ROTATIONAL-TRANSLATIONAL ADDITION THEOREMS FOR VECTOR SPHERoidal WAVE FUNCTIONS

M.F.R. COORAY and I.R. CIRIC
Department of Electrical Engineering, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2

ABSTRACT: Rotational-translational addition theorems for the vector spheroidal wave functions \( M_{mn}^{a(i)}(h; \xi, \eta, \phi) \) and \( N_{mn}^{a(i)}(h; \xi, \eta, \phi) \), i = 1, 2, 3, 4, are derived from those for the corresponding scalar spherical wave functions \( \psi_{mn}^{a(i)}(h; \xi, \eta, \phi) \). A vector spheroidal wave function defined in one spheroidal coordinate system \((h; \xi, \eta, \phi)\) is expressed in terms of a series of vector spheroidal wave functions defined in another spheroidal coordinate system \((h'; \xi', \eta', \phi')\), which is rotated and translated with respect to the first one. These theorems allow a rigorous treatment of boundary value problems relative to time-harmonic vector field waves in the presence of a system of spheroids with arbitrary orientations. As a special case, general rotational-translational addition theorems for vector spherical wave functions are also presented.

1. INTRODUCTION

The problem of radiation and scattering of electromagnetic waves in a system with two or more spheroids of an arbitrary orientation is of particular importance, since real objects with a large variety of shapes can well be approximated by prolate or oblate spheroids. An exact eigenfunction solution to this problem can be obtained on the basis of rotational-translational addition theorems for vector spheroidal vector wave functions. The derivation of such theorems is the purpose of this paper.

Translational addition theorems for the case of spherical scalar wave functions were obtained by the authors of [1], and those for spherical vector wave functions were developed later in [2, 3]. Stein [2] also presented rotational addition theorems for spherical vector wave functions. In [4] the authors used the translational addition theorems for spherical vector wave functions to solve the problem of scattering of a plane electromagnetic wave by two perfectly conducting prolate spheroids with parallel major axes [7-9]. They have also been used for deriving the admittance of a pair of spheroidal dipole antennas with parallel major axes [10]. Recently, the authors of [11] obtained the rotational addition theorem and rotational-translational addition theorems for scalar spheroidal wave functions. In
this paper we extend that work [11], to derive the rotational-translational addition theorems for the vector spheroidal wave functions $M_{mn}^{(i)}(h; \xi, \eta, \phi)$ and $N_{mn}^{(i)}(h; \xi, \eta, \phi)$ with $a = x, y, z; i = 1, 2, 3, 4$, as well as for the vector wave functions $M_{mn}^{(i)}(h; \xi, \eta, \phi)$ and $N_{mn}^{(i)}(h; \xi, \eta, \phi)$ which also intervene in the solution of vector field problems in the presence of a system of spheroids of arbitrary orientation. We also derive the rotational-translational addition theorems for the vector spheroidal wave functions $M_{\omega \cdot m}^{(i)}(h; \xi, \eta, \phi)$ and $N_{\omega \cdot m}^{(i)}(h; \xi, \eta, \phi)$ when $a$ is the radial vector $r$.

The coordinates $\xi, \eta, \phi$ are the spheroidal coordinates and $h = kF$, where $k$ is the wave number of the time-harmonic fields and $F$ is the semi-interfocal distance of the spheroids $\xi = \text{const}$. The vector wave functions $M_{mn}^{(i)}(h; \xi, \eta, \phi)$ and $N_{mn}^{(i)}(h; \xi, \eta, \phi)$ are those defined in [8], having a $e^{j(m \cdot n + 1)\phi}$ dependence, where $j = \sqrt{-1}$. Thus they are equivalent to the functions $M_{mn}^{(i)}(h; \xi, \eta, \phi)$ and $N_{mn}^{(i)}(h; \xi, \eta, \phi)$ used in [12]. The notation in the rest of the text is that adopted in this latter reference unless otherwise stated, with $i = 1, 2, 3, 4$ corresponding to the radial spheroidal functions of the first, second, third and fourth kind, respectively. All the derivations presented in this paper are for prolate spheroidal wave functions. The expressions for the oblate system can easily be obtained from those for the prolate system by the transformation $\xi \rightarrow j\xi, h \rightarrow -jh$ (or $F \rightarrow -jF$).

2. ROTATIONAL-TRANSLATIONAL ADDITION THEOREMS FOR VECTOR SPHEROIDAL WAVE FUNCTIONS

In order to simplify the notation, without any loss of generality, consider only two Cartesian reference frames $(x, y, z)$ and $(x', y', z')$ as shown in Fig. 1. A point $P$ has spheroidal coordinates $\xi, \eta, \phi$ and $\xi', \eta', \phi'$ associated with these two reference frames, respectively. The system $(x', y', z')$ is obtained from $(x, y, z)$ by rotating the latter to $(x_{||}, y_{||}, z_{||})$ which is parallel to $(x', y', z')$ and then by a translation. The origin $O'$ of $(x', y', z')$ has spherical coordinates $d, \theta_d, \phi_d$ with respect to the Cartesian system $(x_{||}, y_{||}, z_{||})$.

The scalar spheroidal wave functions $\psi_{mn}^{(i)}(h; \xi, \eta, \phi)$ in the unprimed system can be expressed as series expansions in terms of scalar spheroidal wave functions in the primed system, by using the rotational-translational addition theorems for these functions [11],

$$
\psi_{mn}^{(i)}(h; \xi, \eta, \phi) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (\mu) Q_{\mu \nu}^{m \nu}(\alpha, \beta, \gamma; d) \psi_{\mu \nu}^{(1)}(h'; \xi', \eta', \phi') \quad r' \leq d ; \quad i = 1, 2, 3, 4,
$$

$$
\psi_{mn}^{(i)}(h; \xi, \eta, \phi) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} P_{\mu \nu}^{m \nu}(\alpha, \beta, \gamma; d) \psi_{\mu \nu}^{(1)}(h'; \xi', \eta', \phi') \quad r' \geq d ; \quad i = 1, 2, 3, 4,
$$

where $(\mu) Q_{\mu \nu}^{m \nu}(\alpha, \beta, \gamma; d)$ and $P_{\mu \nu}^{m \nu}(\alpha, \beta, \gamma; d)$ are the rotational-translational expansion coefficients (see Appendix), with $(\mu) Q_{\mu \nu}^{m \nu}(\alpha, \beta, \gamma; d) = P_{\mu \nu}^{m \nu}(\alpha, \beta, \gamma; d)$, and $\alpha, \beta, \gamma$ are the Euler angles [13] that specify the rotation of the primed system with
Figure 1: Rotation and translation of the Cartesian coordinate system \((x, y, z)\) to the system \((x', y', z')\).

respect to the unprimed one. The relation between the unit vectors \(\hat{x}, \hat{y}, \hat{z}\) and \(\hat{x}', \hat{y}', \hat{z}'\) is given by

\[
\hat{a} = c_{ax}\hat{x}' + c_{ay}\hat{y}' + c_{az}\hat{z}' , \quad \hat{a} = \hat{x}, \hat{y}, \hat{z} ,
\]

where the coefficients \(c_{ax}', c_{ay}', c_{az}'\) are expressed in terms of the Euler angles,

\[
\begin{align*}
    c_{xx'} &= \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma ,
    
    c_{xy'} &= - (\cos \alpha \cos \beta \sin \gamma + \sin \alpha \cos \gamma ) ,
    
    c_{xz'} &= \cos \alpha \sin \beta ,
    
    c_{yx'} &= \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma ,
    
    c_{yy'} &= \cos \alpha \cos \gamma - \sin \alpha \cos \beta \sin \gamma ,
    
    c_{yz'} &= \sin \alpha \sin \beta ,
    
    c_{zx'} &= - \sin \beta \cos \gamma ,
    
    c_{zy'} &= \sin \beta \sin \gamma ,
    
    c_{zz'} &= \cos \beta .
\end{align*}
\]
Spheroidal vector wave functions which are independent elementary solutions of the vector Helmholtz equation can be simply generated as [14]

\[
M_{mn}^{a(i)}(h; \xi, \eta, \phi) = \nabla \psi_{mn}^{(i)}(h; \xi, \eta, \phi) \times a, \\
N_{mn}^{a(i)}(h; \xi, \eta, \phi) = k^{-1} \nabla \times M_{mn}^{a(i)}(h; \xi, \eta, \phi),
\]

where \(a\) is a constant vector or the radial vector. In this paper we consider the vector wave functions given in eqs. (5) and (6) with \(a\) being one of the Cartesian unit vectors \(\hat{x}, \hat{y}, \hat{z}\), their linear combinations which are particularly useful in the analysis of field problems involving spheroids [5, 8, 12], as well as the vector wave functions with \(a\) being the radial vector \(r\).

### 2.1. Functions Defined with the Cartesian Unit Vectors \(\hat{x}, \hat{y}, \text{or} \hat{z}\)

In the following, the coordinate triads \((\xi, \eta, \phi)\) and \((\xi', \eta', \phi')\) will be denoted by \(r\) and \(r'\), respectively, and the arguments of \((^{(i)}Q_{\mu \nu}^{mn}(\alpha, \beta, \gamma; d)\) and \(P_{\mu \nu}^{mn}(\alpha, \beta, \gamma; d)\) will be omitted. Substituting eqs. (1) and (2) in eq. (5) and then eq. (5) in eq. (6), for \(a = x, y, z\), yields

\[
X_{mn}^{a(i)}(h; r) = \sum_{\nu = 0}^{\infty} \sum_{\mu = -\nu}^{\nu} (^{(i)}Q_{\mu \nu}^{mn}) [c_{ax}X_{\mu \nu}^{x(i)}(h'; r') + c_{ay}X_{\mu \nu}^{y(i)}(h'; r')] \\
+ c_{az}X_{\mu \nu}^{z(i)}(h'; r')], \quad r' \leq d; \quad i = 1, 2, 3, 4,
\]

\[
X_{mn}^{a(i)}(h; r) = \sum_{\nu = 0}^{\infty} \sum_{\mu = -\nu}^{\nu} (^{(i)}P_{\mu \nu}^{mn}) [c_{ax}X_{\mu \nu}^{x(i)}(h'; r') + c_{ay}X_{\mu \nu}^{y(i)}(h'; r')] \\
+ c_{az}X_{\mu \nu}^{z(i)}(h'; r')], \quad r' \geq d; \quad i = 1, 2, 3, 4,
\]

where \(X\) is either of the vector spheroidal wave functions \(M\) or \(N\). These simple expressions give the functions \(M\) and \(N\) in one system of spheroidal coordinates in terms of the same type of functions in another system of spheroidal coordinates, rotated and translated with respect to the first one. In the analysis of field problems, the following linear combinations [8] are used:

\[
X_{mn}^{z(i)}(h; r) = \frac{1}{2} [X_{mn}^{x(i)}(h; r) \pm jX_{mn}^{y(i)}(h; r)], \quad i = 1, 2, 3, 4.
\]

From eqs. (7), (8) and (9), the following expressions are derived finally:

\[
X_{mn}^{a(i)}(h; r) = \sum_{\nu = 0}^{\infty} \sum_{\mu = -\nu}^{\nu} (^{(i)}Q_{\mu \nu}^{mn}) [C_{1}X_{\mu \nu}^{x(i)}(h'; r') + C_{2}X_{\mu \nu}^{z(i)}(h'; r')] \\
+ C_{3}X_{\mu \nu}^{z(i)}(h'; r')], \quad r' \leq d; \quad i = 1, 2, 3, 4,
\]

\[
X_{mn}^{a(i)}(h; r) = \sum_{\nu = 0}^{\infty} \sum_{\mu = -\nu}^{\nu} (^{(i)}P_{\mu \nu}^{mn}) [C_{1}X_{\mu \nu}^{x(i)}(h'; r') + C_{2}X_{\mu \nu}^{z(i)}(h'; r')] \\
+ C_{3}X_{\mu \nu}^{z(i)}(h'; r')], \quad r' \geq d; \quad i = 1, 2, 3, 4,
\]
\[ X_{mn}^{(i)}(h; r) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (i) Q_{\mu \nu}^{mn} \left[ C_{2} X_{\mu \nu}^{(1)}(h'; r') + C_{1} X_{\mu \nu}^{(1)}(h'; r') \right] \\
+ C_{3} X_{\mu \nu}^{(1)}(h'; r') \right], \quad r' \leq d; \quad i = 1, 2, 3, 4, \quad (12) \]

\[ X_{mn}^{(i)}(h; r) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (i) P_{\mu \nu}^{mn} \left[ C_{4} X_{\mu \nu}^{(1)}(h'; r') + C_{1} X_{\mu \nu}^{(1)}(h'; r') \right] \\
+ C_{5} X_{\mu \nu}^{(1)}(h'; r') \right], \quad r' \geq d; \quad i = 1, 2, 3, 4, \quad (13) \]

\[ X_{mn}^{(i)}(h; r) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (i) P_{\mu \nu}^{mn} \left[ C_{4} X_{\mu \nu}^{(1)}(h'; r') + C_{4} X_{\mu \nu}^{(1)}(h'; r') \right] \\
+ C_{5} X_{\mu \nu}^{(1)}(h'; r') \right], \quad r' \geq d; \quad i = 1, 2, 3, 4, \quad (14) \]

\[ X_{mn}^{(i)}(h; r) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (i) P_{\mu \nu}^{mn} \left[ C_{5} X_{\mu \nu}^{(1)}(h'; r') \right], \quad r' \geq d; \quad i = 1, 2, 3, 4, \quad (15) \]

where the asterisk denotes the complex conjugate, and

\[ C_{1} = \frac{1}{2} \left[ (c_{xx} + c_{yy}) - j(c_{xy} - c_{yx}) \right], \]

\[ C_{2} = \frac{1}{2} \left[ (c_{xx} - c_{yy}) + j(c_{xy} + c_{yx}) \right], \]

\[ C_{3} = \frac{1}{2} (c_{xz} + jcz_{y}), \]

\[ C_{4} = c_{xz} - jcz_{y}, \]

\[ C_{5} = c_{zz}. \quad (16) \]

The expressions in eqs. (7)-(8) and eqs. (10)-(15) constitute the rotational-translational addition theorems for the vector spheroidal wave functions \( M_{mn}^{a}(i) \), \( M_{mn}^{z}(i) \), \( N_{mn}^{a}(i) \) and \( N_{mn}^{z}(i) \).  

2.2. Functions Defined with the Radial Vector \( r \)

Using eqs. (1) and (A.1), the even and odd spheroidal scalar wave functions in the unprimed system can be expressed in the form of a series expansion in terms of both even and odd spheroidal scalar wave functions in the primed system, for \( r' \leq d \) and \( i = 1, 2, 3, 4 \), as

\[ \psi_{e, omn}^{(i)}(h; r) = \sum_{q=0}^{\infty} \sum_{\mu=-q}^{q} F_{\mu q}^{mn} \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\nu} \sum_{i=\mu, \mu+1}^{\nu} \Gamma_{\mu \nu i} \]

\[ \cdot \left[ e^{o, e}_{c} P_{\mu \nu q}^{mn} \psi_{e, omn}^{(1)}(h'; r') \pm o, e_{s} P_{\mu \nu q}^{mn} \psi_{e, omn}^{(1)}(h'; r') \right], \quad (17) \]

where
\[ e, \nu B_{\mu \nu q} = j_{m+q-n} e^{-i\phi_d} \left\{ (i) a_{\mu \nu}^{j_{m+q}} (d) e^{i\mu \phi_d} \cos [(\mu - \mu) \phi_d + \mu \phi + m \phi] \right\} \]

\[ \pm (-1)^{\mu} \frac{(\nu - \mu)!}{(\nu + \mu)!} \left\{ (i) a_{\mu \nu}^{j_{m+q}} (d) e^{i\mu \phi_d} \cos [(\mu - \mu) \phi_d + \mu \phi + m \phi] \right\} \right], \] (18)

\[ o, \nu B_{\mu \nu q} = j_{m+q-n} e^{-i\phi_d} \left\{ (i) a_{\mu \nu}^{j_{m+q}} (d) e^{i\mu \phi_d} \sin [(\mu - \mu) \phi_d + \mu \phi + m \phi] \right\} \]

\[ \pm (-1)^{\mu} \frac{(\nu - \mu)!}{(\nu + \mu)!} \left\{ (i) a_{\mu \nu}^{j_{m+q}} (d) e^{i\mu \phi_d} \sin [(\mu - \mu) \phi_d + \mu \phi + m \phi] \right\} \right], \] (19)

for \( \mu > 0, \)

\[ o, \nu B_{0 \nu q} = j_{m+q-n} e^{-i\phi_d} \left\{ (i) a_{0 \nu}^{j_{m+q}} (d) \cos (\tilde{\phi}_d + \mu \phi + m \phi) \right\}, \] (20)

\[ o, \nu B_{0 \nu q} = j_{m+q-n} e^{-i\phi_d} \left\{ (i) a_{0 \nu}^{j_{m+q}} (d) \sin (\tilde{\phi}_d + \mu \phi + m \phi) \right\}, \] (21)

for \( \mu = 0, \)

\[ F_{\mu q}^{mn} = d_{\mu q}^{(m+q)} (h)(-1)^{\mu-m} \left[ \frac{N_{m|m+q}}{N_{\mu|m+q}} \right]^{1/2} d_{\mu m}^{(m+q)} (\beta), \] (22)

and

\[ \Gamma_{\mu \nu}(h') = \frac{N_{\mu \nu}^{j_{m+q}}}{N_{\mu \nu}^{j_{m+q}}} d_{\nu \mu}^{(m+q)} (h') \] (23)

Eq. (17) gives the rotational-translational addition theorems for the scalar spheroidal wave functions \( \psi_{e, \nu m q} \) \((i = 1, 2, 3, 4)\), for \( r' \leq d \). Using the relationship between the spheroidal and spherical scalar wave functions [12], we get for \( r' \leq d \)

\[ \psi_{e, \nu m q}(h'; r) = \sum_{q=0}^{\infty} \sum_{\mu=-(|m|+q)}^{\infty} F_{\mu q}^{mn} \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \left[ e, \nu B_{\mu \nu q}^{mn} \psi_{e, \nu m q}^{(1)}(r', \theta', \phi') \right] \]

\[ \pm o, \nu B_{\mu \nu q}^{mn} \psi_{e, \nu m q}^{(1)}(r', \theta', \phi') \] (24)

where

\[ \psi_{e, \nu m q}^{(1)}(r', \theta', \phi') = j_{\nu}(kr')P_{\nu}^{m}(\cos \theta') \sin \mu \phi'. \]

(25)

(The notation in eqs. (17)–(25) is explained in the Appendix.)

Taking the gradient on both sides of eq. (24) and then the cross product with \( \mathbf{r} \) gives

\[ M_{e, \nu m q}^{(1)}(h'; r) = \sum_{q=0}^{\infty} \sum_{\mu=-(|m|+q)}^{\infty} F_{\mu q}^{mn} \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \left[ e, \nu B_{\mu \nu q}^{mn} \nabla \psi_{e, \nu m q}^{(1)}(r', \theta', \phi') \right] \]

\[ \pm o, \nu B_{\mu \nu q}^{mn} \nabla \psi_{e, \nu m q}^{(1)}(r', \theta', \phi') \right] \times \mathbf{r}. \] (26)
Since the gradient of a scalar function is invariant to a transformation of the coordinate system, using the relations

\[ r = r' + d; \quad d = d \sin \theta_d \cos \phi_d \hat{x}' + d \sin \theta_d \sin \phi_d \hat{y}' + d \cos \theta_d \hat{z}' \]

denoting the vector spherical wave function \( \nabla \psi_{e,o \mu \nu}^{(1)}(r', \theta', \phi') \times \hat{a}' \) by \( m_{e,o \mu \nu}^{(1)}(a' = x', y', z', r') \), and omitting the argument of \( M_{e,o \mu \nu}^{(1)}(h; r) \), we get

\[
M_{e,o \mu \nu}^{(1)} = \sum_{q=0}^{\infty} \sum_{\mu = -(|m| + q)}^{\infty} F_{\mu q}^{mn} \sum_{\nu = 0}^{\infty} \{ e^{o \cdot \nabla_{e,o \mu \nu}^{(1)}(m_{e,o \mu \nu}^{(1)} + d \sin \theta_d \cos \phi_d m_{e,o \mu \nu}^{(1)} + d \sin \theta_d \sin \phi_d m_{e,o \mu \nu}^{(1)} + d \cos \theta_d m_{e,o \mu \nu}^{(1)})} \}
\]

\[
+ d \sin \theta_d \sin \phi_d m_{e,o \mu \nu}^{(1)} + d \cos \theta_d m_{e,o \mu \nu}^{(1)} + d \cos \theta_d m_{e,o \mu \nu}^{(1)} \}
\]

Now we express the vector spherical wave functions \( m_{e,o \mu \nu}^{(1)}(a' = x', y', z') \), in terms of the vector spherical wave functions \( m_{e,o \mu \nu}^{(1)}(a' = x', y', z') \), and omitting the argument of \( M_{e,o \mu \nu}^{(1)}(h; r) \), we get

\[
\sum_{\nu = 0}^{\infty} \sum_{\mu = 0}^{\nu} B_{\mu q}^{mn} m_{e,o \mu \nu}^{(1)}(x', y', z') = \sum_{\nu = 1}^{\infty} \sum_{\mu = 0}^{\nu} \{ a_{\mu q}^{mn} m_{e,o \mu \nu}^{(1)} + b_{\mu q}^{mn} n_{e,o \mu \nu}^{(1)} \}
\]

\[
\sum_{\nu = 0}^{\infty} \sum_{\mu = 0}^{\nu} B_{\mu q}^{mn} m_{e,o \mu \nu}^{(1)}(x', y', z') = \sum_{\nu = 1}^{\infty} \sum_{\mu = 0}^{\nu} \{ a_{\mu q}^{mn} m_{e,o \mu \nu}^{(1)} + b_{\mu q}^{mn} n_{e,o \mu \nu}^{(1)} \}
\]

\[
\sum_{\nu = 0}^{\infty} \sum_{\mu = 0}^{\nu} B_{\mu q}^{mn} m_{e,o \mu \nu}^{(1)}(x', y', z') = \sum_{\nu = 1}^{\infty} \sum_{\mu = 0}^{\nu} \{ a_{\mu q}^{mn} m_{e,o \mu \nu}^{(1)} + b_{\mu q}^{mn} n_{e,o \mu \nu}^{(1)} \}
\]

where \( a_{\mu q}^{mn}, b_{\mu q}^{mn}, a_{\mu q}^{mn}, b_{\mu q}^{mn}, a_{\mu q}^{mn}, b_{\mu q}^{mn} \), and \( b_{\mu q}^{mn} \) are given correspondingly by

\[
a_{\mu q}^{mn} = \frac{k}{2 \nu (\nu + 1)} \left\{ \frac{\nu}{(2 \nu + 3)} \left[ (\nu + \mu + 1)(\nu + \mu + 2)B_{\nu + 1, \nu + 1, \mu + 1}^{mn} - B_{\nu + 1, \nu + 1, \mu - 1}^{mn} \right] + \frac{(\nu + 1)}{(2 \nu - 1)} \left[ -(\nu - \mu - 1)(\nu - \mu)B_{\nu + 1, \nu - 1, \mu + 1}^{mn} + B_{\nu + 1, \nu - 1, \mu - 1}^{mn} \right] \right\},
\]

\[
a_{\mu q}^{mn} = \frac{\pm k}{2 \nu (\nu + 1)} \left\{ \frac{-\nu}{(2 \nu + 3)} \left[ (\nu + \mu + 1)(\nu + \mu + 2)B_{\nu + 1, \nu - 1, \mu + 1}^{mn} + B_{\nu + 1, \nu - 1, \mu - 1}^{mn} \right] + \frac{(\nu + 1)}{(2 \nu - 1)} \left[ (\nu - \mu - 1)(\nu - \mu)B_{\nu + 1, \nu - 1, \mu + 1}^{mn} + B_{\nu + 1, \nu - 1, \mu - 1}^{mn} \right] \right\},
\]

\[
a_{\mu q}^{mn} = \frac{k}{\nu (\nu + 1)} \left\{ \frac{(\nu + 1)(\nu - \mu)}{2 \nu - 1} B_{\nu + 1, \nu - 1, \mu + 1}^{mn} + \frac{\nu (\nu + \mu + 1)}{2 \nu + 3} B_{\nu + 1, \nu + 1, \mu + 1}^{mn} \right\},
\]

\[
b_{\mu q}^{mn} = \frac{\pm k}{2 \nu (\nu + 1)} \left\{ (\nu - \mu)(\nu + \mu + 1)B_{\nu + 1, \nu + 1, \mu q}^{mn} + B_{\nu + 1, \nu - 1, \mu q}^{mn} \right\},
\]

\[
b_{\mu q}^{mn} = \frac{k}{\nu (\nu + 1)} \left\{ (\nu - \mu)(\nu + \mu + 1)B_{\nu + 1, \nu + 1, \mu q}^{mn} - B_{\nu + 1, \nu - 1, \mu q}^{mn} \right\},
\]

\[
b_{\mu q}^{mn} = \frac{\pm \mu \nu}{\nu (\nu + 1)} B_{\nu + 1, \nu + 1, \mu q}^{mn}.
\]
Substituting eq. (29) in (28), and taking into account the cancellation of the three components of \( \mathbf{m}_{e,\nu_0\nu}^{(1)} \) and \( \mathbf{n}_{e,\nu_0\nu}^{(1)} \) for \( \nu = \mu = 0 \), yields

\[
M_{e,\nu_0\nu}^{r(i)} = \sum_{q=0}^{\infty} \sum_{q=1}^{\infty} \sum_{\mu = \nu_0}^{\infty} \sum_{\nu = \nu_0}^{\nu = \mu = 0} \left\{ \epsilon_{x} m_{\mu_0\nu_0\nu} m_{e,\nu_0\nu}^{r(i)} + \epsilon_{y} n_{\mu_0\nu_0\nu} n_{e,\nu_0\nu}^{r(i)} \right\} + \epsilon_{z} m_{\mu_0\nu_0\nu}^{r(i)} + \epsilon_{t} n_{\mu_0\nu_0\nu}^{r(i)},
\]

(32)

where

\[
\epsilon_{x} m_{\mu_0\nu_0\nu} = \epsilon_{x} B_{\mu_0\nu_0\nu} + d \sin \theta_d \cos \phi_d \epsilon_{x} a_{\mu_0\nu_0\nu} + d \cos \theta_d \epsilon_{x} a_{\mu_0\nu_0\nu} \pm d \sin \theta_d \sin \phi_d \epsilon_{x} a_{\mu_0\nu_0\nu},
\]

\[
\epsilon_{y} m_{\mu_0\nu_0\nu} = d \sin \theta_d \sin \phi_d \epsilon_{x} b_{\mu_0\nu_0\nu} \mp (d \sin \theta_d \cos \phi_d \epsilon_{x} b_{\mu_0\nu_0\nu} + d \cos \theta_d \epsilon_{x} b_{\mu_0\nu_0\nu}) + d \cos \theta_d \epsilon_{x} b_{\mu_0\nu_0\nu},
\]

(33)

\[
o_{x} m_{\mu_0\nu_0\nu} = d \sin \theta_d \sin \phi_d \epsilon_{x} a_{\mu_0\nu_0\nu} \pm (d \sin \theta_d \cos \phi_d \epsilon_{x} a_{\mu_0\nu_0\nu} + d \cos \theta_d \epsilon_{x} a_{\mu_0\nu_0\nu}),
\]

\[
o_{x} m_{\mu_0\nu_0\nu} = d \sin \theta_d \cos \phi_d \epsilon_{x} b_{\mu_0\nu_0\nu} + d \cos \theta_d \epsilon_{x} b_{\mu_0\nu_0\nu} \pm d \sin \theta_d \sin \phi_d \epsilon_{x} b_{\mu_0\nu_0\nu},
\]

(34)

in which the coefficients \( \epsilon_{x} a_{\mu_0\nu_0\nu}, \epsilon_{x} b_{\mu_0\nu_0\nu}, \ldots \), are evaluated by replacing \( B_{\mu_0\nu_0\nu}^{mn} \) in eqs. (30) and (31) by \( \epsilon_{x} B_{\mu_0\nu_0\nu}^{mn} \) and \( \epsilon_{x} B_{\mu_0\nu_0\nu}^{mn} \), appropriately.

Taking the curl of both sides of eq. (32) and then multiplying by \( k^{-1} \) gives

\[
N_{e,\nu_0\nu}^{r(i)} = \sum_{q=0}^{\infty} \sum_{q=1}^{\infty} \sum_{\mu = \nu_0}^{\infty} \sum_{\nu = \nu_0}^{\nu = \mu = 0} \left\{ \epsilon_{x} m_{\mu_0\nu_0\nu}^{r(i)} m_{e,\nu_0\nu}^{r(i)} + \epsilon_{y} n_{\mu_0\nu_0\nu}^{r(i)} n_{e,\nu_0\nu}^{r(i)} \right\} + \epsilon_{z} m_{\mu_0\nu_0\nu}^{r(i)} + \epsilon_{t} n_{\mu_0\nu_0\nu}^{r(i)},
\]

(34)

where the invariance of the curl operator to a transformation of the coordinate system has been considered. Using the relations [6, 12]

\[
m_{e,\nu_0\nu}^{r(i)}(r', \theta', \phi') = \sum_{l=\mu, \mu + 1}^{\infty} \Gamma_{\mu l} M_{e,\nu_0\nu}^{r(i)}(h; r'),
\]

(35)

\[
n_{e,\nu_0\nu}^{r(i)}(r', \theta', \phi') = \sum_{l=\mu, \mu + 1}^{\infty} \Gamma_{\mu l} N_{e,\nu_0\nu}^{r(i)}(h; r'),
\]

(36)

in eqs. (32) and (34), gives

\[
M_{e,\nu_0\nu}^{r(i)} = \sum_{q=0}^{\infty} \sum_{q=1}^{\infty} \sum_{\mu = \nu_0}^{\infty} \sum_{\nu = \nu_0}^{\nu = \mu = 0} \sum_{\mu = \nu_0}^{\mu = 0} \sum_{\mu = \nu_0}^{\nu = \mu = 0} \left\{ \epsilon_{x} m_{\mu_0\nu_0\nu}^{r(i)} m_{e,\nu_0\nu}^{r(i)} + \epsilon_{y} n_{\mu_0\nu_0\nu}^{r(i)} n_{e,\nu_0\nu}^{r(i)} \right\} + \epsilon_{z} m_{\mu_0\nu_0\nu}^{r(i)} + \epsilon_{t} n_{\mu_0\nu_0\nu}^{r(i)},
\]

(37)
\[
N_{e, omn}^{r(i)} = \sum_{q=0,1}^{\infty} \sum_{\mu=-(|m|+q)}^{\infty} \sum_{\nu=0}^{\mu} \sum' \left( A_{\mu \nu \mu q l}^{mn} N_{e, oml}^{r(i)} + B_{\mu \nu \nu q l}^{mn} M_{e, oml}^{r(i)} \right) + C_{\mu \nu \mu q l}^{mn} N_{e, oml}^{r(i)} + D_{\mu \nu \mu q l}^{mn} M_{e, oml}^{r(i)} ,
\]

where

\[
A_{\mu \nu \mu q l}^{mn} = e_{\omega} x_{\mu \nu \mu q}^{mn}(\alpha, \beta, \gamma; d) \Gamma_{\mu \nu \mu l} , \quad C_{\mu \nu \mu q l}^{mn} = e_{\omega} z_{\mu \nu \mu q}^{mn}(\alpha, \beta, \gamma; d) \Gamma_{\mu \nu \mu l} ,
\]

\[
D_{\mu \nu \mu q l}^{mn} = e_{\omega} y_{\mu \nu \mu q}^{mn}(\alpha, \beta, \gamma; d) \Gamma_{\mu \nu \mu l} , \quad B_{\mu \nu \mu q l}^{mn} = e_{\omega} t_{\mu \nu \mu q}^{mn}(\alpha, \beta, \gamma; d) \Gamma_{\mu \nu \mu l} ,
\]

with

\[
e_{\omega} x_{\mu \nu \mu q}^{mn}(\alpha, \beta, \gamma; d) = e_{\omega} x_{\mu \nu \mu q}^{mn} F_{\mu \nu \mu q}^{mn} , \quad e_{\omega} z_{\mu \nu \mu q}^{mn}(\alpha, \beta, \gamma; d) = e_{\omega} z_{\mu \nu \mu q}^{mn} F_{\mu \nu \mu q}^{mn} ,
\]

\[
e_{\omega} y_{\mu \nu \mu q}^{mn}(\alpha, \beta, \gamma; d) = e_{\omega} y_{\mu \nu \mu q}^{mn} F_{\mu \nu \mu q}^{mn} , \quad e_{\omega} t_{\mu \nu \mu q}^{mn}(\alpha, \beta, \gamma; d) = e_{\omega} t_{\mu \nu \mu q}^{mn} F_{\mu \nu \mu q}^{mn} .
\]

Expressions in eqs. (37) and (38) give the rotational-translational addition theorems for vector spheroidal wave functions \( M_{e, omn}^{r(i)} \) and \( N_{e, omn}^{r(i)} \) for \( i = 1, 2, 3, 4 \) and \( r' \leq d \), which are necessary in the study of multiple scattering of electromagnetic waves by two spheroids of arbitrary orientation.

### 3. SPECIAL CASES

#### 3.1. Translational Addition Theorems

This special case is obtained when \( \alpha \rightarrow 0, \beta \rightarrow 0, \) and \( \gamma \rightarrow 0 \). Referring to eqs. (A.3), (A.4) and (A.5) (see Appendix), we can write

\[
R_{\mu \nu}^{m s}(0, 0, 0) = \delta_{m \mu} ,
\]

where \( \delta \) is the Kronecker delta function. By setting \( s = |m| + q, l = |\mu| + r \) in eqs. (A.1) and (A.2), then substituting \( a^{m,|m|+q}(d) \) in eq. (A.1) from (A.6), and \( b^{m,|m|+q}(d) \) in eq. (A.2) from (A.8) and (A.9), eqs. (A.1) and (A.2) can be rewritten as

\[
Q_{\mu \nu}^{mn} = \frac{2(-1)^{\mu}}{N_{\mu \nu}(h')} \sum _{q=0,1}^{\infty} \sum _{r=0,1}^{\infty} \sum _{p=p_0, p_0+1}^{\infty} \sum' j^{p+\nu-\mu} a(m, |m| + q - \mu, |\mu| + r, |\mu| + r) d_{q}^{mn}(h') Q_{\mu \nu}^{mn} (kd) P_{\mu \nu}^{mn} (\cos \theta_d) e^{i(m-\mu)\phi_d} ,
\]

\[
P_{\mu \nu}^{mn} = \frac{2(-1)^{m-\mu}}{N_{\mu \nu}(h')} \sum _{q=0,1}^{\infty} \sum _{p=|\mu|, |\mu|+1}^{\infty} \sum _{l=l_0, l_0+1}^{\infty} d_{q}^{mn}(h) a(m, |m| + q - \mu, |\mu| + r, |\mu| + r) d_{q}^{mn}(h) P_{\mu \nu}^{mn} (\cos \theta_d) e^{i(m-\mu)\phi_d} .
\]
When $\alpha \to 0$, $\beta \to 0$, $\gamma \to 0$, the coefficients $c_{a',x}$, $c_{a',y}$, $c_{a',z}$ ($a = x, y, z$) defined in eq. (4) are all zero except $c_{x',x}$, $c_{y',y}$ and $c_{z',z}$ which are equal to unity. Substitution of these in eqs. (7)–(8) gives for $a = x, y, z$ and $a' = x', y', z'$

$$X_{mn}^{a(i)}(h; r) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} Q_{\mu \nu}^{mn} X_{\mu \nu}^{a(i)}(h' ; r') , \quad r' \leq d ; \quad i = 1, 2, 3, 4 , \quad (44)$$

$$X_{mn}^{a(i)}(h; r) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} P_{\mu \nu}^{mn} X_{\mu \nu}^{a(i)}(h' ; r') , \quad r' \geq d ; \quad i = 1, 2, 3, 4 , \quad (45)$$

where $X$ is either of the vector spheroidal wave functions $M$ or $N$.

Eq. (44) for $X = M$ and $X = N$ is exactly the same as eqs. (42) and (44) in [5]. However eq. (45) for $X = M$ and $X = N$ is not the same as eqs. (43) and (45) in [5], but can be brought to the same form by the following rearrangement and change of notation. After substituting $P_{\mu \nu}^{mn}$ from eq. (43), eq. (45) becomes

$$X_{mn}^{a(i)}(h ; r) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} 2(-1)^{m-\mu} \sum'_{q=0,1} d^{m q}_{q}(h) \sum'_{p=|\mu|,|\mu|+1} \sum'_{l=l_{0},l_{0}+1} \sum'_{p=|\mu|-l,|\mu|-l+1} \sum'_{q=0,1} d_{q}^{m|\mu|(h')} j^{l+q-n}(2l+1) a(m, |m|+q, \mu, m, l, p) \frac{d^{m|\mu|}(h')}{(2p+1)}$$

$$\cdot \frac{(p+\mu)!}{(p-\mu)!} z_{l}^{(1)}(kd) P_{l}^{m-\mu}(\cos \theta_d) e^{i(m-\mu)\phi_d} X_{\mu \nu}^{a(i)}(h' ; r') . \quad (46)$$

By replacing $\mu$ by $m - \mu$, $\nu$ by $t$, $l$ by $\nu$, and rearranging, we obtain

$$X_{mn}^{a(i)}(h ; r) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum'_{q=0,1} d^{m|\mu|}(h) d_{q}^{m-\mu, t}(h') a(m, |m|+q, -\mu, \nu, p) X_{m-\mu, t}^{a(i)}(h' ; r') . \quad (47)$$

Eq. (47) for $X = M$ and $X = N$ gives the translational addition theorems for the case $r' \geq d$. Also in the limit $\alpha \to 0$, $\beta \to 0$, $\gamma \to 0$, eqs. (37) and (38) give the translational addition theorems for vector spheroidal wave functions $M_{e, o m n}^{r(i)}$ and $N_{e, o m n}^{r(i)}$, for $r' \leq d$. When $i = 3$, their expressions are identical to those in [6].

### 3.2. Addition Theorems for Vector Spherical Wave Functions

The spheroidal coordinate systems $(\xi, \eta, \phi)$ and $(\xi', \eta', \phi')$ reduce to the spherical systems $(r, \theta, \phi)$ and $(r', \theta', \phi')$, respectively, in the limit $h \to 0$ and $h' \to 0$, when the angular and radial spheroidal functions become the associated Legendre functions and spherical Bessel functions, respectively,
\[ S_{mn}(h, \eta) \rightarrow P^n_m(\cos \theta), \]
\[ S_{\mu\nu}(h', \eta') \rightarrow P^n_{\nu}(\cos \theta'), \]
\[ N_{\mu\nu}(h) \rightarrow \frac{2}{2\nu + 1} \frac{(\nu + \mu)!}{(\nu - \mu)!}, \]
\[
\{ R_{mn}^{(1)}(h, \xi), R_{mn}^{(3)}(h, \xi), R_{mn}^{(4)}(h, \xi) \} \rightarrow \{ j_n(kr), h_n^{(1)}(kr), h_n^{(2)}(kr) \},
\]
\[
\{ R_{\mu\nu}^{(1)}(h', \xi'), R_{\mu\nu}^{(3)}(h', \xi'), R_{\mu\nu}^{(4)}(h', \xi') \} \rightarrow j_{\nu}(kr'), h_{\nu}^{(1)}(kr'), h_{\nu}^{(2)}(kr').
\]

Substituting eqs. (18)-(22) in eq. (24), and rearranging, we get in the limit \( h \rightarrow 0, h' \rightarrow 0, \)
\[
\psi^{(i)}_{mn}(r, \theta, \phi) = \sum_{\nu = 0}^{\infty} \sum_{\mu = -\nu}^{\nu} (^{(i)}G^{mn}_{\mu\nu}(\alpha, \beta, \gamma; d) \psi^{(i)}_{\mu\nu}(r', \theta', \phi'), \]
where
\[
(^{(i)}G^{mn}_{\mu\nu}(\alpha, \beta, \gamma; d) = (-1)^{\mu}(2\nu + 1) j^{\nu-n} \sum_{q=0,1}^{\infty} d_q^{mn}(h) \sum_{\mu = -(|m|+q)}^{(|m|+q)} R_{\mu\nu,|m|+q}(\alpha, \beta, \gamma) \]
\[
\times \sum_{p=\rho_0, \rho_0+1}^{\infty} j^p a(\tilde{\mu}, |m| + q - \mu, \nu | p) \psi^{(i)}_{\mu-\mu, p}(d) .
\]

By expanding \( \psi^{(i)}_{\mu-\mu, p}(d) \) in double integrals [1, 14], eq. (50) becomes
\[
(^{(i)}G^{mn}_{\mu\nu}(\alpha, \beta, \gamma; d) = (-1)^{\mu}(2\nu + 1) j^{\nu-n} \sum_{q=0,1}^{\infty} d_q^{mn}(h) \sum_{\mu = -(|m|+q)}^{(|m|+q)} \sum_{p=\rho_0, \rho_0+1}^{\infty} j^p a(\tilde{\mu}, |m| + q - \mu, \nu | p)(4\pi j^p)^{-1} \]
\[
\cdot \int \int \exp(jkd \cos \gamma_{kd}) P_{p}^{\tilde{\mu}-\mu}(\cos \theta_{k}^i) e^{i(\tilde{\mu}-\mu)\phi_k} \sin \theta_k d\theta_k d\phi_k ,
\]
where
\[
\cos \gamma_{kd} = \sin \theta_k^i \sin \theta_d \cos(\phi_d - \phi_k^i) + \cos \theta_k^i \cos \theta_d ,
\]
with \( \int_{c} \) being \( \int_{0}^{\pi} \) for \( i = 1, 2 \int_{0}^{\pi/2-j\infty} \) for \( i = 3, \) and \( 2 \int_{\pi/2-j\infty}^{\pi} \) for \( i = 4. \) Taking into account the linearization expansion [1, 2] of the product \( P_{|m|+q}^{\tilde{\mu}-\mu}(\cos \theta_k^i) P_{\nu}^{-\mu}(\cos \theta_k^i), \)
eq (51) can be written in the limit \( h' \rightarrow 0 \) as
\[
(^{(i)}G^{mn}_{\mu\nu}(\alpha, \beta, \gamma; d) = (-1)^{\mu}(2\nu + 1) j^{\nu-n}(4\pi)^{-1} \]
\[
\sum_{q=0,1}^{\infty} d_q^{mn}(h) \sum_{\mu = -(|m|+q)}^{(|m|+q)} R_{\mu\nu,|m|+q}(\alpha, \beta, \gamma) a(\tilde{\mu}, |m| + q - \mu, \nu | p) \]
\[
\cdot \int \int \exp(jkd \cos \gamma_{kd}) P_{|m|+q}^{\tilde{\mu}-\mu}(\cos \theta_k^i) P_{\nu}^{-\mu}(\cos \theta_k^i) e^{i(\tilde{\mu}-\mu)\phi_k} \sin \theta_k d\theta_k d\phi_k ,
\]
Using the expansion [11],
\[ P_{|m|+q}^m(\cos \theta_k) e^{im\phi_k} = \sum_{\mu=-|m|+q}^{|m|+q} R_{\mu|n|+q}^m(\alpha, \beta, \gamma) P_{|m|+q}^\mu(\cos \theta_k) e^{i\mu\phi_k}, \tag{54} \]
yields, in the limit \( h \to 0 \),
\[ (i) G^{mn}_{\mu \nu}(\alpha, \beta, \gamma; d) = (-1)^n(2\nu + 1)j^{\nu-n}(4\pi)^{-1} \int_0^{2\pi} \int c \exp( jkd \cos \gamma_{kd}) P_n^m(\cos \theta_k) \]
\[ \cdot P_{n}^{-\mu}(\cos \theta_k') e^{im\phi_k} e^{-i\mu\phi_k'} \sin \theta_k' \, d\theta_k' \, d\phi_k'. \tag{55} \]

Applying again the expansion of \( P_n^m(\cos \theta_k) e^{im\phi_k} \) as shown in eq. (54) gives
\[ (i) G^{mn}_{\mu \nu}(\alpha, \beta, \gamma; d) = (-1)^n(2\nu + 1)j^{\nu-n} \sum_{\mu=-n}^n R_{\mu n}^{mn}(\alpha, \beta, \gamma)(4\pi)^{-1} \int_0^{2\pi} \int c \exp( jkd \cos \gamma_{kd}) \]
\[ \cdot P_{n}^{-\mu}(\cos \theta_k') P_{n}^{-\mu}(\cos \theta_k') e^{i(\mu-\mu)\phi_k} \sin \theta_k' \, d\theta_k' \, d\phi_k'. \tag{56} \]

and with the linearization expansion of \( P_{n}^{-\mu}(\cos \theta_k') P_{n}^{-\mu}(\cos \theta_k') \) and the expansion of \( \psi^{(i)}_{\mu-\mu, p}(d) \) in double integrals, we obtain finally
\[ (i) G^{mn}_{\mu \nu}(\alpha, \beta, \gamma; d) = (-1)^n(2\nu + 1)j^{\nu-n} \sum_{\mu=-n}^n R_{\mu n}^{mn}(\alpha, \beta, \gamma) \sum_{p=p_0-p_0+1}^{n+\nu} j^{p} a(\mu, n|-\mu, \nu \mid p) \cdot z^{(i)}(kd) P_{p}^{-\mu}(\cos \theta_k') e^{i(\mu-\mu)\phi_d}. \tag{57} \]

Eq. (57) gives the rotational-translational coefficients for spherical wave functions, when \( r' \leq d \).

From Fig. 1 we have
\[ r = r' + d. \tag{58} \]

Taking the cross product of \( \nabla \psi^{(i)}_{mn}(t) \) with each side of eq. (58) gives
\[ m^{r(i)}_{mn} = \nabla \psi^{(i)}_{mn} \times r' + \nabla \psi^{(i)}_{mn} \times d. \tag{59} \]

Since the gradient of a scalar function is invariant to a transformation of the coordinate system, using eq. (49) we can write
\[ \nabla \psi^{(i)}_{mn} \times r' = \sum_{\nu=0}^{n} \sum_{\mu=-\nu}^{\nu} (i) G^{mn}_{\mu \nu} m^{r(i)}_{\mu \nu}. \tag{60} \]

Also
\[ \nabla \psi^{(i)}_{mn} \times d = d_x m^{r(i)}_{mn} + d_y m^{r(i)}_{mn} + d_z m^{r(i)}_{mn}. \tag{61} \]
The vector spherical wave functions $m^{(i)}_{mn} (a = x, y, z)$ can be expressed in terms of the vector spherical wave functions $m^{r(1)}_{mn}$ and $n^{r(1)}_{mn}$ in the form [3]

$$m^{(i)}_{mn} = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (e^{r}_{\mu\nu} m^{r(1)}_{\mu\nu} + g^{r}_{\mu\nu} n^{r(1)}_{\mu\nu}),$$

$$m^{\gamma(i)}_{mn} = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (e^{\gamma}_{\mu\nu} m^{r(1)}_{\mu\nu} + g^{\gamma}_{\mu\nu} n^{r(1)}_{\mu\nu}),$$

$$m^{\delta(i)}_{mn} = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (e^{\delta}_{\mu\nu} m^{r(1)}_{\mu\nu} + g^{\delta}_{\mu\nu} n^{r(1)}_{\mu\nu}).$$

$e^{r}_{\mu\nu}, g^{r}_{\mu\nu}, \ldots$ have the same form as those of $a^{r}_{\mu\nu\bar{\nu}q}, b^{r}_{\mu\nu\bar{\nu}q}, \ldots$, respectively, with $B^{mn}_{\mu\nu\bar{\nu}q}$ replaced by $(i)G^{mn}_{\mu\nu}$, and $\pm$ in $a^{\gamma}_{\mu\nu\bar{\nu}q}$ and $b^{\gamma}_{\mu\nu\bar{\nu}q}$, and $\mp$ in $b^{\delta}_{\mu\nu\bar{\nu}q}$ by $-j$ and $+j$, respectively.

Finally eq. (59) can be expressed in the form

$$m^{r(i)}_{mn} = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (A^{mn}_{\mu\nu} m^{r(1)}_{\mu\nu} + B^{mn}_{\mu\nu} n^{r(1)}_{\mu\nu}), \quad r' \leq d; \quad i = 1, 2, 3, 4,$$

in which

$$A^{mn}_{\mu\nu} = A^{(i)}_{\mu\nu} + (i)G^{mn}_{\mu\nu},$$

where

$$A^{(i)}_{\mu\nu} = d_{x} e^{(i)}_{\mu\nu} + d_{y} e^{(i)}_{\mu\nu} + d_{z} e^{(i)}_{\mu\nu},$$

$$B^{mn}_{\mu\nu} = d_{x} g^{mn}_{\mu\nu} + d_{y} g^{mn}_{\mu\nu} + d_{z} g^{mn}_{\mu\nu},$$

with the $x, y, z$ components of $d$ given by

$$d_{x} = d(\sin \theta_{d} \cos \phi_{d} c_{xx'}, + \sin \theta_{d} \sin \phi_{d} c_{xy'}, + \cos \theta_{d} c_{xz'}),$$

$$d_{y} = d(\sin \theta_{d} \cos \phi_{d} c_{yx'}, + \sin \theta_{d} \sin \phi_{d} c_{yy'}, + \cos \theta_{d} c_{yz'}),$$

$$d_{z} = d(\sin \theta_{d} \cos \phi_{d} c_{zx'}, + \sin \theta_{d} \sin \phi_{d} c_{zy'}, + \cos \theta_{d} c_{zz'}).$$

Eq. (63) gives the rotational-translational addition theorems for the vector spherical wave functions $m^{r(i)}_{mn}$, when $r' \leq d$. The expressions corresponding to the vector spherical wave functions $n^{r(i)}_{mn}$ [2, 3] have the same form, with $m^{r(1)}_{mn}$ and $n^{r(1)}_{mn}$ in eq. (63) replaced by $n^{r(1)}_{\mu\nu}$ and $m^{r(1)}_{\mu\nu}$, respectively. It is also possible to obtain these theorems starting directly from the corresponding theorems for vector spheroidal wave functions, given in eqs. (37) and (38). However this derivation is more elaborate than that presented above.

The translational addition theorems for vector spherical wave functions $m^{r(i)}_{mn}$ are obtained from eq. (63) as a special case, when $\alpha \to 0, \beta \to 0, \gamma \to 0$. Now, $c_{ax'}, c_{ay'}, c_{az'} (a = x, y, z)$ are all zero except $c_{xx'}, c_{yy'},$ and $c_{zz'}$, which are equal to unity. In this case eq. (57) reduces to
\[(i) G_{\mu \nu}^{mn} = (-1)^{\mu}(2\nu + 1) j^{\nu-n} \times \sum_{p=p_0, p_0+1}^{n+\nu} j^p a(m, n|\mu, \nu|p) z_p^{(i)}(kd) P_{\mu p}^{m-\mu}(\cos \theta_d) e^{i(m-\mu)\phi_d}, \quad (68)\]

and eq. (63) becomes identical to Theorem I, given in [3]. Similarly, for the vector spherical wave functions \(n_{mn}^{r(i)}\) we obtain Theorem II in [3].

4. CONCLUSIONS

Rotational-translational addition theorems for vector spheroidal wave functions \(M^{a(i)} \text{ and } N^{a(i)}\) \((a = x, y, z; i = 1, 2, 3, 4)\), as well as for vector spheroidal wave functions \(M^{x(i)}\text{ and } N^{x(i)}\) \((i = 1, 2, 3, 4)\) have been derived. Translational addition theorems for these vector spheroidal wave functions have been deduced as special cases. Even though translational addition theorems and rotational addition theorems for vector spherical wave functions already exist in the literature [2, 3], they cannot be simply combined to obtain rotational-translational addition theorems for vector spherical wave functions. In this paper, new rotational-translational addition theorems for vector spherical wave functions \(m_{mn}^{r(i)}\text{ and } n_{mn}^{r(i)}\) have also been obtained as special cases.

The rotational-translational addition theorems for vector spheroidal wave functions presented in this paper have been used by the authors to obtain an analytic solution to the problem of electromagnetic field scattering by two spheroids of arbitrary orientation [15]. By selecting appropriate vector wave functions, a unique system matrix can be used for calculating backscattering cross sections, independently of the angle of incidence. Results of a prescribed accuracy, corresponding to the whole range of angles of incidence, are therefore calculated with a better computational efficiency as compared with those obtained by various numerical techniques, for instance, moment methods where the problem has to be solved for each angle of incidence separately.

APPENDIX

Rotational-Translational Coefficients

The rotational-translational coefficients \((^{(i)}Q_{\mu \nu}^{mn}(\alpha, \beta, \gamma; d)\text{ and } P_{\mu \nu}^{mn}(\alpha, \beta, \gamma; d)\) given in eqs. (1) and (2) are those obtained in [11]:

\[^{(i)}Q_{\mu \nu}^{mn}(\alpha, \beta, \gamma; d) = \sum_{s=|m|, |m|+1}^{\infty} d_{s-|m|}(h) \sum_{s=-s}^{s} R_{\mu \nu}^{ms}(\alpha, \beta, \gamma) \times \sum_{l=|\mu|, |\mu|+1}^{\infty} a_{\mu \nu}^{s}(d) j^{s-n+\nu-l} \frac{N_{\mu l}(h)}{N_{\mu \nu}(h')} d_{l-|\mu|}(h'), \quad (A.1)\]
\[ P_{\mu \nu}^{mn}(\alpha, \beta, \gamma; d) = \sum_{s=|m|, |m|+1}^{\infty} d_{s-|m|}^m(h) \sum_{\tilde{\mu} = -s}^{s} R_{\tilde{\mu} s}^{m} (\alpha, \beta, \gamma) \cdot \sum_{p = |\mu|, |\mu|+1}^{\infty} b_{\mu p}^{\tilde{\mu} s}(d) j^{s-n+v-p} \frac{N_{\mu p}}{N_{\mu v}(h')} d_{p-|\mu|}^{\mu v}(h'), \]  

in which \( d_{q}^m(h) \), \( d_{q}^{\mu v}(h') \) are the spheroidal expansion coefficients, and \( N_{\mu v}(h') \) is the normalization constant \([12]\). In these expressions the following notation is used:

\[ N_{ml} = \frac{2}{(2l+1)} \frac{(l+m)!}{(l-m)!}, \]  

\[ R_{ml}^m(\alpha, \beta, \gamma) = (-1)^{m'-m} \left[ \frac{N_{ml}}{N_{m'l}} \right]^{1/2} e^{i m \alpha} \varphi_{m'm}^{(l)}(\beta) e^{i m \beta}, \]  

\[ a_{m'm}^{(l)}(\beta) = \left[ \frac{(l+m')!(l-m')!}{(l+m)!(l-m)!} \right]^{1/2} \left( \cos \frac{\beta}{2} \right)^{m'+m} \left( \sin \frac{\beta}{2} \right)^{m-m'} \cdot P_{l-m'}^{(m'-m,m'+m)}(\cos \beta), \]  

with \( P_{l-m'}^{(m'-m,m'+m)}(\cos \beta) \) being the Jacobi polynomial of argument \( \cos \beta \);

\[ a^{\tilde{\mu} s}_{\mu l}(d) = (-1)^{\mu} \sum_{p = p_0, p_0 + 1}^{l+s} j^{l+p-s}(2l+1)a(\tilde{\mu}, s|\mu, l|p) \psi_{\mu - \mu, p}^{(l)}(d), \]  

in which \( a(\tilde{\mu}, s|\mu, l|p) \) are the linearization expansion coefficients \([1, 2]\), the first term in the series being \( p_0 = \max(|l-s|, |\tilde{\mu} - \mu|) \) or \( p_0 + 1 \) so that its last term is \( l+s \), and

\[ \psi_{\mu - \mu, p}^{(l)}(d) = z_{p}^{(l)}(kd) P_{p}^{\mu - \mu}(\cos \theta_d) e^{i(\mu - \mu) \phi_d}, \]  

where \( z_{p}^{(i)}, i = 1, 2, 3, 4, \) are the spherical Bessel functions \( j_{p}, n_{p}, h_{p}^{(1)}, \) and \( h_{p}^{(2)}, \) respectively, and \( P_{p}^{\mu - \mu} \) is the Legendre function of the first kind;

\[ b_{\mu p}^{\tilde{\mu} s}(d) = \sum_{l = l_0, l_0 + 1}^{p+s} b_{\tilde{\mu} - \mu, l, p}^{\tilde{\mu} s}(d), \]  

\[ b_{\tilde{\mu} - \mu, l, p}^{\tilde{\mu} s}(d) = (-1)^{\mu - \mu} j^{l+p-s}(2l+1)a(\tilde{\mu}, s|\mu - \tilde{\mu}, l|p) \psi_{\mu - \mu, l}^{(1)}(d), \]  

with \( l_0 = \max(|p-s|, |\tilde{\mu} - \mu|) \).

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