac{\partial A}{\partial \varphi} \bigg|_{\varphi = \pm \alpha} = 0, \]

\[ \frac{\partial A}{\partial r} \bigg|_{r = c} = \mu_0 J_m \sin p(\varphi - \gamma), \]

the vector potential in the three subdomains may be written as follows

\[ A_i(r, \varphi) = \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^{p(2n-1)} \left[ M'_n \cos p(2n-1)\varphi + N'_n \sin p(2n-1)\varphi \right], \]

\[ A_o(r, \varphi) = \sum_{n=1}^{\infty} \left[ \left( \frac{r}{c} \right)^{p(2n-1)} + \left( \frac{c}{r} \right)^{p(2n-1)} \right] \left[ M''_n \cos p(2n-1)\varphi + \right. \]

\[ + N''_n \sin p(2n-1)\varphi \right] + \frac{\mu_0 c}{p} \left( \frac{r}{c} \right)^p J_m \sin p(\varphi - \gamma), \]

\[ A_s(r, \varphi) = \sum_{k=1}^{\infty} C_k \left( \frac{r}{b} \right)^{k\pi \alpha} + D_k \left( \frac{b}{r} \right)^{k\pi \alpha} \left[ \varepsilon_o \sin \frac{k\pi \varphi}{2\alpha} + \varepsilon_s \cos \frac{k\pi \varphi}{2\alpha} \right] + \]

\[ + \left( C_0 + D_0 \ln \frac{b}{r} \right), \]

where

\[ \varepsilon_o = \begin{cases} 1 & \text{for odd } k, \\ 0 & \text{for even } k, \end{cases} \]

\[ \varepsilon_s = \begin{cases} 0 & \text{for odd } k, \\ 1 & \text{for even } k. \end{cases} \]
The $M_n, N_n, M'_n, N'_n, C_k, D_k, C_0, D_0$ constants of integration are to be determined by imposing the passing conditions at the surfaces $r = a$ and $r = b$, i.e.:

-at $r = a$,

$$\frac{\partial A_i}{\partial r} = \begin{cases} \frac{\partial A_s}{\partial r} & \text{for } \varphi \in (-\alpha, \alpha), \\ 0 & \text{for } \varphi \in \left(\alpha, \frac{\pi}{p} - \alpha\right); \end{cases}$$  \hspace{1cm} (10)

$$A_i = A_i \text{ for } \varphi \in (-\alpha, \alpha);$$  \hspace{1cm} (11)

-at $r = b$,

$$\frac{\partial A_o}{\partial r} = \begin{cases} \frac{\partial A_s}{\partial r} & \text{for } \varphi \in (-\alpha, \alpha), \\ 0 & \text{for } \varphi \in \left(\alpha, \frac{\pi}{p} - \alpha\right); \end{cases}$$  \hspace{1cm} (12)

$$A_o = A_o \text{ for } \varphi \in (-\alpha, \alpha).$$  \hspace{1cm} (13)

Relations (10) and (12) express the continuity of tangential components of the magnetic field strength at $r = a$ and $r = b$, respectively.

Having different eigenvalues for the Fourier series in the expressions (6), (7), on the one hand, and (8) on the other, it is therefore necessary to expand $\frac{\partial A_i}{\partial r} \bigg|_{r=a}$ and $\frac{\partial A_o}{\partial r} \bigg|_{r=b}$ into Fourier series of period corresponding to $[-\frac{\pi}{p}, \frac{\pi}{p}]$ in terms of $\frac{\partial A_s}{\partial r} \bigg|_{r=a}$ and $\frac{\partial A_s}{\partial r} \bigg|_{r=b}$, respectively, as well as $A_i \bigg|_{r=a}$ and $A_o \bigg|_{r=b}$ into Fourier series of period corresponding to $[-\alpha, \alpha]$ in terms of $A_i \bigg|_{r=a}$ and $A_o \bigg|_{r=b}$, respectively (Fig. 2).

The identification of homologous terms in the series obtained from (6) — (8) and the corresponding series deriving from (10) — (13) finally
results in the following infinite set of equations, the unknowns being
the \( M'_n, \ N'_n, \ M''_n, \ N''_n, \ C_k, \ D_k, \ C_0, \ D_0 \) coefficients

\[
\sum_{n=1}^{\infty} (p_{kn}'s_n)N''_n + (-1)^{k} \frac{\alpha}{2} \left[ C_{2k-1} + D_{2k-1} \right] = -p_{k1}'i_c, \quad k = 1, 2, 3, \ldots,
\]

\[
\sum_{n=1}^{\infty} p_{kn}'N'_n + (-1)^{k} \frac{\alpha}{2} \left[ \left( \frac{a}{b} \right)^{\frac{(2k-1)\pi}{2\alpha}} \ C_{2k-1} + \left( \frac{b}{a} \right)^{\frac{(2k-1)\pi}{2\alpha}} \ D_{2k-1} \right] = 0, \quad k = 1, 2, 3, \ldots,
\]

\[
N'_n + \sum_{k=1}^{\infty} (-1)^{k} q_{kn}' \left[ \left( \frac{a}{b} \right)^{\frac{(2k-1)\pi}{2\alpha}} \ C_{2k-1} - \left( \frac{b}{a} \right)^{\frac{(2k-1)\pi}{2\alpha}} \ D_{2k-1} \right] = 0, \quad n = 1, 2, 3, \ldots,
\]

\[
d_n N''_n + \sum_{k=1}^{\infty} (-1)^{k} q_{kn}' [C_{2k-1} - D_{2k-1}] = \begin{cases} -i_c, & n = 1, \\ 0, & n = 2, 3, 4, \ldots; \end{cases}
\]

\[
\sum_{n=1}^{\infty} (p_{on}'s_n)M''_n + \alpha C_0 = -p_{01}'i_c,
\]

\[
\sum_{n=1}^{\infty} (p_{on}'s_n)M''_n + (-1)^{k} \frac{\alpha}{2} \left[ C_{2k} + D_{2k} \right] = -p_{k1}'i_c, \quad k = 1, 2, 3, \ldots,
\]

\[
\sum_{n=1}^{\infty} \left( p_{on}'M'_n + \alpha \left[ C_0 + \ln \frac{b}{a} \cdot D_0 \right] \right) = 0
\]

\[
\sum_{n=1}^{\infty} p_{kn}'M'_n + (-1)^{k} \frac{\alpha}{2} \left[ \left( \frac{a}{b} \right)^{\frac{k\pi}{\alpha}} C_{2k} + \left( \frac{b}{a} \right)^{\frac{k\pi}{\alpha}} D_{2k} \right] = 0, \quad k = 1, 2, 3, \ldots,
\]

\[
M'_n = \frac{4}{\pi} - \frac{p_{on}''}{2n - 1} D_0 + \sum_{k=1}^{\infty} (-1)^{k} q_{kn}' \left[ \left( \frac{a}{b} \right)^{\frac{k\pi}{\alpha}} C_{2k} - \left( \frac{b}{a} \right)^{\frac{k\pi}{\alpha}} D_{2k} \right] = 0,
\]

\[
d_n M''_n = \frac{4}{\pi} - \frac{p_{on}''}{2n - 1} D_0 + \sum_{k=1}^{\infty} (-1)^{k} q_{kn}' \left[ C_{2k} - D_{2k} \right] = \begin{cases} -i_s, & n = 1, \\ 0, & n = 2, 3, 4, \ldots; \end{cases}
\]
where the following notations have been used

\[ e_n = \left( \frac{b}{c} \right)^{n(2n-1)} + \left( \frac{c}{b} \right)^{n(2n-1)}, \quad d_n = \left( \frac{b}{c} \right)^{n(2n-1)} - \left( \frac{c}{b} \right)^{n(2n-1)}, \]

\[ p_{kn}' = \frac{p(2n-1)\cos p(2n-1)x}{\left[ \frac{(2k-1)\pi}{2\alpha} \right]^2 - [p(2n-1)]^2}, \quad q_{kn}' = \frac{2}{\alpha} \frac{2k-1}{2n-1} p_{kn}', \]

\[ p_{kn}'' = \frac{p(2n-1)\sin p(2n-1)x}{\left( \frac{k\pi}{\alpha} \right)^2 - [p(2n-1)]^2}, \quad q_{kn}'' = \frac{4}{\alpha} \frac{k}{2n-1} p_{kn}'', \]

\[ i_c = \frac{\mu_0 c}{p} \left( \frac{b}{c} \right)^p J_m \cos p\gamma, \quad i_s = -\frac{\mu_0 c}{p} \left( \frac{b}{c} \right)^p J_m \sin p\gamma. \]

The first four groups of equations, (14), refer to the vector potential of the magnetic field relative to the direct-axis of the machine (the corresponding current sheet on the stator surface having the linear density equal to \( J_m \cos p\gamma \sin p\varphi \)); the last four groups of equations, (15), refer to the magnetic field relative to the quadrature-axis of the machine (the corresponding current sheet having the linear density equal to \( -J_m \sin p\gamma \cos p\varphi \)).

By eliminating the \( C_k, D_k, C_0 \) and \( D_0 \) unknowns one obtains the infinite set of equations involving only the infinite sets of coefficients referring to the \((g')\) and \((i)\) subdomains

\[ d_n N''_n - \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} (a'_{km} s_m) N''_m + \sum_{m=1}^{\infty} \sum_{k=-1}^{\infty} b'_{km} N'_m = \]

\[ = \left( \sum_{k=1}^{\infty} a'_{1k1} \right) i_c, \quad n = 1, \]

\[ = \left( \sum_{k=1}^{\infty} a'_{nk1} \right) i_c, \quad n = 2, 3, 4, \ldots, \] \hfill (17)

\[- \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} (b'_{km} s_m) N''_m - N'_n + \sum_{m=1}^{\infty} \sum_{k=-1}^{\infty} a'_{km} N'_m = \]

\[ = \left( \sum_{k=1}^{\infty} b'_{1k1} \right) i_c, \quad n = 1, 2, 3, \ldots; \]
\[
\begin{align*}
\frac{d_u}{M''} &= \sum_{m=1}^{\infty} \left( \sum_{k=1}^{\infty} a_{nkm}'' + c_{nm}' \right) s_m M''' + \sum_{m=1}^{\infty} \left( \sum_{k=1}^{\infty} b_{nkm}'' + c_{nm}' \right) M'_m = \\
&= \left( \sum_{k=1}^{\infty} a_{n1k1}'' + c_{11}' \right) i_s, \quad n = 1, \\
&= \left( \sum_{k=1}^{\infty} a_{n1k1}'' + c_{11}' \right) i_s, \quad n = 2, 3, 4, \ldots,
\end{align*}
\]

\[
\begin{align*}
- \sum_{m=1}^{\infty} \left( \sum_{k=1}^{\infty} b_{nkm}'' + c_{nm}' \right) s_m M''' + M'_n + \sum_{m=1}^{\infty} \left( \sum_{k=1}^{\infty} a_{nkm}'' + c_{nm}' \right) M'_m = \\
= \left( \sum_{k=1}^{\infty} b_{n1k1}'' + c_{11}' \right) i_s, \quad n = 1, 2, 3, \ldots,
\end{align*}
\]

where the following notations have been used

\[
\begin{align*}
a_{nkm}' &= \frac{1}{2} \left[ \left( \frac{b}{a} \right)^{\frac{(2k-1)\pi}{2\alpha}} + \left( \frac{a}{b} \right)^{\frac{(2k-1)\pi}{2\alpha}} \right] b_{nkm}', \\
b_{nkm}' &= \frac{4}{\alpha} \frac{p'_{km} q'_{km}}{\left( \frac{b}{a} \right)^{\frac{(2k-1)\pi}{2\alpha}} - \left( \frac{a}{b} \right)^{\frac{(2k-1)\pi}{2\alpha}}}, \\
a_{nkm}'' &= \frac{1}{2} \left[ \left( \frac{b}{a} \right)^{\frac{k\pi}{\alpha}} + \left( \frac{a}{b} \right)^{\frac{k\pi}{\alpha}} \right] b_{nkm}'', \\
b_{nkm}'' &= \frac{4}{\alpha} \frac{p''_{km} q''_{km}}{\left( \frac{b}{a} \right)^{\frac{k\pi}{\alpha}} - \left( \frac{a}{b} \right)^{\frac{k\pi}{\alpha}}}, \\
c_{nm}' &= \frac{4}{\pi} \frac{1}{\alpha} \frac{\ln \frac{b}{a}}{2n - 1}.
\end{align*}
\]

The general method of solving the systems of equations (17) and (18) consists in retaining a finite number of equations and the same number of unknowns, according to the desired accuracy of the solution.

Once these systems of equations solved, the vector potential (6) — (8), the magnetic induction and the magnetic field strength are completely determined at any point inside the stator surface.

\[ \text{2.2. ACTIVE AND REACTIVE POWERS TRANSMITTED TO THE ROTOR} \]

In order to calculate the instantaneous electromagnetic power transmitted from the stator to the rotor, it is necessary to have the field values expressed in a system of coordinates fixed with the stator.
Through the transformation of coordinates

$$r \rightarrow r, \varphi \rightarrow \varphi - \frac{\omega}{p} t,$$

in which \( t \) is the time, expression (7) becomes

\[
A_\varphi(r, \varphi, t) = \sum_{n=1}^{\infty} \left[ \left( \frac{r}{c} \right)^{n(2n-1)} + \left( \frac{c}{r} \right)^{n(2n-1)} \right] \{ M_n'' \cos [(2n - 1) (p \varphi - \omega t)] + \\
+ N_n'' \sin [(2n - 1) (p \varphi - \omega t)] \} + \frac{\mu_0 c}{p} \left( \frac{r}{c} \right)^p J_m \sin [p(\varphi - \gamma) - \omega t],
\]

(21)

representing the vector potential in the airgap in the stator coordinates system.

The instantaneous electromagnetic power transmitted towards the rotor may be expressed by using the Poynting vector integral over the internal surface of the stator, [5]:

\[
p(t) = lc \left. \frac{\partial A_\varphi}{\partial t} \right|_{r=c} d\varphi = lc \int_0^{2\pi} \frac{\partial A_\varphi}{\partial t} \bigg|_{r=c} J_s^t d\varphi,
\]

(22)

where the linear density of the current sheet in the stator coordinate system is

\[
J_s^t = J_m \sin [p(\varphi - \gamma) - \omega t].
\]

(23)

The active power transmitted from the stator to the rotor may be written as, [6]

\[
P = \frac{\omega}{2\pi} \int_0^{2\pi} p(t) dt.
\]

(24)

The reactive power transmitted from the stator to the rotor is proportional with the magnetic energy, \( W_m \), from the inside of the stator surface, [6]

\[
Q = 2\omega W_m = 2\omega \left. -\frac{1}{2} lc \int_0^{2\pi} (A_\varphi H_\varphi) \mid_{r=c} d\varphi \right] = \omega lc \int_0^{2\pi} A_\varphi \mid_{r-c} J_s d\varphi.
\]

(25)

It should be noticed that only the first harmonic of the magnetic field in the airgap contributes to the transmission of power (formulae (22), (24), (25)), since the linear density of the current sheet has been supposed to have the expressions (1), (23).
By integrating in (22), (24) and (25), one obtains

\[ P = \omega (2\pi c l) \frac{\mu_0 e}{2p} J_m^2 \left[ (k_{ad} - k_{aq}) \sin p\gamma \cos p\gamma \right] \]  \hspace{1cm} (26)\]

\[ Q = \omega (2\pi c l) \frac{\mu_0 e}{2p} J_m^2 \left[ k_{ad} \cos^2 p\gamma + k_{aq} \sin^2 p\gamma \right] \]  \hspace{1cm} (27)\]

where it has been denoted:

\[ k_{ad} \equiv 1 + \frac{N_2'}{\mu_0 e J_m \cos p\gamma}, \quad k_{aq} \equiv 1 + \frac{M_2'}{\mu_0 e J_m \sin p\gamma} \]  \hspace{1cm} (28)\]

2.3. EQUIVALENT CIRCUIT

a) The synchronous reactances of the machine due to the magnetic field inside the stator surface corresponding to the direct-axis and quadrature-axis will be, respectively:

\[ x_{ad} = \frac{Q|_{\gamma = 0}}{m'T'^2} = (2\pi c l) \frac{\mu_0 e}{2pm'} \left( \frac{J_m}{I} \right)^2 \omega k_{ad} \]  \hspace{1cm} (29)\]

\[ x_{aq} = \frac{Q|_{\gamma = \pi}}{m'T'^2} = (2\pi c l) \frac{\mu_0 e}{2pm'} \left( \frac{J_m}{I} \right)^2 \omega k_{aq} \]  \hspace{1cm} (30)\]

where \( I \) is the current in the stator winding and \( k_{ad} \) and \( k_{aq} \) are the dimensionless longitudinal and transverse synchronous reactances, respectively

\[ k_{ad} = x_{ad}/X_0, \]  \hspace{1cm} (31)\]

\[ k_{aq} = x_{aq}/X_0, \]  \hspace{1cm} (32)\]

with

\[ X_0 \equiv (2\pi c l) \frac{\mu_0 e}{2pm'} \left( \frac{J_m}{I} \right)^2 \omega. \]  \hspace{1cm} (33)\]

The \( \left( \frac{J_m}{I} \right)^2 \) ratio depends on the arrangement of the stator winding [7].

The complex impedance per phase due to the electromagnetic field is

\[ Z = \frac{1}{m'T'^2} (P + jQ) = \]

\[ = (x_{ad} - x_{aq}) \sin p\gamma \cos p\gamma + j(x_{ad} \cos^2 p\gamma + x_{aq} \sin^2 p\gamma). \]  \hspace{1cm} (34)\]
The equivalent complex impedance of the circuit in Fig. 3 is, therefore,

\[ Z_{eq} = R_s + jX_\sigma + Z = \]

\[ = R_s + (x_d - x_q) \sin p\gamma \cos p\gamma + j(x_d \cos^2 p\gamma + x_q \sin^2 p\gamma), \]  \hspace{1cm} (35)

having the modulus

\[ Z_{eq} = \sqrt{R_s^2 + R_s(x_d - x_q) \sin 2p\gamma + x_d^2 \cos^2 p\gamma + x_q^2 \sin^2 p\gamma}, \]  \hspace{1cm} (36)

in which

\[ x_d = X_\sigma + x_{ad}, \] \hspace{1cm} (37)

\[ x_q = X_\sigma + x_{aq}. \]  \hspace{1cm} (38)

b) The input current for the synchronous performance may be written in complex as

\[ I = U/Z_{eq}, \] \hspace{1cm} (39)

where \( U \) is the phase supply voltage.
The reference current used for drawing the curves in Fig. 9 is

$$I_0 = U/X_e,$$  \hspace{1cm} (40)

$X_e$ being the phase reactance corresponding to a plain cylindrical rotor

$$X_e = (2\pi cl) \frac{\mu_0 c}{2 pm'} \left( \frac{J_m}{I} \right)^2 \omega \frac{c}{pg} = X_0 \frac{c}{pg},$$  \hspace{1cm} (41)

c) The input power factor may be determined by using the relation

$$\cos \varphi = \left[ R_s + (x_d - x_q) \sin p\gamma \cos p\gamma \right] \frac{1}{Z_q}.$$  \hspace{1cm} (42)

2.4. SYNCHRONOUS TORQUES

The torque upon the reluctance motor rotor may be calculated by using either Maxwell’s stress tensor, or the generalized forces theorem [6], as well as directly from the active power (26):

$$M = \frac{P}{\omega}, \quad P = (2\pi cl) \mu_0 c \frac{J_m^2}{2} \left[ (k_{ad} - k_{a_q}) \sin p\gamma \cos p\gamma \right].$$  \hspace{1cm} (43)

Making use of the notation

$$M_l \equiv (2\pi cl) \mu_0 c \frac{J_m^2}{2} = \frac{P}{\omega} m' X_0 I^2,$$  \hspace{1cm} (44)

Fig. 6. - The maximum adimensional torque at constant current, $g/c = 0.015, \Delta/c = 0.15.$

the dimensionless torque at constant current may be written under the form

$$m_1 \equiv \frac{M}{M_l} = (k_{ad} - k_{a_q}) \sin p\gamma \cos p\gamma.$$  \hspace{1cm} (45)
The maximum value of \( m_1 \) is obtained when \( p \gamma = \frac{\pi}{4} \),

\[
m_{1 \text{max}} = \frac{1}{2} (k_{ad} - k_{aq}).
\]  

(46)

With (31), (32), (33) and (39), the torque (43) may be written under the form

\[
M = \frac{p}{\omega} m' \frac{U^2}{Z_{eq}^2} [(x_d - x_q) \sin p \gamma \cos p \gamma],
\]

(47)

and the dimensionless torque at constant terminal voltage will have the expression

\[
m_u \equiv M_\mu = \frac{X_0}{Z_{eq}^2} [(x_d - x_q) \sin p \gamma \cos p \gamma] = m_1 \left( \frac{X_0}{Z_{eq}} \right)^2,
\]

(48)

where one has denoted

\[
M_\mu \equiv \frac{p}{\omega} m' \frac{U^2}{X_0}.
\]

(49)

The maximum value of \( m_u \) is obtained when

\[
tg p \gamma = \sqrt{\frac{R_s^2 + x_d^2}{R_s^2 + x_q^2}}.
\]

(50)

3. NUMERICAL COMPUTATION

The geometry of the system in Fig. 1 is characterized by four dimensionless parameters: \( p \), \( g/c \), \( \Delta/c \) and \( \beta \).

It is noticeable from (28) and the following relations that the \( N_1' \) and \( M_1' \) coefficients, i.e. the constants of integration corresponding to the fundamental harmonic of field in the airgap (s. (21)), are the only unknowns in the system of equations (17) and (18) whose numerical values must be calculated.

Numerical computation has been carried out for the following values of the parameters

\[
p : 1, 2, 3;
\]

\[
g/c : 0.005, 0.010, 0.015, 0.020, 0.025, 0.030, 0.050;
\]

\[
\Delta/c : 0.15, 0.20, 0.25, 0.30, 0.35, 0.40;
\]

\[
\beta : 0.165 - 0.965.
\]

To improve the convergence of \( \sum_{k=1}^{\infty} a'_{nkm} \) and \( \sum_{k=1}^{\infty} a''_{nkm} \) summations in (17) and (18), respectively, these series have been expressed with the aid of the logarithmic derivative of the gamma function and its derivative as is shown in the Appendix.
Fig. 7. - The maximum adimensional torque at constant terminal voltage, $g/c = 0.015$, $\Delta/c = 0.15$; $R_s = 3.04 \Omega$, $X_\sigma = 4.00 \Omega$, $X_0 = 5.475 / \mu \Omega$.

Fig. 8. - Effects of varying airgap length for $p = 1$, $\Delta/c = 0.15$; $R_s = 3.04 \Omega$, $X_\sigma = 4.00 \Omega$, $X_0 = 5.475 \Omega$.

Fig. 9. - $I^{(max)}/I_0$ for $g/c = 0.015$, $\Delta/c = 0.15$; $R_s = 3.04 \Omega$, $X_\sigma = 4.00 \Omega$, $X_0 = 5.475 / \mu \Omega$.

Fig. 10. - $\cos \varphi^{(max)}$ for $g/c = 0.015$, $\Delta/c = 0.15$; $R_s = 3.04 \Omega$, $X_\sigma = 4.00 \Omega$, $X_0 = 5.475 / \mu \Omega$. 
In order to establish the accuracy of the numerical results, $k_{ad}$ and $k_{aq}$ have been computed by keeping in turn the first 4, 6, 8 and 10 equations and a corresponding equal number of terms in the respective summations in each of the two groups of equations in the two systems (17) and (18). The errors (in percentages) for $k_{ad}$ and $k_{aq}$ do not exceed the values shown in the table.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Number of equations in each group</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
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<tr>
<td>$p = 1$</td>
<td>$g = \frac{c}{c}$</td>
<td>0.010</td>
<td>-15</td>
<td>-7</td>
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<tr>
<td></td>
<td></td>
<td>0.015</td>
<td>-12</td>
<td>-5</td>
</tr>
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<td>-4</td>
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<td></td>
<td></td>
<td>0.025</td>
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<td>-3.9</td>
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<tr>
<td></td>
<td></td>
<td>0.030</td>
<td>-8.79</td>
<td>-3.58</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.050</td>
<td>-7.75</td>
<td>-3.35</td>
</tr>
<tr>
<td>$p = 2$</td>
<td>$g = \frac{c}{c}$</td>
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<td>-8.6</td>
<td>-3.7</td>
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<tr>
<td></td>
<td></td>
<td>0.015</td>
<td>-7.5</td>
<td>-2.7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.020</td>
<td>-5.7</td>
<td>-2.12</td>
</tr>
<tr>
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<td>0.025</td>
<td>-4.5</td>
<td>-1.8</td>
</tr>
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<td></td>
<td></td>
<td>0.030</td>
<td>-3.75</td>
<td>-1.55</td>
</tr>
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<td>0.050</td>
<td>-3.45</td>
<td>-1.1</td>
</tr>
<tr>
<td>$p = 3$</td>
<td>$g = \frac{c}{c}$</td>
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<td>-7.5</td>
<td>-2.8</td>
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<tr>
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<td></td>
<td>0.015</td>
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<td>-1.7</td>
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<td></td>
<td></td>
<td>0.050</td>
<td>-1.85</td>
<td>-0.76</td>
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</table>

The $\Delta/c$ and $\beta$ parameters have a very little influence upon these errors in comparison with that of the $p$ and $g/c$ parameters.

The only numerical results given in this paper are those represented under the form of diagrams in Figs. 4 - 10. All these results were computed with an error less than 0.5 %.

4. CONCLUSIONS

The general shape of the curves in Fig. 4 and Fig. 5 is that characteristic of the usual reluctance motor [8]. $k_{ad}$ is practically independent of $\Delta/c$ and $k_{aq}$ increases with $\Delta/c$ so that $k_{ad}/k_{aq}$ decreases approximately linear by $(10 - 35)\%$ when $\Delta/c$ increases from 0.15 to 0.40.
The highest values of torque are obtained for a motor with two poles, the maximum value of the torque at constant terminal voltage showing a peak as in Fig. 7; the smaller is the airgap length the greater will be the peak value of this torque (Fig. 8). It should be noted that for $\beta$ within 0.6 and 0.9 $m_{\text{max}}^p$ is a little greater when $g/c$ is greater. Taking into account that the input current increases very much when $\beta$ decreases and $g/c$ increases, the reluctance motors with segmental rotors will be designed with $p = 1$, $\beta = 0.7 \div 0.9$ and $g/c$ as small as possible. Within these limits the power factor has a reasonable value of about 0.6. The input current and the torque decrease by $(10 \div 15)%$ when $\Delta/c$ increases from 0.15 to 0.40.

The numerical results obtained by the authors of this paper correspond to the experimental tests carried through by Lawrenson and Gupta [3].

The procedure developed above can be applied for a great deal of cases regarding the field of electrical machines and apparatuses, as well as the other branches of engineering and physics, enlarging the sphere of problems which can be solved exactly.

\textbf{APPENDIX}

The computation of the $\sum_{k=1}^{\infty} a'_{km}$ summation in (17) may be carried out by writing it under the form

$$\sum_{k=1}^{\infty} a'_{km} = \frac{2}{\pi} \sum_{k=1}^{\infty} \left( \frac{b}{a} \right)^{(2k-1)\pi/2\alpha} + \left( \frac{a}{b} \right)^{(2k-1)\pi/2\alpha} p'_{km} q'_{kn} =$$

$$= \frac{2}{\alpha} \left[ \frac{2}{\pi} \sum_{k=1}^{\infty} \left( \frac{a}{b} \right)^{(2k-1)\pi/2\alpha} p'_{km} q'_{kn} + \sum_{k=1}^{\infty} p'_{km} q'_{kn} \right], \quad (A.1)$$

in which the first series is very quickly convergent for $\frac{a}{b} < 1$ and $\alpha \ll \frac{\pi}{2}$.

To evaluate the last series in the above expression, one can write the identity

$$\frac{(2k - 1) \pi}{2\alpha} \sum_{k=1}^{\infty} \frac{1}{(2k - 1)^2} - \frac{1}{(2k - 1)^2} \sum_{k=1}^{\infty} \frac{1}{(2k - 1)^2}$$

$$= \left\{ \frac{(2k - 1) \pi}{2\alpha} \sum_{k=1}^{\infty} \left( \frac{1}{k + \alpha_m} + \frac{1}{k - \alpha_m} \right) \right\} \sum_{k=1}^{\infty} \left( \frac{1}{k + \alpha_m} \right)^{1/2} \sum_{k=1}^{\infty} \left( \frac{1}{k + \alpha_m} \right)^{1/2}, \quad \text{for } n \neq m,$$

$$= \left( \frac{2\alpha}{\pi} \right)^{3/4} \sum_{k=1}^{\infty} \left( \frac{1}{(k - \alpha_m)^2} - \frac{1}{(k + \alpha_m)^2} \right), \quad \text{for } n = m, \quad (A.2)$$
where
\[ \alpha_l \equiv p(2l - 1) \frac{2\alpha}{\pi}, \quad l = n, m. \] (A.3)

Making use of the digamma function (logarithmic derivative of the gamma function) \( \psi(z) \) and its derivative \( \psi'(z) \), which are tabulated in [9], [10] and satisfy the relations
\[ \sum_{k=1}^{\infty} \frac{1}{k + z} = - \psi(z) + \frac{1}{2} \psi\left( \frac{z}{2} \right) + \frac{1}{2} \left( - C + \sum_{k=0}^{\infty} \frac{1}{k + 1} \right), \] (A.4)

where \( C \) is Euler's constant,
\[ \psi(z) + \psi(-z) = 2\psi(z) + \pi \text{ctg} \pi z + \frac{1}{z}, \] (A.5)
\[ \sum_{k=1}^{\infty} \frac{1}{(k + z)^2} = \psi'(z) - \frac{1}{4} \psi'\left( \frac{z}{2} \right), \] (A.6)
\[ \psi'(z) - \psi'(-z) = 2\psi'(z) - \left( \frac{\pi}{\sin \pi z} \right)^2 - \frac{1}{z^2}, \] (A.7)

the following results can be easily obtained

\[ \sum_{k=1}^{\infty} p'_{km} q'_{kn} = \]
\[ \left( \frac{2\alpha}{\pi} \right)^2 \frac{\alpha_m}{\alpha_n} \cos \frac{\pi \alpha_m}{2} \cos \frac{\pi \alpha_n}{2} \left[ \psi(\alpha_n) - \psi(\alpha_m) \right] - \]
\[ = - \frac{1}{2} \left[ \psi\left( \frac{\alpha_m}{2} \right) - \psi\left( \frac{\alpha_m}{2} \right) \right] - \frac{\pi}{4} \left[ \text{ctg} \frac{\pi \alpha_n}{2} - \text{ctg} \frac{\pi \alpha_m}{2} \right], \quad \text{for } n \neq m, \quad (A.8)\]
\[ \left( \frac{2\alpha}{\pi} \right)^2 \cos^2 \frac{\alpha_m}{2} \left\{ - \psi'(\alpha_m) + \frac{1}{4} \psi'\left( \frac{\alpha_m}{2} \right) + \frac{1}{2} \left( \frac{\pi}{2 \cos \frac{\pi \alpha_m}{2}} \right)^2 \right\}, \quad \text{for } n = m. \]

Similarly, it can be shown that \( \sum_{k=1}^{\infty} a'_{knm} \) in (18) is
\[ \sum_{k=1}^{\infty} a'_{knm} = \frac{2}{\alpha} \sum_{k=1}^{\infty} \frac{\left( \frac{b}{a} \right)^{k\pi \alpha}}{\alpha} \frac{\left( \frac{a}{b} \right)^{k\pi \alpha}}{\alpha} p''_{km} q''_{kn} = \]
\[ = \frac{2}{\alpha} \left[ 2 \sum_{k=1}^{\infty} \left( \frac{a}{b} \right)^{k\pi \alpha} \right] p''_{km} q''_{kn} + \sum_{k=1}^{\infty} p'_{km} q'_{kn}, \] (A.9)
with

\[
\sum_{k=1}^{\omega} p_{km} q_{km} = \\
\left( \frac{2\alpha}{\pi} \right)^{\frac{1}{2}} \frac{1}{2} \sin^{2} \frac{\pi \alpha_m}{2} \left\{ - \frac{1}{4} \psi' \left( \frac{\alpha_m}{2} \right) + \frac{1}{2} \left( \frac{\pi}{2 \sin \frac{\pi \alpha_m}{2}} \right)^{\frac{1}{2}} + \right. \\
\left. + \frac{1}{2 \alpha_m^2} \right\}, \text{ for } n = m
\]

(A.10)

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REFERENCES


