INVERSION TRANSFORMATION FOR THE FINITE-ELEMENT SOLUTION OF THREE-DIMENSIONAL EXTERIOR-FIELD PROBLEMS

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ABSTRACT: A simple and efficient method for the finite-element solution of three-dimensional unbounded region field problems is presented in this paper. The proposed technique consists of a global mapping of the original unbounded region onto a bounded domain by applying a standard inversion transformation to the spatial coordinates. Same numerical values of the potential function are assigned to the transformed points. The functional associated to the field problem, which incorporates the boundary conditions, has the same structure in the transformed domain as that in the original one. This allows the implementation of the standard finite-element method in the bounded transformed domain.

The finite-element solution is obtained on the basis of a complete discretization of the bounded, transformed domain by standard finite elements, with no approximate assumption made for the behaviour of the field at infinity, other than that introduced by the finite-element idealization. This leads to improved accuracy of the numerical results, compared to those obtained in the original region, for the same number of nodes. Application to three test problems illustrates the high efficiency of the proposed method in terms of both accuracy and computational effort. The technique presented is particularly recommended for exterior-field problems in the presence of material inhomogeneities and anisotropies.

1. INTRODUCTION

Finite-element analysis of exterior-field problems is usually performed by applying the following techniques: the truncation of the unbounded region at a sufficiently large distance; the coupling of finite elements within a finite region with integral equations [1, 2], and various local or global mapping techniques [3–7]. In the truncated mesh technique, approximate conditions are imposed on the terminating boundaries and, consequently, this procedure is computationally inefficient. Improved accuracy is obtained by coupling finite elements with integral equations, but specific hybrid algorithms have to be used and no standard finite-element computer program can be implemented directly.

Local mapping techniques have been elaborated based on properly formulated infinite elements for modelling the anticipated decay of the field quantities in the far region [3–5]. A different approach involves mapping the entire unbounded region of the problem onto a bounded domain in which the numerical solution is obtained without any approximate assumptions on the behaviour of the field quantities at infinity. Conformal mappings through complex analytic functions have been considered for two-dimensional, parallel-plane scalar Laplacian fields [6] and for general axisymmetric exterior-field problems [7]. For Laplacian field problems in unbounded
three-dimensional regions, the classical Kelvin transformation has also been applied, in conjunction with the finite difference method, to their solution in the transformed domain [8].

In this paper, we consider three-dimensional exterior-field problems and apply the inversion transformation to map the unbounded region onto a bounded one, with the same potential function at the corresponding transformed points. Subsequently, the finite-element method is applied to obtain the solution in the transformed domain, based on the expression of the functional determined from that associated to the original field problem by using the same inversion transformation. Unlike the Kelvin transformation, which is easily applicable to the Laplace equation only, the proposed method is a general one, in the sense that it can be applied in a standard, simple manner to a large variety of field problems, including those with material inhomogeneities and anisotropies.

As in the conventional finite-element analysis, the solution in the bounded transformed domain is interpolated by standard polynomial functions within the finite elements. A-priori knowledge of the type of decay of the exact solution in the far region is not required. Preservation of the exact boundary conditions of the original problem in the transformed domain, including the regularity conditions at infinity, significantly improves the accuracy of the numerical results. The examples given as an illustration show that, even with a very coarse mesh, highly accurate results are produced not only for global quantities but also for the local field quantities.

2. PROPOSED METHOD

Let the three-dimensional unbounded region $D$ be the exterior of a bounded domain $D_b$. To illustrate the theoretical formulation of the method, consider the following boundary-value problem for a scalar field $\Phi$ in $D$.

\[ \nabla \cdot (\kappa \cdot \nabla \Phi) = g \quad \text{in } D \quad (1) \]
\[ \Phi(s) = \Phi_0(s), \quad s \in S_1 \quad (2) \]
\[ (\kappa \cdot \nabla \Phi) \cdot n|_{S_2} + \sigma(s)\Phi(s) = h(s), \quad s \in S_2 \quad (3) \]

where $g$, $\Phi_0(s)$, $\sigma(s) \geq 0$ and $h(s)$ are given functions of point, $S_1 + S_2$ is the boundary of $D_b$, and $n$ denotes the unit vector along the outward normal. The symmetric tensor $\kappa$ represents the material properties of the anisotropic and inhomogeneous medium in $D$.

Let the quadratic form $V \cdot (\kappa \cdot V)$ be positive definite for any related physical vector field $V$ in $D$. Then the finite-element solution to the above boundary-value problem, as any other variational method solution, can be obtained through the minimization of the functional [9]

\[ F = \int_D \left[ (\nabla \Phi) \cdot (\kappa \cdot \nabla \Phi) + 2g\Phi \right] dD + \int_{S_2} (\sigma \Phi^2 - 2h\Phi) dS \quad (4) \]

with trial functions which satisfy the Dirichlet boundary condition in eq. (2).

Instead of using directly the functional in eq. (4), associated to the original
Inversion transformation for the finite-element solution of an unbounded region field problem, we apply the inversion transformation to obtain an equivalent transformed functional relative to a transformed, bounded domain. With the transformed cartesian coordinates denoted by $x'_1, x'_2, x'_3$, the inversion transformation of the original cartesian coordinates $x_1, x_2, x_3$ with respect to a sphere of radius $a$ and centered at $(x_1, x_2, x_3) = (b_1, b_2, b_3)$ is given by

$$x'_i = \frac{a^2}{r^2} (x_i - b_i) + b'_i, \quad i = 1, 2, 3$$  \hspace{1cm} (5)

where

$$r \equiv \left[ \sum_{i=1}^{3} (x_i - b_i)^2 \right]^{1/2}$$  \hspace{1cm} (6)

Correspondingly the inverse transformation is,

$$x_i = \frac{a^2}{r'^2} (x'_i - b'_i) + b_i, \quad i = 1, 2, 3$$  \hspace{1cm} (7)

with

$$r' \equiv \left[ \sum_{i=1}^{3} (x'_i - b'_i)^2 \right]^{1/2}$$  \hspace{1cm} (8)

and

$$rr' = a^2$$  \hspace{1cm} (9)

By applying the transformation in eqs. (5) and (6), with $(b_1, b_2, b_3) \in D_b$, the original problem region $D + S_1 + S_2$ is mapped conformally onto the bounded domain $D' + S'_1 + S'_2$, and the functional in eq. (4) is transformed in the form (see Appendix)

$$F = \int_D \left[ (\nabla' \Phi') \cdot (\kappa' \cdot \nabla' \Phi') + 2g' \left( \frac{a}{r'} \right)^6 \Phi' \right] \, dD' + \int_{S'_1} \left( \sigma' \Phi'^2 - 2h' \Phi' \right) \left( \frac{a}{r'} \right)^4 \, dS'$$  \hspace{1cm} (10)

$\Phi', g', \sigma'$ and $h'$ are, respectively, $\Phi, g, \sigma$ and $h$ as functions of $x'_1, x'_2$ and $x'_3$, obtained through eq. (7). $\nabla'$ is the del operator in primed coordinates. In a matrix form, the tensor $\kappa'$ is related to $\kappa$ by

$$\kappa' = \left( \frac{a}{r'} \right)^2 T \kappa T^{-1}$$  \hspace{1cm} (11)

where the square matrix $T$ has the element $T_{ij} = (\partial x'_i / \partial x_j)$ and $T^{-1}$ denotes its inverse. $\kappa'$ is a symmetric tensor, as it is $\kappa$, and can be interpreted as representing the fictitious material properties of the medium in the transformed domain.

The transformed functional (10) has a form which can be obtained directly from
that in eq. (4) by replacing $g$ by $g'(a/r')^6$, and $\sigma$ and $h$ by $\sigma'(a/r')^4$ and $h'(a/r')^4$, respectively. This functional is now minimized in the transformed domain by using trial functions which satisfy the Dirichlet boundary condition for $\Phi'$ given in eq. (2), taking into account the point transformation in eqs. (7) and (8). The solution in the transformed domain yields, through this transformation, the original field problem solution. It should be noted that global quantities related to the functional associated to the problem can be obtained directly from its transformed form, without being necessary to recover the original problem solution through mapping.

A special case, important from a practical point of view, is that of an inhomogeneous but isotropic medium, when $\kappa$ is a scalar, i.e., $\kappa = \kappa(x_1, x_2, x_3)$. Equation (11) shows that $\kappa' = (a/r')^2 \kappa'(x'_1, x'_2, x'_3)$ is now also a scalar, with $\kappa'(x'_1, x'_2, x'_3)$ being $\kappa$ expressed as a function of the primed coordinates.

The simple form of the transformed functional in eq. (10) is due to the fact that the inversion transformation is conformal. Since the transformed functional is similar to the original one, a general finite-element computer program appropriate for the original field problem can also be used in a direct manner in the transformed domain. In addition, the complete discretization of the bounded, transformed domain, with no supplementary approximate assumption, yields accurate numerical results. Consequently, the proposed method is highly efficient and its implementation is very simple and straightforward.

It should be noted that in order to obtain a bounded transformed domain $D' + S'_1 + S'_2$, the center of inversion must always be positioned inside the bounded domain $D_b$, $(b_1, b_2, b_3) \in D_b$. In most applications, it is convenient to locate the center of inversion at the origin, as shown in the next section. However, in the case where the unbounded domain $D + S_1 + S_2$ contains the origin, the inversion center will have to be chosen at a different point, always inside $D_b$.

3. TEST EXAMPLES

To illustrate the implementation and the efficiency of the proposed method, let us consider the calculation of the electrostatic field of a conducting body in three different situations. The electric potential distribution and the electric capacitance can be determined by minimizing the functional in eq. (4) [10], with $g = \sigma = h = 0$ and constant Dirichlet boundary condition $\Phi = \Phi_0 \neq 0$ on the conductor surface $S$. For a three-dimensional electrostatic field the potential can be chosen to be equal to zero at infinity. In all the examples, second order tetrahedral elements have been used in the finite element analysis.

3.1. Conducting cube in an unbounded space

Consider a conducting cube of side $c$ centered at the coordinate system origin, as shown in Fig. 1, and assume a linear and isotropic dielectric outside the cube, i.e., $\kappa = \kappa(x_1, x_2, x_3)$. The unbounded region $D + S$ is mapped onto the bounded domain $D' + S'$, shown in Fig. 2, by applying the inversion transformation with respect to a unit sphere centered at the origin $(a = 1, b_i = b'_i = 0, i = 1, 2, 3$, in eqs. (5)-(11)). The transformed functional in eq. (10) is now
Figure 1: First octant of a cube of side $c$.

Figure 2: First octant of the transformed domain $D'$ for cube problem.

$$F = \int_{D'} \frac{\kappa'(x'^1_1, x'^1_2, x'^1_3)}{r'^2} (\nabla' \Phi')^2 \, dD'$$

(12)

and must be minimized under the following constraints for $\Phi'$

$$\Phi' = \Phi_0 \text{ on } S$$

(13)

and

$$\Phi' = 0 \text{ at } x'^1_1 = x'^1_2 = x'^1_3 = 0$$

(14)
since the point at infinity in the original region is mapped onto the origin in the transformed domain.

Two cases, corresponding to two kinds of dielectric medium surrounding the cube, have been analyzed.

**Case 1. Homogeneous outside medium**

For a homogeneous dielectric outside the cube

\[
\kappa(x_1, x_2, x_3) = \varepsilon_0 = \text{const}
\]  

(15)

where \(\varepsilon_0\) is the constant permittivity of the medium. Consequently, in eq. (12)

\[
\kappa'(x'_1, x'_2, x'_3) = \varepsilon_0 = \text{const}
\]  

(16)

**Case 2. Inhomogeneous outside medium**

As an illustrative example, we have chosen for the permittivity \(\kappa\) of the dielectric outside the cube the following expression

\[
\kappa(x_1, x_2, x_3) = \varepsilon_0 (5c/\sqrt{x_1^2 + x_2^2 + x_3^2 + 1})
\]  

(17)

where \(\varepsilon_0\) is, for instance, the permittivity of free space. Using eq. (9), with \(a = 1\), yields

\[
\kappa'(x'_1, x'_2, x'_3) = \varepsilon_0 (5c\sqrt{x_1'^2 + x_2'^2 + x_3'^2 + 1})
\]  

(18)

Due to the system symmetry, only 1/48 of the transformed domain \(D'\) was actually discretized by finite elements. Various meshes of second order tetrahedral elements were used in the analysis. The finite-element method was also applied in a similar manner to the original region by truncating it with a larger, symmetrically placed cube set at zero potential (see Fig. 4). The normalized capacitance \(C = F_{\min}/(4\pi\varepsilon_0 c\Phi_0^2)\), where \(F_{\min}\) is the minimum value of the functional in eqs. (12) or (4), obtained in a few representative trials is given in Table 1. Indications about the meshes used in Trial 3 and Trial 5 are presented in Figs. 3 and 4, respectively.

Accurate numerical results obtained by the boundary integral equation method, as well as upper and lower bounds for the capacitance of a conducting cube in a homogeneous dielectric are available in the literature [11–13]. The exact value of the

| Table 1 |
|---|---|---|---|---|---|
| Item | Transformed domain | | | Original domain | |
| | Trial 1 | Trial 2 | Trial 3 | Trial 4 | Trial 5 |
| Number of elements | 27 | 64 | 104 | 66 | 127 |
| Number of unknowns | 55 | 119 | 179 | 122 | 225 |
| Artificial boundary | | | | 10c × 10c × 10c | 20c × 20c × 20c |
| Normalized capacitance, Case 1 | 0.643 | 0.6595 | 0.658 | 0.786 | 0.738 |
| Normalized capacitance, Case 2 | 2.125 | 2.225 | 2.212 | 3.367 | 2.8595 |
normalized capacitance is within the range $0.654 < C < 0.668$ [14]. Our result in Trial 3, $C = 0.658$, is in good agreement with the values reported in the literature. The capacitance determined in Trial 2, by using 64 elements, differs from that calculated in Trial 3, with 104 elements, by less than 0.23%. Even for a very coarse mesh used in Trial 1 the result of 0.643 obtained for the normalized capacitance is still quite acceptable, demonstrating the efficiency and the accuracy of the proposed method. As
shown in Table 1 and Fig. 5, it is much better than the truncated mesh technique. The potential distribution along the z-axis in Fig. 5 is that calculated in Trial 3 and Trial 5.

The results obtained for the normalized capacitance in Case 2, in the bounded, transformed domain and in the unbounded, original region, corresponding to the same meshes as in the previous case, are presented in Table 1.

It should be noted that the value of normalized capacitance corresponding to Trial 1 in Table 1 is less than the exact one, which corresponds theoretically to the minimum value of the functional in eq. (12). This is due to the fact that the numerical integration formulas used [15] do not evaluate exactly the integral in eq. (12), since its integrand contains the factor $K'/r^2$. These integration formulas are nevertheless adequate and the calculated capacitance approaches its “exact” value as the mesh is refined.

3.2. Conducting sphere in a homogeneous dielectric

In order to check the accuracy of the presented method from point of view of the local field quantities, we applied it to the calculation of the capacitance and potential distribution of a conducting sphere of diameter $c$, placed in an infinitely extended homogeneous dielectric, for which the exact solution is well known. Inversion transformation with respect to a concentric unit sphere was used, and, as in Trial 3 for Example 3.1, $1/48$ of the transformed domain was discretized by 104 elements. The percentage error in the calculated capacitance is 1.4%. The computed normalized potential distribution along the radial direction is represented in Fig. 5, being found in good agreement with its exact solution. Within a radial distance of 10 times the radius
from the sphere center, the maximum percentage error in potential values is 5.4%. For the mesh considered the accuracy of the calculated values of nodal potential increases substantially towards the sphere surface.

4. CONCLUSION

The method proposed and tested in this paper is extremely simple and straightforward, and standard three-dimensional finite-element computer programs can be used for its implementation. Approximations regarding the behaviour at infinity of the exterior-field problem solution are eliminated by reformulating the variational problem in a bounded, transformed domain, with the preservation of the exact boundary conditions. Consequently, this method provides a better accuracy of the numerical results as compared to those corresponding to the solution in the original, unbounded region.

As shown in the illustrative examples in Section 3, the method presented is also very efficient, good accuracy being obtained even with a relatively small number of elements in the transformed domain. It is applicable in the same standard manner independently of the nature of the linear material properties, with anisotropies and inhomogeneities that may extend theoretically to infinity.

The method presented can easily be generalized to n-dimensional exterior-field problems by replacing the powers of 2, 4 and 6 of \( a/r' \) in eqs. (10), (11), (22) and (25)–(27) by \( 2n - 4, 2n - 2 \) and \( 2n \), respectively.

The general idea of transforming the functional associated to an unbounded region field problem and obtaining the solution in a bounded domain can be applied to the analysis of scalar fields other than those considered for illustration in this paper, and even to the analysis of vector fields.

APPENDIX

The transformed functional in eq. (10) can be derived from the original one in eq. (4) either by a direct calculation [16] or by using the matrix form of the corresponding vectors and tensors. In matrix form, the vector \( \nabla \Phi \) can be written as

\[
\nabla \Phi = T' \nabla' \Phi'
\]

where the column matrices \( \nabla \Phi \) and \( \nabla' \Phi' \) have the elements \( \partial \Phi / \partial x_i \) and \( \partial \Phi' / \partial x'_i \), respectively, and the square matrix \( T \) has the element \( T_{ij} = \partial x'_i / \partial x_j \) with \( i, j = 1, 2, 3 \), \( T' \) being its transpose. From eqs. (5)–(9) one can easily see that the transformation matrix \( T \) is a symmetric one, i.e.

\[
T' = T
\]

and elementary calculation yields the inverse of \( T \)

\[
T^{-1} = (\frac{a}{r'})^4 T
\]
the Jacobian of the transformation

\[ J = |T|^{-1} = \left( \frac{a}{r'} \right)^6 \]  

(22)

and the elements of length, surface and volume

\[ dl = \left( \frac{a}{r'} \right)^2 dl', \quad dS = \left( \frac{a}{r'} \right)^4 dS', \quad dD = \left( \frac{a}{r'} \right)^6 dD' \]  

(23)

The scalar \((\nabla \Phi') \cdot (\kappa \cdot \nabla \Phi) dD\) in eq. (4) is therefore obtained in matrix form, with eqs. (19)–(21) and (23), as

\[
(\nabla \Phi')'\kappa\nabla \Phi' dD = (\nabla' \Phi')'T\kappa'T(\nabla' \Phi')'\left( \frac{a}{r'} \right)^6 dD' \\
= (\nabla' \Phi')'\kappa'(\nabla' \Phi')' dD'
\]

(24)

where

\[
\kappa' = \left( \frac{a}{r'} \right)^6 T\kappa'T = \left( \frac{a}{r'} \right)^2 T\kappa' T^{-1}
\]

(25)

The matrix form obtained in eq. (24) corresponds to the scalar \((\nabla' \Phi') \cdot (\kappa' \cdot \nabla' \Phi') dD'\) in eq. (10). This and eqs. (23) yield the transformed functional in eq. (10).

The Euler–Lagrange equation and the natural boundary conditions associated to the functional in eq. (10) are [16]

\[
\nabla' \cdot (\kappa' \cdot \nabla' \Phi') = \left( \frac{a}{r'} \right)^6 g' \text{ in } D'
\]

(26)

and

\[
(\kappa' \cdot \nabla' \Phi') \cdot n'|_{S_2} + \left( \frac{a}{r'} \right)^4 \sigma'(s') \Phi'(s') = \left( \frac{a}{r'} \right)^4 h'(s'), \quad s' \in S_2'
\]

(27)

and represent the transformed form of eqs. (1) and (3), respectively.

**ACKNOWLEDGEMENT**

This work was supported by a grant from the Natural Sciences and Engineering Research Council (NSERC) of Canada, whose financial support is gratefully acknowledged.
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