Scattering of Electromagnetic Waves by a System of Two Conducting Spheroids of Arbitrary Orientation

M. FRANCIS R. COORAY and IOAN R. CIRIC, SENIOR MEMBER, IEEE

Abstract—An exact solution to the problem of scattering of a plane electromagnetic wave by two perfectly conducting prolate spheroids arbitrarily oriented is obtained by expanding the incident and scattered electric fields in terms of an appropriate set of vector spheroidal eigenfunctions. The incident wave is considered to be a monochromatic, uniform plane electromagnetic wave of arbitrary polarization and angle of incidence. In order to impose the boundary conditions, the field scattered by one spheroid is expressed in terms of the spheroidal coordinates attached to the other spheroid, by using the rotational-translational addition theorems for vector spheroidal wave functions. The column matrix of the scattered field expansion coefficients is equal to the product of a square matrix which is independent of the direction and polarization of the incident wave, and the column matrix of the known incident field expansion coefficients. The unknown scattered field expansion coefficients are obtained by solving the associated set of simultaneous linear equations. Numerical results for the bistatic and backscattering cross sections for prolate spheroids with various axial ratios and orientations are presented.

I. INTRODUCTION

The analysis of electromagnetic wave scattering by spheroids has been of increasing interest during the past few decades, due to the possibility of applying exact analytical methods and to the fact that a large number of real system objects can be modeled by spheroids with appropriate axial ratios.

An exact solution to the problem of scattering of a plane electromagnetic wave by a system of two spheres was obtained by Brumming and Lo [1], using the translational addition theorems for vector spherical wave functions given by Cruzan [2], which is essentially an extension of the formulation previously presented by Stein [3]. Exact solutions for scattering of a plane electromagnetic wave by two perfectly conducting prolate spheroids with parallel major axes were developed by Sinha and MacPhie [4], and by Dalmas and Deleuil [5]. The problem was solved by using different types of vector spheroidal wave functions and the corresponding translational addition theorems [6], [7].

This paper presents an exact solution to the problem of scattering of a plane electromagnetic wave by a system of two perfectly conducting prolate spheroids, with arbitrary orientation. This solution has been obtained by applying rotational-translational addition theorems for vector spheroidal wave functions derived recently by the authors [8], on the basis of the theorems for scalar spheroidal wave functions [9]. As in the case of two spheroids with parallel major axes, the solution is given in the form \( S = [G]I \), where \( S \) is the column matrix of the unknown coefficients in the total scattered field expansion, \( I \) is the column matrix of the known coefficients in the incident field expansion, and \( [G] \) is the system matrix, whose elements depend only on the scattering system geometry and the frequency of the incident field. The solution for oblate spheroids can be obtained from that for prolate spheroids by using the transformation \( \xi - j \xi \) and \( h = -j h \) (or \( F = -jF \)), where \( \xi \) is the radial spheroidal coordinate, \( h = kF \), with \( F \) being the semi-interfocal distance and \( k \) the wavenumber.

II. INCIDENT AND SCATTERED FIELD EXPANSIONS IN TERMS OF VECTOR SPHEROIDAL WAVE FUNCTIONS

Consider two prolate spheroids \( A \) and \( B \) as shown in Fig. 1. Unprimed coordinates refer to the spheroid \( A \) and primed coordinates refer to the spheroid \( B \). The major axis of \( A \) is along the \( z \) axis of the Cartesian system \( Oxyz \) and that of \( B \) is along the \( z' \) axis of \( O'x'y'z' \). The system \( Ox'y'z' \) is parallel to \( Ox'y'z \), and is rotated with respect to \( Oxyz \) through the Euler angles \( \alpha, \beta, \gamma \) as defined in [12]. The center \( O' \) of \( B \) has spherical coordinates \( d, \theta_d, \phi_d \) with respect to \( Ox'y'z' \) and \( d, \theta_0, \phi_0 \) with respect to \( Oxyz \). A linearly polarized, monochromatic plane electromagnetic wave, with an electric field of unit amplitude, is incident at an angle \( \theta_i \) with respect to the major axis of \( A \), the plane of incidence being the \( x-z \) plane (\( \phi_i = 0 \)), as shown in Fig. 1. The polarization angle \( \gamma_k \) is the angle between the direction of the incident electric field intensity vector and the direction of the normal to the plane of incidence. \( \gamma_k \) is zero for transverse electric (TE) polarization and \( \pi/2 \) for transverse magnetic (TM) polarization.

The incident electric field in the unprimed coordinate system \( E_{iA} \) can be expanded in a series of prolate spheroidal vector wave functions and written in matrix form as [4]

\[
E_{iA} = \mathbf{M}_{iA}^{(1)} \mathbf{T} \mathbf{I}_A
\]

with the overbar denoting a column matrix and \( \mathbf{T} \) the transpose of a matrix; the elements of \( \mathbf{M}_{iA}^{(1)} \) and \( \mathbf{I}_A \) are prolate spheroidal vector wave functions of the first kind, expressed in terms of the unprimed spheroidal coordinates \( \xi, \eta, \phi \), and the corresponding known expansion coefficients, respectively, as defined in the Appendix I.

The electromagnetic field scattered by the spheroid \( B \) represents a nonplane wave whose electric field intensity \( E_{sB} \) can...
spheroidal vector wave functions of the fourth kind, expressed in terms of unprimed spheroidal coordinates, and the corresponding unknown expansion coefficients, respectively, which are defined in Appendix I.

The total electric field seen from the spheroid \( A \) can be written as

\[
E_A = E_{IA} + E_{IB} + E_{sA}
\]

\[
= \tilde{M}^{(1)T}_{IA} \bar{I}_A + \tilde{M}^{(1)T}_{IB} [\Gamma']^T \tilde{\beta} + \tilde{M}^{(4)T}_{sA} \tilde{\alpha}.
\]

Similarly, the total electric field seen from the spheroid \( B \) can be expressed in terms of appropriate prolate spheroidal vector wave functions in the primed coordinate system as

\[
E_B = E_{IB} + E_{LAB} + E_{sB}
\]

\[
= \tilde{M}^{(1)T}_{IB} \bar{I}_B + \tilde{M}^{(1)T}_{LAB} [\Gamma']^T \tilde{\alpha} + \tilde{M}^{(4)T}_{sB} \tilde{\beta}
\]

where \( \tilde{M}^{(1)}_{IB}, \tilde{I}_B, \tilde{M}^{(1)}_{LAB}, \) and [\( \Gamma' \)] are given in Appendix I.

III. Boundary Conditions

The tangential components (\( \eta \) and \( \phi \)) of the total electric field intensity must be equal to zero on the surface of each perfectly conducting spheroid \( \xi = \xi_A \) and \( \xi = \xi_B \). Thus from (6) and (7) we have

\[
(M^{(1)T}_{IA} \bar{I}_A + M^{(1)T}_{IB} [\Gamma']^T \tilde{\beta} + M^{(4)T}_{sA} \tilde{\alpha}) \times \xi = 0 (8)
\]

\[
(M^{(1)T}_{IB} \bar{I}_B + M^{(1)T}_{LAB} [\Gamma']^T \tilde{\alpha} + M^{(4)T}_{sB} \tilde{\beta}) \times \xi = 0. (9)
\]

Taking the scalar product of both sides of (8) and (9) by

\[
\left\{ \begin{array}{c}
\tilde{\eta} e^{i\phi} \\
\tilde{\phi} e^{i\phi}
\end{array} \right\} S_{m,|m|+N(h, \eta)} e^{i(m+1)\phi}
\]

and

\[
\left\{ \begin{array}{c}
\tilde{\eta} e^{i\phi} \\
\tilde{\phi} e^{i\phi}
\end{array} \right\} S_{m,|m|+N(h, \eta')} e^{i(m+1)\phi'},
\]

\( \alpha \) \( \text{and} \) \( \beta \) are column matrices whose elements are prolate spheroidal vector wave functions of the fourth kind, expressed in terms of incoming vector wave functions in unprimed coordinates \( M^{(4)}_{sA} \) (see the Appendix I) in the form

\[
M^{(4)}_{sA} = [\Gamma] M^{(1)}_{sA}
\]

\( \text{where} \) the elements of the matrix [\( \Gamma \)] are the rotational-translational coefficients defined in the Appendices I and II. If we denote this secondary incident field by \( E_{sA} \), taking the transpose of both sides of (3) and then substituting \( M^{(4)}_{sA} \) in (2) gives

\[
E_{sA} = \tilde{M}^{(1)T}_{sA} [\Gamma']^T \tilde{\beta}.
\]

The two electric fields \( E_{IA} \) and \( E_{IB} \), which are incident on the spheroid \( A \), determine an electric field \( E_{IA} \), scattered by \( A \), which can also be expanded in a series of prolate spheroidal vector wave functions as

\[
E_{IA} = \tilde{M}^{(4)T}_{IA} \tilde{\alpha}.
\]

\( \tilde{M}^{(4)}_{IA} \) and \( \tilde{\alpha} \) are column matrices whose elements are prolate spheroidal vector wave functions of the fourth kind, expressed in terms of unprimed spheroidal coordinates, and the corresponding unknown expansion coefficients, respectively, which are defined in Appendix I.
of the scattering system and the frequency of the incident wave.

The matrix form (13) gives the coefficients in the expressions (5) and (2) of the electric fields scattered by the two spheroids. Once these coefficients are calculated, the resultant electric field at any point is determined

$$E = E_t + E_{xA} + E_{yB}.$$  \hspace{1cm} (16)

IV. NORMALIZED SCATTERING CROSS SECTIONS

Let the point of observation have spherical coordinates $r, \theta, \phi$ and $r’, \theta’, \phi’$ with respect to the two systems $Oxyz$ and $O’x’y’z’$, respectively. The scattering cross sections are calculated in the far zone ($r \to \infty, r’ \to \infty$), where

$$\lim_{r \to \infty} h \xi \to kr, \quad \lim_{r \to \infty} \eta \to \cos \theta, \quad \lim_{r \to \infty} \hat{\eta} \to -\hat{\theta}$$

$$\lim_{r' \to \infty} h' \xi' \to kr', \quad \lim_{r' \to \infty} \eta' \to \cos \theta’, \quad \lim_{r' \to \infty} \hat{\eta}' \to -\hat{\theta}’. \hspace{1cm} (17)$$

Using these asymptotic values, we have the following asymptotic expressions [11]:

$$lim_{r \to \infty} R_{mn}^{(4)}(h, \xi) \to j^{n+1} \frac{e^{-jk r}}{kr},$$

$$lim_{r \to \infty} \frac{d}{dh} R_{mn}^{(4)}(h, \xi) \to j^{n+1} \frac{e^{-jk r}}{kr} e^{i k \nu_{idd}}, \hspace{1cm} (18)$$

$$lim_{r' \to \infty} R_{mn}^{(4)}(h’, \xi’) \to j^{n+1} \frac{e^{-jk r’}}{kr’} e^{i k \nu_{idd}},$$

$$lim_{r' \to \infty} \frac{d}{dh'} R_{mn}^{(4)}(h’, \xi’) \to j^{n+1} \frac{e^{-jk r’}}{kr’} e^{i k \nu_{idd}} \hspace{1cm} (19)$$

where $F$ and $F’$ are the semi-interfocal distances of spheroids $A$ and $B$, respectively, and

$$k_{s} = k(\sin \theta \cos \phi + \sin \phi \cos \theta). \hspace{1cm} (20)$$

The asymptotic forms of various vector spheroidal wave functions are obtained on the basis of the expressions in (17)–(20), and, subsequently, the scattered electric field intensity in the far zone can be written as

$$E_t = E_{xA} + E_{yB}$$

$$= \frac{e^{-jk r}}{kr} [F_{xA}(\theta, \phi)\hat{\theta} + F_{xA}(\theta, \phi)\hat{\phi}]$$

$$+ F_{yB}(\theta’, \phi’)[g_{1} \hat{\theta} + g_{2} \hat{\phi}]$$

$$+ F_{yB}(\theta’, \phi’)[g_{3} \hat{\theta} + g_{4} \hat{\phi}]$$

$$= \frac{e^{-jk r}}{kr} [F_{\theta}(\theta, \phi)\hat{\theta} + F_{\phi}(\theta, \phi)\hat{\phi}]. \hspace{1cm} (21)$$

where

$$F_{\theta}(\theta, \phi) = F_{\theta A}(\theta, \phi) + F_{\theta B}(\theta, \phi)$$

$$F_{\phi}(\theta, \phi) = F_{\phi A}(\theta, \phi) + F_{\phi B}(\theta, \phi)$$

$$[g_{1}, g_{2}]^T = [\Omega]{[C]}[\cos \theta’ \cos \phi’ \cos \theta’ \sin \phi’ - \sin \theta’]^T \hspace{1cm} (22)$$

$$[g_{3}, g_{4}]^T = [\Omega]{[C]}[-\sin \phi’ \cos \phi’ 0]^T \hspace{1cm} (23)$$

with

$$F_{\theta A}(\theta, \phi) = - \sum_{m=0}^{\infty} \sum_{n=-m}^{m} \frac{j^{n+1}}{2} \frac{S_{mn}}{\alpha_{mn} - \alpha_{mn}^*} \sin (m+1)\phi$$

$$\cdot \{(\alpha_{mn}^* + \alpha_{mn}^*) \cos (m+1)\phi + j(\alpha_{mn}^* - \alpha_{mn}) \sin (m+1)\phi \}$$

$$- \sum_{n=1}^{\infty} \frac{j^{n+1}}{2} \frac{S_{mn}}{\alpha_{mn} - \alpha_{mn}^*} \sin (m+1)\phi$$

$$\cdot \{(\alpha_{mn}^* + \alpha_{mn}^*) \cos (m+1)\phi + j(\alpha_{mn}^* - \alpha_{mn}) \sin (m+1)\phi \}$$

$$+ j(\alpha_{mn}^* - \alpha_{mn}^*) \sin (m+1)\phi \}$$

$$\cdot \{\cos \theta \cos \phi \cos \theta \sin \phi - \sin \theta \} \hspace{1cm} (24)$$

$$\{0\} = \begin{bmatrix} \cos \theta \cos \phi \cos \theta \sin \phi - \sin \theta \\ - \sin \phi \cos \phi 0 \end{bmatrix}, \hspace{1cm} \begin{bmatrix} S_{mn} \end{bmatrix} = \begin{bmatrix} C_{xx} & C_{xy} & C_{xz} \\ C_{yx} & C_{yy} & C_{yz} \\ C_{zx} & C_{zy} & C_{zz} \end{bmatrix} \hspace{1cm} (25)$$

in which

$$\alpha_{mn}^* = k\alpha_{mn}^*, \quad \alpha_{mn}^* = k\alpha_{mn}^*, \quad S_{mn} = S_{mn}(h, \cos \theta). \hspace{1cm} (26)$$

The expressions of $F_{\theta B}(\theta’, \phi’)$ and $F_{\phi B}(\theta’, \phi’)$ in primed coordinates are obtained from those of $F_{\theta A}(\theta, \phi)$ and $F_{\phi A}(\theta, \phi)$, respectively, by replacing $\alpha$ by $\beta$, and multiplying each expression by an overall phase factor $\exp (j k_s \cdot d)$. The expressions of $F_{\theta B}(\theta, \phi)$ and $F_{\phi B}(\theta, \phi)$ are obtained from those of $[g_{1}, F_{\theta A}(\theta', \phi') + g_{2} F_{\phi A}(\theta', \phi')]$ and $[g_{3}, F_{\theta B}(\theta', \phi') + g_{4} F_{\phi B}(\theta', \phi')]$, respectively, by substituting all the functions in primed variables $\theta’, \phi’$ in terms of the unprimed variables $\theta, \phi$. Since the direction of the scattered wave vector $k$, in the far field with respect to the primed system is specified by the
angular spherical coordinates \((\theta', \phi')\) (see Fig. 1), we have
\[ k_r = k(x' \sin \theta' \cos \phi' + y' \sin \theta' \sin \phi' + z' \cos \theta'). \] (29)
Substituting \(x', y', z'\) in (20) in terms of \(x', y', z'\) (see Appendix I), and identifying the corresponding coefficients of \(x', y', z'\) with those in (29) gives
\[
\begin{align*}
[\sin \theta' \cos \phi' & \sin \theta' \sin \phi' \cos \theta']^T \\
= [C]^T [\sin \theta \cos \phi \sin \theta \sin \phi \cos \theta]^T
\end{align*}
\] (30)
which is the required relationship between the primed and unprimed angular coordinates.

The bistatic radar cross section is defined as
\[ \sigma(\theta, \phi) = \lim_{r \to \infty} 4\pi^2 \left| \frac{E_r \cdot \ell}{|E_r|^2} \right|^2 \] (31)
with the unit vector \(\ell\) denoting the direction of polarization of the receiver at the point of observation. When \(\ell\) has the same direction as \(E_r\), the normalized bistatic cross section is
\[ \frac{\pi \sigma(\theta, \phi)}{\lambda^2} = |F_\phi(\theta, \phi)|^2 + |F_\theta(\theta, \phi)|^2. \] (32)
The normalized bistatic cross sections in the \(E-\) and \(H-\)planes are obtained by substituting \(\phi = \pi/2\) and \(\phi = 0\), respectively, in (32).

The normalized backscattering cross section is obtained from (32) for \(\theta = \theta_1\) and \(\phi = \phi_1 = 0\)
\[ \frac{\pi \sigma(\theta_1)}{\lambda^2} = |F_\theta(\theta_1, 0)|^2 + |F_\phi(\theta_1, 0)|^2. \] (33)

V. Numerical Results

Results of numerical computation are presented for the bistatic and backscattering cross sections in the far field. Prolate spheroids of axial ratios 2 and 10 have been chosen to observe the effects of fat and thin spheroids on the scattering cross sections, for various displacements of their centers, and for relative orientations corresponding to the Euler angles \(\alpha = 0^\circ, \beta = 90^\circ, \gamma = 0^\circ\) and \(\alpha = 45^\circ, \beta = 90^\circ, \gamma = 45^\circ\). The backscattering cross sections calculated for two prolate spheroids with very small values of the three Euler angles have been found in very good agreement with those calculated by considering the two spheroids as being parallel.

The series expansions of the incident and scattered electric fields in terms of vector spheroidal wave functions are infinite in extent. Thus all the matrices introduced in Sections II and III are infinite in size (see Appendixes I and III). In order to obtain numerical results, it is necessary to truncate all series and corresponding matrices according to the required accuracy. By performing systematically numerical experiments on the expressions (80)–(82) of the rotational—translational addition theorems for vector spheroidal wave functions (see Appendix I), we have found that it is sufficient to take \(-2 \leq \mu \leq 2\) and \(v = |m|, |m| + 1, \ldots, |m| + 5\) in the double summations on the right-hand sides in order to obtain an accuracy of at least two significant digits, when compared with the values of the corresponding left-hand sides of the equations, for various values of \(m\) and \(n\). All the vector spheroidal wave functions and the rotational—translational coefficients have been evaluated with an accuracy of five significant digits. Therefore, in order to ensure a two significant digit accuracy in the computed bistatic and backscattering cross sections, it is sufficient to consider only the \(\phi\) harmonics \(e^{j\alpha}, e^{j\beta}, e^{j\gamma}\), and \(e^{j\phi}\). All the results given in this paper are obtained with \(\alpha\) corresponding to the \(\phi\) harmonics \(e^{j\alpha}, e^{j\beta},\) and \(e^{j\gamma}\). With \(n = |m|, |m| + 1, \ldots, |m| + 5\), and \(N = 0, 1, 2, 3, 4\), in truncating the matrices \([Q_A], [Q_B], [R_A],\) and \([R_B]\), and with \(n = |m|, |m| + 1, \ldots, |m| + 5\), and \(N = 0, 1, 2, 3, 4\), in truncating the matrices \([T_A], [T_B], [R_{AB}],\) and \([R_{BA}]\); the matrix elements in (113) and (114) (see Appendix III) are all evaluated with an accuracy of five significant digits.

Fig. 2 shows the normalized bistatic cross section for TE polarization of the incident wave, as a function of the scattering angle for two identical prolate spheroids and for two axial ratios, with \(a_a = a_g = \lambda/4\), Euler angles \(\alpha = 0^\circ, \beta = 90^\circ, \gamma = 0^\circ\) and displaced along the \(z\) axis. (a) \(d = \lambda/2\). (b) \(d = \lambda\).
specification by the Euler angles spheroids as in Fig. 2, but with a different orientation, significant cross section with the angle of incidence, for both TE and TM polarizations is increased from \( d = A/2 \). (b) \( d = \lambda \).

Fig. 3. Normalized backscattering cross section as a function of \( \theta_i \), for two identical prolate spheroids and for two axial ratios, with \( a_A = a_B = \lambda/4 \), Euler angles \( \alpha = 0^\circ, \beta = 90^\circ, \gamma = 0^\circ \) and displaced along the \( z \) axis. (a) \( d = \lambda/2 \). (b) \( d = \lambda \).

Fig. 4. Normalized backscattering cross section versus \( \theta_i \), for two identical prolate spheroids and for two axial ratios, with \( a_A = a_B = \lambda/4 \), Euler angles \( \alpha = 45^\circ, \beta = 90^\circ, \gamma = 45^\circ \) and displaced along the \( z \) axis. (a) \( d = \lambda/2 \). (b) \( d = \lambda \).

Fig. 4 gives the variation of the normalized backscattering cross section as a function of \( \theta_i \), for two identical prolate spheroids and for two axial ratios, with \( a_A = a_B = \lambda/4 \), Euler angles \( \alpha = 45^\circ, \beta = 90^\circ, \gamma = 45^\circ \) and displaced along the \( z \) axis. (a) \( d = \lambda/2 \). (b) \( d = \lambda \).

Fig. 5 shows the behavior of the backscattering cross section for TM polarization is almost the same in both cases. However this is not so for TE polarization.

If the centers of the two spheroids are displaced by \( \lambda/2 \) in a direction perpendicular to the \( z \) axis of the spheroid \( A \), with the same Euler angles as for the previous configuration, the plots of the normalized backscattering cross section versus angle of incidence are shown in Fig. 5. When the axial ratio of the spheroids \( a/b \) is 2, the curve for TE polarization shows more oscillations than that for \( a/b = 10 \). The minima for TM polarization occur near \( \theta_i = 30^\circ \) and \( \theta_i = 150^\circ \), the lower minimum being for \( a/b = 10 \).

In Fig. 6, the variation of the backscattering cross section versus \( \theta_i \), for two identical prolate spheroids and for two axial ratios 2 and 10, having the same Euler angles as in Figs. 4 and 5. The center \( O' \) of the spheroid \( B \) has spherical coordinates \( d = \lambda/2 \), \( \theta_0 = 60^\circ \), and \( \phi_0 = 20^\circ \) with respect to the system \( Oxyz \) attached to the spheroid \( A \).
The incident electric field in the unprimed coordinate system is expressed as a sum of the incident field outside the spheroidal region and the fields inside the spheroidal region. The solution inside the spheroidal region is given by the spheroidal wave functions, which are solutions to the Helmholtz equation in the prolate spheroidal coordinates. The scattered field inside the spheroidal region is given by the superposition of the scattered fields from each spheroid, which are calculated using the scattering coefficients and the spheroidal wave functions.

VI. Conclusion

On the basis of the rotational-translational addition theorems for vector spheroidal wave functions recently derived [8], an exact solution to the problem of scattering of electromagnetic waves by a system of two spheroids of arbitrary orientation has been obtained for the first time. The exact boundary conditions are imposed by expanding the resultant field seen from a system of coordinates attached to each spheroid in terms of an appropriate set of vector spheroidal eigenfunctions. Numerical results are presented for prolate spheroids having axial ratios 2 and 10, rotated with respect to each other, in various configurations. Results obtained by the exact method developed in this paper are also important for evaluating the accuracy of the approximate methods which can be applied for the analysis of electromagnetic scattering by similar configuration systems. The formulation presented for perfectly conducting spheroids can be extended to dielectric spheroids and also to excitations which are not plane waves.

APPENDIX I

Incident and Scattered Electric Field

Expansions in Matrix Notation

The following prolate spheroidal vector wave functions are used [11], [4]:

\[
M_{mn}^{\pm(i)} = \frac{1}{2} \left[ M_{mn}^{(i)} \pm j M_{mn}^{*(i)} \right]
\]

\[
M_{mn}^{(i)} = \nabla \Psi_{mn}^{(i)} \times \hat{a}, \quad \hat{a} = \hat{x}, \hat{y}, \hat{z}
\]

\[
\Psi_{mn}^{(i)} = R_{mn}^{(i)}(h, \xi) S_{mn}(h, \eta) e^{j m \phi}, \quad i = 1, 2, 3, 4
\]

where \( R_{mn}^{(i)}(h, \xi) \) and \( S_{mn}(h, \eta) \) are the spheroidal radial function of the \( i \)th kind and the spheroidal angle function, respectively.

The incident electric field in the unprimed coordinate system (see Fig. 1) \( E_{iA} \) can be expanded in the form [11], [4]

\[
E_{iA} = \sum_{m = -\infty}^{\infty} \sum_{n = |m|}^{\infty} \left( p_{mn}^{+} M_{mn}^{+(i)} + p_{mn}^{-} M_{mn}^{-(i)} \right)
\]

in which

\[
p_{mn}^{\pm} = \frac{2}{kN_{mn}(h)} \frac{1}{\cos \theta_i} \left( \frac{\cos \gamma_k}{\cos \theta_i} \mp j \sin \gamma_k \right)
\]
with $N_{mn}(h)$, $\gamma$ and $\theta_i$ being the normalization constant of the spheroidal angle function, the polarization angle, and the incident wave angle (see Fig. 1), respectively. The expression of $E_{id}$ when $\gamma = 0$ and $\theta_i = \pi/2$ is given in [10]. If the terms in the expansion of $E_{id}$ are arranged in the $\phi$ sequence $e^{i\phi}$, $e^{i2\phi}$, ..., then it can be written in matrix form as given in (1), in which

$$\mathbf{M}_{id}(\hat{\mathbf{r}}) = \begin{bmatrix} \tilde{M}_{10} \\ \tilde{M}_{11} \\ \vdots \\ \tilde{M}_{12} \end{bmatrix}, \quad \mathbf{\tilde{I}}_A = \begin{bmatrix} \tilde{p}_0 \\ \tilde{p}_1 \\ \vdots \\ \tilde{p}_2 \end{bmatrix}$$

(39)

where

$$\mathbf{M}_{\phi}^T = [\tilde{M}_{\phi-1}^+(1)T \tilde{M}_{\phi}^-(1)T]$$

$$\mathbf{M}_{\phi}^T = [\tilde{M}_{\phi+1}^+(1)T \tilde{M}_{\phi}^-(1)T \tilde{M}_{\phi+1}^+(1)T \tilde{M}_{\phi+1}^-(1)T], \quad \sigma \geq 1$$

(40)

with

$$\mathbf{M}_{\phi}^T = [\tilde{M}_{\phi0}^+(1)T \tilde{M}_{\phi1}^+(1)T \tilde{M}_{\phi2}^+(1)T \cdots]$$

(41)

and

$$\mathbf{\tilde{p}}_0^T = [\tilde{p}_{-1}^+ T \tilde{p}_{-1}^- T]$$

$$\mathbf{\tilde{p}}_\phi^T = [\tilde{p}_\phi^+ T \tilde{p}_\phi^- T], \quad \sigma \geq 1$$

(42)

with

$$\mathbf{\tilde{p}}_\phi^T = [\tilde{p}_{\phi0}^+ T \tilde{p}_{\phi1}^+ T \tilde{p}_{\phi2}^+ T \cdots].$$

(43)

Let us consider now the incident electric field in the primed coordinate system $E_{ib}$. Since the incident wave vector $\mathbf{k}$ is assumed to be in the $x$-$z$ plane of the unprimed coordinate system, its direction is given by the angular spherical coordinates $\theta_i$, $\phi_i = 0$, as shown in Fig. 1. Thus

$$\mathbf{k} = -k (\sin \theta_i \hat{x} + \cos \theta_i \hat{z})$$

(44)

If the direction of $\mathbf{k}$ with respect to the primed system is specified by the angular spherical coordinates $\theta_i$, $\phi_i$ (see Fig. 1), then

$$\mathbf{k} = -k (\sin \theta_i \cos \phi_i \hat{x} + \sin \theta_i \sin \phi_i \hat{y} + \cos \theta_i \hat{z})$$

(45)

The unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$ can be expressed in terms of $\hat{x}'$, $\hat{y}'$, $\hat{z}'$ as

$$\hat{\mathbf{a}} = c_{xx} \hat{x}' + c_{xy} \hat{y}' + c_{xz} \hat{z}', \quad \hat{\mathbf{a}} = \hat{x}, \hat{y}, \hat{z}$$

(46)

where

$$c_{xx'} = \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma$$

$$c_{yy'} = -\cos \alpha \cos \beta \sin \gamma + \sin \alpha \cos \gamma$$

$$c_{zz'} = \cos \alpha \sin \beta$$

$$c_{xy'} = \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma$$

$$c_{yx'} = \cos \alpha \cos \gamma - \sin \alpha \cos \beta \sin \gamma$$

$$c_{zx'} = \sin \alpha \sin \beta$$

$$c_{xz'} = -\sin \alpha \cos \gamma$$

(47)

with $\alpha, \beta, \gamma$ being the Euler angles. Substituting $\hat{x}$ and $\hat{z}$ from (46) in (44) and identifying the corresponding coefficients of $\hat{x}'$, $\hat{y}'$, $\hat{z}'$ with those in (45) yields

$$\sin \theta_i \cos \phi_i = c_{xx}, \sin \theta_i + c_{xx}, \cos \theta_i$$

$$\sin \theta_i \sin \phi_i = c_{xy}, \sin \theta_i + c_{xy}, \cos \theta_i$$

$$\cos \theta_i = c_{xz}, \sin \theta_i + c_{xz}, \cos \theta_i$$

(48)

which allows the evaluation of $\theta_i'$ and $\phi_i'$, since $\theta_i$ and $c_{xx}$, $c_{xy}$, $c_{xz}$ ($\alpha = x, y, z$) are known. The incident field $E_{ib}$ can be expressed as

$$E_{ib} = E_{ib}^{TE} \cos \gamma_k + E_{ib}^{TM} \sin \gamma_k$$

(49)

where

$$E_{ib}^{TE} = \hat{y} e^{-jkr}$$

$$E_{ib}^{TM} = (-\cos \theta_i \hat{x} + \sin \theta_i \hat{z}) e^{-jkr}$$

(50)

(51)

Considering the relationship between the vectors $r$, $r'$, and $d$, we can write (see Fig. 1)

$$e^{-jkr'} = e^{-jkr} e^{-j\mathbf{k} \cdot \mathbf{d}}$$

(52)

Taking first the gradient on both sides of (52), and then the cross product with $\hat{x}$ gives

$$e^{-jkr'} \hat{y} = (jk \cos \theta_i)^{-1} e^{-jkr} \nabla (e^{-jkr'}) \times \hat{x}$$

(53)

Substituting $\hat{x}$ from (46) and applying the expansion [11]

$$e^{-jlrr'} = 2 \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} j^n N_{mn}(h') S_{mn}(h', \cos \theta_i')$$

(54)

gives

$$E_{ib}^{TE} = 2(jk \cos \theta_i)^{-1} e^{-jkr} \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} j^n N_{mn}(h') \cdot S_{mn}(h', \cos \theta_i') e^{-jlrr'} [c_{xx'} \mathbf{M}_{xx'}^{(1)}(h'; r') + c_{xy'} \mathbf{M}_{xy'}^{(1)}(h'; r') + c_{xz'} \mathbf{M}_{xz'}^{(1)}(h'; r') + c_{yy'} \mathbf{M}_{yy'}^{(1)}(h'; r') + c_{yz'} \mathbf{M}_{yz'}^{(1)}(h'; r') + c_{zz'} \mathbf{M}_{zz'}^{(1)}(h'; r')]$$

(55)

where the coordinate triad ($\xi', \eta', \phi'$) is denoted by $r'$. 349
Using the vector functions (see (34)-(36))

\[ \mathbf{M}_m^{(4)'} = \frac{1}{2} \left[ \mathbf{M}_m^{(4)} + j \mathbf{M}_m^{(4)}' \right], \quad i = 1, 2, 3, 4 \]  

(56)
yields

\[ \mathbf{E}_{ib}^T = e^{-jk \cdot d} \sum_{m=-m}^m \sum_{n=-|m|}^{|m|} q_{mn}^{TE} \left[ \left( c_{xx}, -j c_{xy} \right) \mathbf{M}_m^{(4)'} (h', r') + \left( c_{xx}, j c_{xy} \right) \mathbf{M}_m^{(4)'} (h'; r') \right] \]

(57)

\[ + \left( c_{xx}, j c_{xy} \right) \mathbf{M}_m^{(4)'} (h'; r') + c_{xx}, \mathbf{M}_m^{(4)'} (h'; r') \right] \]

Thus the incident field in (49) can be written in the form

\[ \mathbf{E}_{ib} = e^{-jk \cdot d} \sum_{m=-m}^m \sum_{n=-|m|}^{|m|} \left( \mathbf{p}_{mn}^{+} \mathbf{M}_m^{(1)'} + \mathbf{p}_{mn}^{-} \mathbf{M}_m^{(1)'} + \mathbf{p}_{mn}^{z} \mathbf{M}_m^{(1)'} \right) \]

(65)

where

\[ p_{mn}^{\pm} = \frac{2j^{n-1}}{kN_{mn}(h')} S_{mn}(h', \cos \theta_i') e^{-j \phi_i} \left[ \left( c_{xx}, \pm j c_{xy} \right) \sin \gamma_k \right] \]

(58)

This expansion is valid for \( \theta_i \neq \pi/2 \). To obtain an expansion which can be used for \( \theta_i = \pi/2 \), we take the gradient on both sides of (52) and then the cross product with \( \mathbf{n'} \), which gives

\[ \mathbf{E}_{ib}^T = - (jk \sin \theta_i)^{-1} e^{-jk \cdot d} \nabla (e^{-jk \cdot r'}) \times \mathbf{z} \]

(59)

Substituting \( e^{-jk \cdot r'} \) from (54) and \( \mathbf{z} \) from (46), and then using (56), we have

\[ \mathbf{E}_{ib}^T = e^{-jk \cdot d} \sum_{m=-m}^m \sum_{n=-|m|}^{|m|} *q_{mn}^{TE} \left[ \left( c_{xx}, -j c_{xy} \right) \mathbf{M}_m^{(4)'} (h'; r') + \left( c_{xx}, j c_{xy} \right) \mathbf{M}_m^{(4)'} (h'; r') \right] \]

(60)

\[ + \left( c_{xx}, j c_{xy} \right) \mathbf{M}_m^{(4)'} (h'; r') + c_{xx}, \mathbf{M}_m^{(4)'} (h'; r') \right] \]

with

\[ *q_{mn}^{TE} = - \frac{2j^{n-1}}{kN_{mn}(h')} S_{mn}(h', \cos \theta_i') e^{-j \phi_i} \sin \theta_i \]

(61)

which is valid for \( \sin \theta_i 
eq 0 \). By taking now the gradient on both sides of (52) and then the cross product with \( \mathbf{y} \), we have

\[ (- \cos \theta, \sin \theta \mathbf{z}) = - j k^{-1} e^{-j k \cdot d} \nabla (e^{-jk \cdot r'}) \times \mathbf{y} \]

(62)

Substituting \( e^{-jk \cdot r'} \) from (54) and \( \mathbf{y} \) from (46), and then using (56), yields

\[ \mathbf{E}_{ib}^T = e^{-jk \cdot d} \sum_{m=-m}^m \sum_{n=-|m|}^{|m|} q_{mn}^{TM} \left[ \left( c_{xx}, -j c_{xy} \right) \mathbf{M}_m^{(4)'} (h'; r') + \left( c_{xx}, j c_{xy} \right) \mathbf{M}_m^{(4)'} (h'; r') \right] \]

(63)

\[ + \left( c_{xx}, j c_{xy} \right) \mathbf{M}_m^{(4)'} (h'; r') + c_{xx}, \mathbf{M}_m^{(4)'} (h'; r') \right] \]

Thus

\[ q_{mn}^{TM} = - \frac{2j^{n-1}}{kN_{mn}(h')} S_{mn}(h', \cos \theta_i') e^{-j \phi_i} \]

(64)

It should be noted that if the coordinate system \( O'x'y'z' \) is brought to coincide with \( Oxyz \), and \( h' = h \), then the expressions in (65)–(67) reduce to those corresponding to the incident field \( E_{id} \) (see (37)–(38)). By arranging the terms in the \( \phi' \) sequence \( e^{i \phi'}, e^{i 2 \phi'}, e^{i 3 \phi'}, \ldots \), we can write the expansion of the incident electric field in the primed coordinate system in matrix form as given in (7), in which

\[ \mathbf{M}^{(1)}_{ib} = \begin{bmatrix} \mathbf{M}_0' \\ \mathbf{M}_1' \\ \mathbf{M}_2' \\ \vdots \end{bmatrix}, \quad \mathbf{J}_B = \begin{bmatrix} \mathbf{P}_0' \\ \mathbf{P}_1' \\ \mathbf{P}_2' \\ \vdots \end{bmatrix} e^{-jk \cdot d} \]

(68)

where

\[ \mathbf{M}_i^{(1)} = \begin{bmatrix} \mathbf{M}_i^{(1)T} \\ \mathbf{M}_i^{(1)T} \\ \mathbf{M}_i^{(1)T} \\ \vdots \end{bmatrix}, \quad \mathbf{J}_B = \begin{bmatrix} \mathbf{J}_{i0}^{(1)} \\ \mathbf{J}_{i1}^{(1)} \\ \mathbf{J}_{i2}^{(1)} \end{bmatrix} \]

(69)

with

\[ \mathbf{M}_i^{(1)T} = \begin{bmatrix} \mathbf{M}_i^{x(1)T} \\ \mathbf{M}_i^{x(1)T} \\ \mathbf{M}_i^{x(1)T} \\ \vdots \end{bmatrix}, \quad \mathbf{J}_{i0}^{(1)} = \begin{bmatrix} \mathbf{J}_{i0}^{(1)} \\ \mathbf{J}_{i1}^{(1)} \end{bmatrix} \]

(70)
and
\[ \hat{\rho}_0^T = [\hat{\rho}_{-1}^T \hat{\rho}_{-1}^T \hat{\rho}_{0}^T] \]
\[ \hat{\rho}_s^T = [\hat{\rho}_{-1}^T \hat{\rho}_{-1}^T \hat{\rho}_s^T \hat{\rho}_{-1}^T \hat{\rho}_{0}^T \hat{\rho}_{s}^T], \quad s \geq 1 \]

(71)

with
\[ \hat{\rho}_s^T = [\hat{\rho}_{-1}^T \hat{\rho}_{-1}^T \hat{\rho}_s^T \hat{\rho}_{-1}^T \hat{\rho}_{s}^T], \quad s \geq 1 \]

(72)

The electric field scattered by the spheroid \( A \), \( E_{sA} \), can be expanded in terms of prolate spheroidal vector wave functions as [4]

\[ E_{sA} = \sum_{m=0}^{\infty} \sum_{n=-m}^{m} (\alpha_{mn}^2 M_{mn}^{(4)} + \alpha_{mn}^1 M_{m+1,n+1}^{(4)} + \alpha_{mn}^0 M_{mn}^{(4)} + \alpha_{mn}^{-1} M_{-m+1,-n+1}^{(4)}) \]

(73)

Arranging the terms in the expansion of \( E_{sA} \) in the same \( \phi \) sequence as that in the expansion of \( E_{sA} \), we can write \( E_{sA} \) in matrix form as given in (5), in which

\[ \tilde{M}_{sA}^{(4)} = \begin{bmatrix} \tilde{M}_{s0}^T \\ \tilde{M}_{s1}^T \\ \vdots \\ \tilde{M}_{s\infty}^T \end{bmatrix}, \quad \tilde{\alpha} = \begin{bmatrix} \tilde{\alpha}_0 \\ \tilde{\alpha}_1 \\ \vdots \\ \tilde{\alpha}_{\infty} \end{bmatrix} \]

(74)

where
\[ \tilde{M}_{s0}^T = [\tilde{M}_{-1}^{(4)T} \tilde{M}_{0}^{(4)T}] \]
\[ \tilde{M}_{s0}^T = [\tilde{M}_{s-1}^{(4)T} \tilde{M}_{s0}^{(4)T} \tilde{M}_{s-1}^{(4)T} \tilde{M}_{-1}^{(4)T}], \quad s \geq 1 \]

(75)

with
\[ \tilde{M}_{s0}^{(4)T} = [M_{r-1}^{(4)} M_{r-1}^{(4)} M_{r+1}^{(4)} M_{r+1}^{(4)} \cdots] \]
\[ \tilde{M}_{s0}^{(4)T} = [M_{r-1}^{(4)} M_{r-1}^{(4)} M_{r+1}^{(4)} M_{r+1}^{(4)} \cdots] \]

(76)

and
\[ \tilde{\alpha}_0^T = [\tilde{\alpha}_0^T \tilde{\alpha}_0^T \tilde{\alpha}_0^T \tilde{\alpha}_0^T \cdots] \]
\[ \tilde{\alpha}_s^T = [\tilde{\alpha}_{s-1}^T \tilde{\alpha}_{s-1}^T \tilde{\alpha}_{s-1}^T \tilde{\alpha}_{s-1}^T \cdots], \quad s \geq 1 \]

(77)

with
\[ \tilde{\alpha}_s^T = [\alpha_{r-1}^{+T} \alpha_{r-1}^{+T} \alpha_{r+1}^{+T} \alpha_{r+1}^{+T} \cdots] \]
\[ \tilde{\alpha}_s^T = [\alpha_{r-1}^{+T} \alpha_{r-1}^{+T} \alpha_{r+1}^{+T} \alpha_{r+1}^{+T} \cdots] \]

(78)

Similarly the expansion of the electric field scattered by the spheroid \( B \), \( E_{sB} \), can be written in a matrix form as given in (2) and (7), with

\[ \tilde{M}_{sB}^{(4)} = \begin{bmatrix} \tilde{M}_{s0} \\ \tilde{M}_{s1} \\ \vdots \\ \tilde{M}_{s\infty} \end{bmatrix}, \quad \tilde{\beta} = \begin{bmatrix} \tilde{\beta}_0 \\ \tilde{\beta}_1 \\ \vdots \end{bmatrix} \]

(79)

where \( \tilde{M}_{s0} \) and \( \tilde{\beta}_s \) \((s = 0, 1, 2, \cdots)\) have the same format as that of \( \tilde{M}_{s0} \) and \( \tilde{\alpha}_s \), respectively, but with the vector wave functions evaluated in primed coordinates, and \( \alpha \) replaced by \( \beta \).

To express the outgoing (scattered) wave from the spheroid \( B \) as an incoming wave with respect to the spheroid \( A \), we use the rotational–translational addition theorems for vector spheroidal wave functions [8] in the expansion of \( E_{sB} \):

\[ M_{mn}^{(4)\ast}(h'; r') - \sum_{r=0}^{\infty} \sum_{\mu=-r}^{r} (4) Q_{mn}^{\mu \ast}(\alpha, \beta, \gamma; d)(C_1 M_{\mu}^{(1)}(h; r) + C_2 M_{\mu}^{(1)}(h; r)) \]

(80)

\[ M_{mn}^{(4)\ast}(h'; r') = \sum_{r=0}^{\infty} \sum_{\mu=-r}^{r} (4) Q_{mn}^{\mu \ast}(\alpha, \beta, \gamma; d)(C_1 M_{\mu}^{(1)}(h; r) + C_2 M_{\mu}^{(1)}(h; r)), \quad r \leq d \]

(81)

\[ M_{mn}^{(4)\ast}(h'; r') = \sum_{r=0}^{\infty} \sum_{\mu=-r}^{r} (4) Q_{mn}^{\mu \ast}(\alpha, \beta, \gamma; d)(C_1 M_{\mu}^{(1)}(h; r) + C_2 M_{\mu}^{(1)}(h; r)), \quad r \leq d \]

(82)

where \( r \) and \( r' \) represent the coordinate triads \((\xi, \eta, \phi)\) and \((\xi', \eta', \phi')\), respectively, and

\[ C_1 = \frac{1}{2} [(c_{xx'} + c_{yy'}) + j(c_{xy'} - c_{yx'})] \]
\[ C_2 = \frac{1}{2} [(c_{xx'} - c_{yy'}) + j(c_{xy'} + c_{yx'})] \]
\[ C_3 = \frac{1}{2} (c_{xx'} + jc_{yy'}) \]
\[ C_4 = c_{xx'} - jc_{yy'} \]
\[ C_5 = c_{zz'} \]

with \( c_{xx'}, c_{yy'}, c_{zz'} \) \((\alpha = x, y, z)\) defined in (47); the asterisk denotes the complex conjugate and \((4) Q_{mn}^{\mu \ast}(\alpha, \beta, \gamma; d)\) are given in Appendix II. If the terms in the series expansions (80)–(82) are arranged in the \( \phi \) sequence \( \phi^0, \varepsilon \phi^0, \varepsilon^2 \phi^0, \cdots \), then we obtain the matrices \( \tilde{M}_{sA}^{(4)} \) and \([T]\) introduced in (3).
The transpose of \( \mathbf{M}_{BA}^{(1)} \) is
\[
\mathbf{M}_{BA}^{(1)*T} = [\mathbf{M}_{BA,0}^{(1)*T} \mathbf{M}_{BA,1}^{(1)*T} \mathbf{M}_{BA,2}^{(1)*T} \cdots]
\]
with the rotational-translational coefficients \((4)Q''_{m,n}\) derived in the Appendix II.

The elements of the matrix \( \mathbf{[\Gamma]} \) introduced in (7) are obtained from the corresponding elements of the matrix \( \mathbf{[\Gamma]} \) by replacing \((4)Q''_{m,n}\) by \((4)Q''_{m,n}\) (see the Appendix II) and \(C_i\) by \(C_i\).

The matrix \( \mathbf{[\Gamma]} \) can be written as
\[
[\Gamma]_{\sigma} = \begin{bmatrix}
[\Gamma_{1,0}]_{\sigma} & [\Gamma_{1,1}]_{\sigma} & [\Gamma_{1,2}]_{\sigma} \\
[\Gamma_{2,0}]_{\sigma} & [\Gamma_{2,1}]_{\sigma} & [\Gamma_{2,2}]_{\sigma} \\
[\Gamma_{3,0}]_{\sigma} & [\Gamma_{3,1}]_{\sigma} & [\Gamma_{3,2}]_{\sigma}
\end{bmatrix}
\]
with
\[
[\Gamma_{1,0}]_{\sigma} = \begin{bmatrix}
[\Gamma_{1,1}]_{\tau}^{-1} & [\Gamma_{2,1}]_{\tau}^{-1} & [\Gamma_{3,1}]_{\tau}^{-1} \\
[\Gamma_{4,1}]_{\tau}^{-1} & [\Gamma_{4,2}]_{\tau}^{-1} & [\Gamma_{3,3}]_{\tau}^{-1}
\end{bmatrix}, \quad \tau \geq 1
\]
\[
[\Gamma_{1,1}]_{\sigma} = \begin{bmatrix}
[\Gamma_{1,1}]_{(r-1)}^{-1} & [\Gamma_{2,1}]_{(r-1)}^{-1} & [\Gamma_{3,1}]_{(r-1)}^{-1} \\
[\Gamma_{4,1}]_{(r-1)}^{-1} & [\Gamma_{4,2}]_{(r-1)}^{-1} & [\Gamma_{3,3}]_{(r-1)}^{-1}
\end{bmatrix}, \quad \tau \geq 1
\]
\[
[\Gamma_{1,2}]_{\sigma} = \begin{bmatrix}
[\Gamma_{1,1}]_{(r+1)}^{(r+1)} & [\Gamma_{2,1}]_{(r+1)}^{(r+1)} & [\Gamma_{3,1}]_{(r+1)}^{(r+1)} \\
[\Gamma_{4,1}]_{(r+1)}^{(r+1)} & [\Gamma_{4,2}]_{(r+1)}^{(r+1)} & [\Gamma_{3,3}]_{(r+1)}^{(r+1)}
\end{bmatrix}, \quad \tau \geq 1, \sigma \geq 1
\]

The submatrices \([\Gamma_{1,\sigma}]^*\) and \([*\Gamma_{1,\sigma}]^*\) for \(\tau, \sigma = \ldots -2, -1, 0, 1, 2, \ldots\) and \(i = 1, 2, 3, 4, 5\) are \(C_i[\Gamma]_\sigma\) and \(C_i[*[\Gamma]]_\sigma\), respectively, where \(C_i\) are defined in (83) and

\[
[\Gamma]_\sigma = \begin{bmatrix}
(4)Q''_{r,m} & (4)Q''_{r,m} & (4)Q''_{r,m} & (4)Q''_{r,m} \\
(4)Q''_{r,m}^{(r+1)} & (4)Q''_{r,m}^{(r+1)} & (4)Q''_{r,m}^{(r+1)} & (4)Q''_{r,m}^{(r+1)} \\
(4)Q''_{r,m}^{(r+2)} & (4)Q''_{r,m}^{(r+2)} & (4)Q''_{r,m}^{(r+2)} & (4)Q''_{r,m}^{(r+2)} \\
- & - & - & -
\end{bmatrix}
\]

352
following relations are used [11]:

\[ S_{mn}(h,\eta) = K_{mn} S_{mn}(h,\eta) \]

\[ R_{mn}^{(i)}(h,\xi) = R_{mn}^{(i)}(h,\xi), \quad i = 1, 2, 3, 4 \]

\[ d_q^{mn}(h) = (-1)^{|m| - |m|/2} \frac{(|m| - m + q)!}{q!} K_{mn} d_q^{mn}(h) \]

\[ N_{mn}(h) = K_{mn}^2 N_{mn}(h) \quad (94) \]

where

\[ K_{mn} = (-1)^{|m| - |m|/2} \frac{(n + m)!}{(n + |m|)!} \]

APPENDIX II

ROTATIONAL-TRANSLATIONAL COEFFICIENTS

The coefficients in the expansion of scalar spheroidal wave functions in unprimed coordinates in terms of same functions in primed coordinates, for \( r' \leq d \) and \( i = 1, 2, 3, 4 \), are [9]

\[ (i) Q_{mn}^{(i)}(\alpha, \beta, \gamma; d) \]

\[ = \sum_{s=|m|,|m|+1}^\infty d_q^{mn}(h) \sum_{k=-s}^s R_{mn}^{(i)}(\alpha, \beta, \gamma) \]

\[ \cdot \sum_{l=|\mu|,|\mu|+1}^\infty (i) a_{nl}(d) j^{s-n+r-1} \frac{N_{\nu l}}{N_{\nu l}(h')} d_{\nu l}^{mn}(h') \quad (96) \]

where \( d_q^{mn}(h) \) and \( d_{\nu l}^{mn}(h') \) are the spheroidal expansion coefficients, and \( N_{\nu l}(h') \) is the normalization constant [11]. In this expansion the following notation is used:

\[ N_{ml} = \frac{2}{(2l + 1)} \frac{(l + m)!}{(l - m)!} \]

\[ R_{m-l}^{(i)}(\alpha, \beta, \gamma) = (-1)^{|m|-m} \frac{N_{ml}}{N_{m-l}} \frac{1}{(2l + 1)!} \frac{2}{(l + m)!} \frac{(l + m)!}{(l - m)!} \]

\[ d_{m-l}^{(i)}(\beta) = \left[ \frac{(l + m)!}{(l - m)!} \right]^{1/2} \left( \cos \frac{\beta}{2} \right)^{-m-m} \]

\[ \cdot \left( \sin \frac{\beta}{2} \right)^{-m-m} \cdot P_{m-l}^{m-m}(\cos \beta) \]

(98)

\[ (i) a_{nl}(d) = (-1)^l \sum_{p=|p|,|p|+1}^\infty j^{l+p-1} (2l + 1) \]

\[ \cdot a(s,|\mu|,|\mu|,d) \psi_{i-\mu,\mu}(d) \]

(100)

in which \( a(s,|\mu|,|\mu|,d) \) are the linearization expansion coefficients [2], [3], the first term in the series being \( p_0 = \max \left(|l - s|, |x - \mu|\right) \), or \( p_0 + 1 \), so that its last term is \( l + s \) and \( s = |l| = |l'| \).

\[ \psi_{i-\mu,\mu}(d) = z_h^{(i)}(k|d|) P_{i-\mu}^{x-\mu}(\cos \theta|d|) e^{i(k-\nu)r} \]

(101)

where \( z_h^{(i)}(k|d|) \) are the spherical Bessel functions \( j_p, n_p, h_p^{(1)} \), and \( h_p^{(2)} \), respectively, and \( P_{i-\mu}^{x-\mu} \) is the associated Legendre function of the first kind.

Considering the translation from the system \( O'x'y'z'z' \) to \( Oxy_{z}z_{z} \) and then the rotation of the system \( Ox_{z}y_{z}z_{z} \) about \( \Omega \) through the Euler angles \( \gamma, \beta, -\alpha \), we derive the rotational-translation coefficients \( (i) Q_{mn}^{(i)} \) for the expansion of spheroidal wave functions in primed coordinates in terms of functions in unprimed coordinates, for \( r \leq d \) and \( i = 1, 2, 3, 4 \), as

\[ (i) Q_{mn}^{(i)}(\alpha, \beta, \gamma; d) \]

\[ = \sum_{s=|m|,|m|+1}^\infty d_q^{mn}(h') \sum_{s=|m|,|m|+1}^\infty j^{s-n+r-1} \frac{N_{\nu l}}{N_{\nu l}(h')} d_{\nu l}^{mn}(h') \]

\[ \cdot \sum_{c=-l}^l R_{n-l}^{(i)}(-\gamma, -\beta, -\alpha) (i) b_{cl}^{mn}(d) \quad (102) \]

with

\[ (i) b_{cl}^{mn}(d) = (-1)^c \sum_{p=|p|,|p|+1}^\infty (-1)^j j^{l+p-s} \]

\[ \cdot (2l + 1) a(s,|\mu|,|\mu|,d) \psi_{i-\mu,\mu}(d). \]

(103)

Substituting \( s = |m| + q \), \( l = |\mu| + r \) in (96) and (102) we get, respectively,

\[ (i) Q_{mn}^{(i)}(\alpha, \beta, \gamma; d) \]

\[ = \sum_{q=0,1}^\infty d_q^{mn}(h) \sum_{s=|m|+q}^\infty R_{s,|m|+q}(\alpha, \beta, \gamma) \]

\[ \cdot \sum_{r=0,1}^\infty j^{s-n+r-1} \frac{N_{\nu l}}{N_{\nu l}(h')} d_{\nu l}^{mn}(h') \]

(104)

and

\[ (i) Q_{mn}^{(i)}(\alpha, \beta, \gamma; d) \]

\[ = \sum_{q=0,1}^\infty d_q^{mn}(h') \sum_{s=|m|+q}^\infty j^{s-n+r-1} \frac{N_{\nu l}}{N_{\nu l}(h')} d_{\nu l}^{mn}(h') \]

\[ \cdot \sum_{c=-|l|+|\mu|+r}^{|l|+|\mu|+r} R_{c,|l|+|\mu|+r}(-\gamma, -\beta, -\alpha) (i) b_{cl}^{mn}(d) \]

(105)

APPENDIX III

MATRICES [Q] AND [R] RESULTING FROM IMPOSING THE BOUNDARY CONDITIONS

The elements of the matrices \([Q_A], [R_A], [R_{BA}], [Q_B], [R_B], \) and \([R_{AB}]\) can be grouped in submatrices, such that
all these matrices are quasi-diagonal in the sense that only the diagonal submatrices are different from zero. All null off-diagonal submatrices have the same size as the corresponding diagonal submatrices. The diagonal submatrices of \([Q_A]\), \([R_A]\), and \([R_{BA}]\) can be written as \([4]\)

\[
[Q_A]_0 = \begin{bmatrix}
[\tau X^+(4)] & [\tau X^-(4)] \\
[\phi X^+(4)] & [\phi X^-(4)] 
\end{bmatrix},
\]

\[
[Q_A]_m = \begin{bmatrix}
[\tau X^+(m+1)] & [\tau X^-(m+1)] \\
[\phi X^+(m+1)] & [\phi X^-(m+1)] 
\end{bmatrix}, \quad m \geq 1.
\]

\[
[R_A]_0 = \begin{bmatrix}
[\tau X^+(1)] & [\tau X^-(1)] \\
[\phi X^+(1)] & [\phi X^-(1)] 
\end{bmatrix},
\]

\[
[R_A]_m = \begin{bmatrix}
[\tau X^+(m+1)] & [\tau X^-(m+1)] \\
[\phi X^+(m+1)] & [\phi X^-(m+1)] 
\end{bmatrix}, \quad m \geq 1.
\]

\[
[R_{BA}]_0 = \begin{bmatrix}
[\tau X^+(1)] & [\tau X^-(1)] & [\tau X^+0] \\
[\phi X^+(1)] & [\phi X^-(1)] & [\phi X^+0] 
\end{bmatrix},
\]

\[
[R_{BA}]_m = \begin{bmatrix}
[\tau X^+(m+1)] & [\tau X^-(m+1)] & [\tau X^+0] \\
[\phi X^+(m+1)] & [\phi X^-(m+1)] & [\phi X^+0] 
\end{bmatrix}, \quad m \geq 1.
\]

The submatrices \([X_m]\) have the form

\[
[X_m] = \begin{bmatrix}
X_{m,0} & X_{m,1} & X_{m,2} & \cdots & X_{m,0} & X_{m,1} & X_{m,2} & \cdots \\
X_{m,1} & X_{m,1} & X_{m,2} & \cdots & X_{m,2} & X_{m,2} & X_{m,2} & \cdots \\
X_{m,2} & X_{m,2} & X_{m,2} & \cdots & X_{m,2} & X_{m,2} & X_{m,2} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]

where the elements are given by

\[
(\tau X^+(m,N))_{m,N} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 (\tau X^+) M^+(m,N,s) e^{-jm(m+1)\phi} d\eta d\phi
\]

These integrals are evaluated in [10]. The submatrices of \([Q_B]\), \([R_B]\), and \([R_{BA}]\) have the same form as those of \([Q_A]\), \([R_A]\), and \([R_{BA}]\), respectively, but with the corresponding elements evaluated in the primed system, attached to the spheroid B.

**References**


waves from two infinitely conducting prolate spheroids which are centered in a plane perpendicular to their axes of revolution," Radio Sci., vol. 20, pp. 575-581, May-June 1985.


