Electromagnetic scattering by an arbitrary configuration of
dielectric spheres

A-K. Hamid, I. R. Cric, and M. Hamid

Department of Electrical and Computer Engineering, University of Manitoba, Winnipeg, Man., Canada, R3T 2N2

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An analytic solution is obtained for the problem of plane electromagnetic-wave scattering by an arbitrary configuration of \( N \) dielectric spheres. The multipole expansion method is employed, and the boundary condition is imposed using the translational addition theorem for vector spherical wave functions. A system of simultaneous linear equations is given in matrix form for the scattering coefficients. An approximate solution, which has been developed and employed by the authors for the scattering by \( N \) conducting spheres, is extended to the dielectric spheres case. Plots for the normalized backscattering, bistatic, and forward-scattering cross sections are presented over wide ranges of permittivity, size, and electrical separations between the neighbouring spheres. The results show a reduction in the normalized backscattering and bistatic cross sections for certain choices of permittivity relative to conducting arrays of spheres of the same dimensions and separations.

Fig. 1. Arbitrary configuration of \( N \) dielectric spheres.

1. Introduction

The multiple scattering of a plane electromagnetic wave by an arbitrary configuration of \( N \) dielectric spheres has various practical applications such as the modelling of complex dielectric scatterers and simulation of human or animal bodies using inhomogeneous dielectric spheres. A novel application is that of loading the aperture of an antenna with a linear array of dielectric spheres to enhance the gain along preferred directions. To date, there are no analytical, approximate, or purely numerical solutions available in the literature for scattering by more than two dielectric spheres. Therefore, the goal of this paper is to present analytic and approximate solutions for the scattering by \( N \) dielectric spheres, since purely numerical techniques would require very large computer storage and hence limit the usefulness of these techniques.

King and Harrison studied (1) the problem of a plane electromagnetic-wave scattering by an imperfectly conducting, dielectric, or plasma sphere. Tabulated results for the backscattering cross section of a conducting or dielectric sphere were obtained by Adler and Johnson (2). On the other hand, a numerical solution based on the moment method was presented by Mautz and Harrington (3) to solve for the scattering by a conducting or dielectric body of revolution in terms of equivalent electric and magnetic currents over the surface of the body, while Kishk and Shafai (4) extended the analysis to the scattering by two bodies of revolution excited by a plane wave or infinitesimal electric dipole. An analytic solution to the scattering by two dielectric spheres was obtained by Bruning and Lo (5) using the translational addition theorem for vector spherical wave functions given by Cruzan (6). Recently, the authors

1 Present address: Department of Electrical Engineering, University of South Alabama, Mobile, AL 36688, U.S.A.
for the vector spherical wave functions. In addition, an approximate method (7) is extended to solve for the scattering by a linear array of \( N \) relatively small dielectric spheres. Numerical results are given graphically for the normalized backscattering, bistatic, and forward-scattering cross sections for systems of spheres with different separations, radii, compositions, and angles of incidence.

Examination of the geometry indicates that an array of dielectric spheres could have lower normalized backscattering and bistatic cross sections, and weaker sphere to sphere coupling than a similar array of perfectly conducting spheres. On the other hand, the forward-scattering cross section could be significantly enhanced with an increase in the number of spheres. Experimental results to improve the gain of antennas loaded with a single dielectric sphere have been reported (10).

2. Formulation

Consider a plane electromagnetic wave incident on an arbitrary configuration of dielectric spheres as shown in Fig. 1, where the radius of the \( p \)th sphere is \( a_p (p = 1, 2, \ldots , N) \) and the permittivity is \( \varepsilon_p \). The spheres are centered at \( d_p \) with local Cartesian coordinates \((x_p, y_p, z_p)\). The incident plane wave has a unit electric-field intensity whose propagation vector \( \mathbf{k} \) lies in the \( xy \) plane and makes an angle \( \alpha \) with the \( z \) axis. The incident electric field is considered to be in the \( y \) direction. The incident fields are

\[ E^i = e^{ikr_\hat{y}} \]

\[ H^i = -\frac{1}{\eta} e^{ikr} (\cos \alpha \hat{x} - \sin \alpha \hat{z}) \]

where \( H^i \) is the incident magnetic-field intensity and \( \eta \) is the surrounding medium intrinsic impedance. The incident plane wave is expressed with reference to the spherical coordinate system of the \( p \)th sphere as

\[ e^{ik(rp + dp)} = e^{ikd_p} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} j^{n+1} \frac{(n-m)!}{(n+m)!} P^m_n(\cos \alpha) \left( \begin{array}{c} \rho_p \\ \phi_p \end{array} \right) J_n(kr_p) e^{jmp} \]

where \( J_n \) is the spherical Bessel function, \( P^m_n \) is the associated Legendre function of the first kind, and

\[ \zeta_p = \sin \Theta_p \cos \Phi_p \sin \alpha + \cos \Theta_p \cos \alpha \]

For the special case of a linear array of \( N \) dielectric spheres spaced along the \( z \) axis, \( \Theta_p = 0 \) \((p = 1, 2, \ldots , N)\).

Expanding the incident electric and magnetic fields in terms of the spherical vector wave functions with the \( e^{-j\omega t} \) time dependence suppressed throughout (7, 8), we obtain

\[ E^i(r_p, \theta_p, \phi_p) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[ P_m(m,n)N_m^{(1)}(r_p, \theta_p, \phi_p) + Q_m(m,n)M_m^{(1)}(r_p, \theta_p, \phi_p) \right] \]

\[ \eta H^i(r_p, \theta_p, \phi_p) = j \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[ P_m(m,n)M_m^{(1)}(r_p, \theta_p, \phi_p) + Q_m(m,n)N_m^{(1)}(r_p, \theta_p, \phi_p) \right] \]

where \( p \) takes integer values from 1 to \( N \), while \( M_m^{(1)} \) and \( N_m^{(1)} \) are the spherical vector wave functions of the first kind, which represent incoming waves associated with the spherical Bessel function. The incident-field expansion coefficients may be obtained from [4] and [5] in ref. 7 with \( \cos \alpha \) replaced by \( \zeta_p \) in the phase factor.

The scattered field from the \( p \)th sphere \( (r_p > a_p) \) can be expanded in terms of spherical wave functions as

\[ E_s^s(r_p, \theta_p, \phi_p) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[ A^s_{pE}(m,n)N_m^{(3)}(r_p, \theta_p, \phi_p) + A^s_{pM}(m,n)M_m^{(3)}(r_p, \theta_p, \phi_p) \right] \]

\[ \eta H_s^s(r_p, \theta_p, \phi_p) = j \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[ A^s_{pE}(m,n)M_m^{(3)}(r_p, \theta_p, \phi_p) + A^s_{pM}(m,n)N_m^{(3)}(r_p, \theta_p, \phi_p) \right] \]

where \( A^s_{pE} \) and \( A^s_{pM} \) are the unknown scattering coefficients to be determined. \( M_m^{(3)} \) and \( N_m^{(3)} \) are the spherical vector wave functions of the third kind, which represent outgoing waves associated with the spherical Hankel function. Similarly, the transmitted electric and magnetic fields into the \( p \)th sphere \( (r_p < a_p) \) may be written as

\[ E_t^s(r_p, \theta_p, \phi_p) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[ A^t_{pE}(m,n)N_m^{(1)}(r_p, \theta_p, \phi_p) + A^t_{pM}(m,n)M_m^{(1)}(r_p, \theta_p, \phi_p) \right] \]

\[ \eta H_t^s(r_p, \theta_p, \phi_p) = j \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[ A^t_{pE}(m,n)M_m^{(1)}(r_p, \theta_p, \phi_p) + A^t_{pM}(m,n)N_m^{(1)}(r_p, \theta_p, \phi_p) \right] \]
where \( \eta_p \) is the medium's intrinsic impedance of the \( p \)th sphere. To impose the boundary condition at \( r_p = a_p \) \( (p = 1, 2, \ldots, N) \), the outgoing scattered fields from the \( q \)th sphere must be expressed in terms of fields incoming on the \( p \)th sphere and vice versa, hence we apply the spherical vector translational addition theorem (6), i.e.,

\[
M_{mn}^{(3)}(r_q, \theta_q, \phi_q) = \sum_{v=1}^{\infty} \sum_{\mu=-\nu}^{\nu} \left[ A_{\mu \nu}^{mn}(d_{pq}, \theta_{pq}, \phi_{pq})M_{\mu \nu}^{(1)}(r_p, \theta_p, \phi_p) + B_{\mu \nu}^{mn}(d_{pq}, \theta_{pq}, \phi_{pq})N_{\mu \nu}^{(1)}(r_p, \theta_p, \phi_p) \right]
\]

\[
N_{mn}^{(3)}(r_q, \theta_q, \phi_q) = \sum_{v=1}^{\infty} \sum_{\mu=-\nu}^{\nu} \left[ A_{\mu \nu}^{mn}(d_{pq}, \theta_{pq}, \phi_{pq})N_{\mu \nu}^{(1)}(r_p, \theta_p, \phi_p) + B_{\mu \nu}^{mn}(d_{pq}, \theta_{pq}, \phi_{pq})M_{\mu \nu}^{(1)}(r_p, \theta_p, \phi_p) \right]
\]

where \( A_{\mu \nu}^{mn} \) and \( B_{\mu \nu}^{mn} \) are the translational addition theorem coefficients given in the Appendix and

\[
d_{pq} = d_q - d_p = (x_q - x_p)\hat{x} + (y_q - y_p)\hat{y} + (z_q - z_p)\hat{z}, \quad \theta_{pq} = \cos^{-1} \left( \frac{z_q - z_p}{d_{pq}} \right), \quad \phi_{pq} = \tan^{-1} \left( \frac{y_q - y_p}{x_q - x_p} \right)
\]

The boundary condition on the surface of the \( p \)th dielectric sphere requires continuity of the tangential electric and magnetic field, i.e.,

\[
\hat{n}_p \times [E_p^{\text{out}}(r_p, \theta_p, \phi_p) - E_p^{\text{in}}(r_p, \theta_p, \phi_p)]|_{r_p = a_p} = 0, \quad p = 1, 2, \ldots, N
\]

\[
\hat{n}_p \times [H_p^{\text{out}}(r_p, \theta_p, \phi_p) - H_p^{\text{in}}(r_p, \theta_p, \phi_p)]|_{r_p = a_p} = 0, \quad p = 1, 2, \ldots, N
\]

where \( \hat{n}_p \) is the outward unit normal to the surface of the \( p \)th dielectric sphere, and the superscript “out” refers to the region external to the sphere \( (r_p > a_p) \). Hence

\[
E_p^{\text{out}}(r_p, \theta_p, \phi_p) = E_p^{\text{in}}(r_p, \theta_p, \phi_p) + \sum_{q=1}^{N} \left( \sum_{m=-n}^{m} \sum_{n=-m}^{m} [A_{qE}^m(m, n)N_{mn}^{(3)}(r_q, \theta_q, \phi_q) + A_{qM}^s(m, n)M_{mn}^{(3)}(r_q, \theta_q, \phi_q)] \right)
\]

while the total magnetic field can be written similarly. It should be noted that \( E_p^1 \) and \( H_p^1 \) are zero for the special case of perfectly conducting spheres (7). Substituting the appropriate forms of the translational addition theorem [10]–[13] into [14] and [15] leads to the following equations for the expansion coefficients

\[
r_p \times \sum_{m=-n}^{m} \sum_{n=-m}^{m} \left[ P_p(m, n)N_{mn}^{(1)}(a_p, \theta_p, \phi_p) + Q_p(m, n)M_{mn}^{(1)}(a_p, \theta_p, \phi_p) \right.

+ A_{pk}^s(m, n)N_{mn}^{(1)}(a_p, \theta_p, \phi_p) + A_{pkM}^s(m, n)M_{mn}^{(3)}(a_p, \theta_p, \phi_p)

+ \sum_{q=1}^{N} \left( A_{qE}^s(m, n) \sum_{v=1}^{\nu} \sum_{\mu=-\nu}^{\nu} [A_{\mu \nu}^{mn}(d_{pq}, \theta_{pq}, \phi_{pq})N_{\mu \nu}^{(1)}(a_p, \theta_p, \phi_p) + B_{\mu \nu}^{mn}(d_{pq}, \theta_{pq}, \phi_{pq})M_{\mu \nu}^{(1)}(a_p, \theta_p, \phi_p)] \right.

+ A_{pkM}^s(m, n) \sum_{v=1}^{\nu} \sum_{\mu=-\nu}^{\nu} [A_{\mu \nu}^{mn}(d_{pq}, \theta_{pq}, \phi_{pq})M_{\mu \nu}^{(1)}(a_p, \theta_p, \phi_p) + B_{\mu \nu}^{mn}(d_{pq}, \theta_{pq}, \phi_{pq})N_{\mu \nu}^{(1)}(a_p, \theta_p, \phi_p) \bigg] \bigg]

\]

and similarly from continuity of the tangential magnetic field, we obtain
\[ r_p \times \frac{1}{\eta} \sum_{n=\pm 1}^{\infty} \sum_{m=\pm n}^{\infty} \left[ P_p(m, n) M_{mn}^1(a_p, \theta_p, \phi_p) + Q_p(m, n) N_{mn}^1(a_p, \theta_p, \phi_p) \right. \\
+ A_{pE}^s(m, n) M_{mn}^3(a_p, \theta_p, \phi_p) + A_{pM}^s(m, n) N_{mn}^3(a_p, \theta_p, \phi_p) \\
+ \sum_{q=1 \neq p}^{N} \left\{ A_{qE}^s(m, n) \sum_{\mu=\pm 1}^{\mu=\pm \nu} \sum_{\mu=\pm 1}^{\mu=\pm \nu} [A_{\mu\nu}^m(d_{pq}, \theta_{pq}, \phi_{pq}) M_{\mu\nu}^1(a_p, \theta_p, \phi_p) + B_{\mu\nu}^m(d_{pq}, \theta_{pq}, \phi_{pq}) N_{\mu\nu}^1(a_p, \theta_p, \phi_p)] \\
+ A_{qM}^s(m, n) \sum_{\mu=\pm 1}^{\mu=\pm \nu} \sum_{\mu=\pm 1}^{\mu=\pm \nu} [A_{\mu\nu}^m(d_{pq}, \theta_{pq}, \phi_{pq}) N_{\mu\nu}^1(a_p, \theta_p, \phi_p) + B_{\mu\nu}^m(d_{pq}, \theta_{pq}, \phi_{pq}) M_{\mu\nu}^1(a_p, \theta_p, \phi_p)] \right\} \\
\left. - \frac{1}{\eta_p} \sum_{n=\pm 1}^{\infty} \sum_{m=\pm n}^{\infty} [A_{pE}^s(m, n) M_{mn}^{(1)}(a_p, \theta_p, \phi_p) + A_{pM}^s(m, n) N_{mn}^{(1)}(a_p, \theta_p, \phi_p)] = 0 \right] \\
\]

Since our main interest is to obtain the scattered field, a solution for the scattering coefficients \( A_{pE}^s(m, n) \) and \( A_{pM}^s(m, n) \) will therefore be given here. Applying the orthogonality properties of the spherical wave functions yields

\[ A_{pE}^s(m, n) = v_n(\rho_p) \left\{ P_p(m, n) + \sum_{q=1 \neq p}^{N} \sum_{\mu=\pm 1}^{\mu=\pm \nu} [A_{qE}^{(\mu \nu)}(d_{pq}, \theta_{pq}, \phi_{pq}) A_{qE}^s(\mu, \nu) + B_{qE}^{(\mu \nu)}(d_{pq}, \theta_{pq}, \phi_{pq}) A_{qM}^s(\mu, \nu)] \right\} \\
A_{pM}^s(m, n) = u_n(\rho_p) \left\{ Q_p(m, n) + \sum_{q=1 \neq p}^{N} \sum_{\mu=\pm 1}^{\mu=\pm \nu} [A_{qM}^{(\mu \nu)}(d_{pq}, \theta_{pq}, \phi_{pq}) A_{qM}^s(\mu, \nu) + B_{qM}^{(\mu \nu)}(d_{pq}, \theta_{pq}, \phi_{pq}) A_{qE}^s(\mu, \nu)] \right\} \\
\]

for \( p = 1, 2, \ldots, N \), where \( v_n(\rho_p) \) and \( u_n(\rho_p) \) are the electric and magnetic scattering coefficients for a single dielectric sphere (8), which are given by

\[ v_n(\rho_p) = - \frac{j_n(\rho_p) [\xi_p j_n(\xi_p)]'}{h_n^{(1)}(\rho_p) [\xi_p j_n(\xi_p)]'} - N_p^2 j_n(\xi_p) [\rho_p j_n(\rho_p)]' \\
u_n(\rho_p) = - \frac{j_n(\xi_p) [\rho_p h_n^{(1)}(\rho_p)]'}{j_n(\xi_p) [\rho_p h_n^{(1)}(\rho_p)]'} - h_n^{(1)}(\rho_p) [\xi_p j_n(\xi_p)]'
\]

Here \( k_p = N_p k, \rho_p = k \rho_p, \xi_p = k_p \rho_p, N_p = \sqrt{k_p \epsilon}, \) while \( \epsilon_p \) and \( \epsilon \) are the permittivities of the \( p \)th sphere and of the surrounding medium, respectively. \( h_n^{(1)} \) is the Hankel function of the first kind of order \( n \) and the prime denotes differentiation with respect to the total argument. In the case of \( N \) lossy dielectric spheres, \( N_p \) is complex.

Equations [19] and [20] can be written in matrix form as

\[ A_{pE}^s = [v_p] P_p + [v_p] \sum_{q=1 \neq p}^{N} \{ [A^{pq}] A_{qE}^s + [B^{pq}] A_{qM}^s \}, \quad p = 1, 2, \ldots, N \\
A_{pM}^s = [u_p] Q_p + [u_p] \sum_{q=1 \neq p}^{N} \{ [A^{pq}] A_{qM}^s + [B^{pq}] A_{qE}^s \}, \quad p = 1, 2, \ldots, N \\
\]

The above system of matrices may be rewritten in the following form

\[ \begin{bmatrix} A_{pE}^s \\ A_{pM}^s \end{bmatrix} = \begin{bmatrix} v_p \\ u_p \end{bmatrix} \begin{bmatrix} P_p \\ Q_p \end{bmatrix} + \sum_{q=1 \neq p}^{N} \begin{bmatrix} [A^{pq}] \\ [B^{pq}] \end{bmatrix} \begin{bmatrix} A_{qE}^s \\ A_{qM}^s \end{bmatrix} \]

where \( A_{pE}^s, A_{pM}^s, A_{qE}^s, A_{qM}^s \) are column matrices for the unknown scattering coefficients of the \( p \)th and \( q \)th sphere, respectively. \( [v_p] \) and \( [u_p] \) are diagonal matrices containing the scattering coefficients of a single sphere, while \( P_p \) and \( Q_p \) are column matrices for the incident field coefficients. Finally, \( [A^{pq}] \) and \( [B^{pq}] \) are matrices associated with the translation addition coefficients.

Equations [23] and [24] are a coupled set of complex linear algebraic equations, and should be solved simultaneously to yield the unknown scattering coefficients. In addition, the infinite series must be truncated to a finite number of terms \( n = \nu = M \) and...
Equation [25] may be written in a convenient matrix form as

\[ A = L + TA \]

where

\[
L = \begin{bmatrix} [v_p] & 0 \\ 0 & [u_p] \end{bmatrix}, \quad T = \begin{bmatrix} [v_p] & 0 \\ 0 & [u_p] \end{bmatrix} \sum_{q=1}^{N} \begin{bmatrix} [A^{pq}] & [B^{pq}] \\ [B^{pq}] & [A^{pq}] \end{bmatrix}
\]

The solution of [26] yields the scattered coefficients in [6] and [7], i.e.,

\[ A = (I - T)^{-1}L \]

For the special case of a linear array of dielectric spheres (9), \( \mu = m \) and the above system is solved for each \( m \) independently, since there is no coupling between azimuthal modes. Once the scattered coefficients are computed, the total scattered field can be determined everywhere.

The expressions for the normalized backscattering and bistatic cross sections are similar to those given in [29] and [30] of ref. 7.

3. Approximate solution

A high-frequency solution was obtained by Bruning and Lo (5) for the electromagnetic scattering by two identical and electrically large conducting spheres using geometric optics and modified geometric diffraction theory. The latter solution becomes more complicated and tedious in the case of two or more dielectric spheres since the creeping, reflected, and transmitted waves are multiply scattered. Recently, the authors (7) proposed an approximate method for the scattering by a linear array of \( N \) conducting spheres. The method proved to be valid for relatively small spheres, and preserves its simplicity and efficiency for the arrays considered, and yields satisfactory accuracy even for a large number of spheres.

In this paper, we extend the analysis to the scattering by a linear array of \( N \) dielectric spheres. The solution is based on the assumption that the field scattered by each dielectric sphere is due to the incident field as well as the fields scattered from the other spheres approximated by plane waves of unknown magnitudes. The two components of the scattered electric field from the \( p \)th sphere due to end-fire plane-wave incidence (\( \alpha = 0 \)) are similar to the ones in ref. 7 with the coefficients \( \Psi_{pE} \) and \( \Psi_{pM} \) given for \( p \)th dielectric sphere as

\[ \Psi_{pE}(1, n) = j^n \frac{2n + 1}{2(n+1)} \frac{j_n(\xi_{pE})(\xi_{pE})'}{-N_j^2 j_n(\xi_{pE})[p_n h_n(\xi_{pE})]'} \]

\[ \Psi_{pM}(1, n) = j^n \frac{2n + 1}{2(n+1)} \frac{j_n(\xi_{pM})(\xi_{pM})'}{-j_n(\xi_{pM})[p_n h_n(\xi_{pM})]'} \]

The total scattered field from the \( p \)th sphere is expressed as

\[ E_p(r_p, \theta_p, \phi_p) = [f_{op}(r_p, \theta_p, \phi_p)\hat{\Theta} + f_{op}(r_p, \theta_p, \phi_p)\hat{\Phi}] \left[ 1 + \sum_{q=1}^{p-1} C_q e^{-jkd_q} \right] + \sum_{q=p+1}^{N} C_q e^{jkd_q} [f_{op}(r_p, \pi - \theta_p, \phi_p)\hat{\Theta} + f_{op}(r_p, \pi - \theta_p, \phi_p)\hat{\Phi}] \]

To determine the unknown \( C_q \) in the above equation, we follow steps analogous to those that have been used for the case of scattering by a linear array of \( N \) perfectly conducting spheres (7), i.e.,

\[ (N - 1)C_p = \sum_{l=1}^{N} \sum_{i=p}^{N} e^{-jkd_{il}} \left\{ g_{pl}(d_{pl}, \theta_{pl}) \left[ 1 + \sum_{q=1}^{p-1} C_q e^{-jkd_q} \right] + g_{pl}(d_{pl}, \theta_{pl}) \sum_{q=p+1}^{N} C_q e^{jkd_q} \right\} \]

Finally, the total scattered electric field expression in the far zone is expressed in a similar way as that in the case of a system of conducting spheres.

4. Numerical results

Numerical results for the normalized backscattering cross section for a linear array of dielectric spheres of equal and unequal radii are plotted as a function of the electrical distance (\( kd \)), angle of incidence (\( \alpha \)), and dielectric constant (\( \varepsilon_r \)). The normalized bistatic cross section is presented for systems of identical spheres as a function of the scattering angle (\( \theta \)), corresponding to end-fire incidence (\( \alpha = 0 \)). In the case of end-fire incidence, the system of matrices is solved only for \( m = 1 \) owing to the rotational symmetry with respect to the \( z \) axis, while in the case of an arbitrary angle of incidence it is sufficient to consider \( m = 0, 1, 2, 3 \). The dielectric constant \( \varepsilon_r \) is the same for all spheres and equals 3.0 in most of the results.
Fig. 2. Normalized backscattering cross section versus electrical separation $kd$ for end-fire incidence and a linear array of (a) three spheres of $\varepsilon_r = 3.0$ and $ka_1 = 0.5$, $ka_2 = 0.25$, $ka_3 = 0.1$; (b) three spheres of $\varepsilon_r = 3.0$, $\varepsilon_r = \infty$, and $\varepsilon_r = 3.0$, and $ka_1 = 0.5$, $ka = 0.25$, and $ka_3 = 0.1$; (c) five equal spheres of $\varepsilon_r = 3.0$ and $ka = 0.5$; (d) five spheres of $\varepsilon_r = 3.0$ and $ka_1 = 0.5$, $ka_2 = 0.4$, $ka_3 = 0.3$, $ka_4 = 0.2$, and $ka_5 = 0.1$. (-analytic, ... approximate)

presented. Moreover, the formulation and computer program are valid for arrays containing a mixture of conducting and dielectric spheres.

Figure 2a presents the normalized backscattering cross section for a linear array of three dielectric spheres with different radii, namely, $ka_1 = 0.5$, $ka_2 = 0.25$, and $ka_3 = 0.1$, as a function of $kd$ ($1 \leq kd \leq 11$) for end-fire incidence, and compares the exact solutions (solid curve) with the approximate solutions (dotted curve). The discrepancy between the two curves occurs at $kd < 2.5$, and decreases as $kd$ increases relative to the electrical size of the spheres. On the other hand, the magnitude of the backscattering cross section varies between a minimum of 0.0283 and a maximum of 0.0481, which is of about 76.7% more than the normalized backscattered field of a single dielectric sphere of the same size (Table 1), while there is a reduction of about 6.9% relative to a similar array of conducting spheres (7). The curve behaves sinusoidally as in the case of conducting spheres with approximately half-wavelength periods. Figure 2b is another example for the same array except the second sphere is perfectly conducting. This leads to a better agreement between the two methods, even for small $kd$, along with an increase in the magnitude of the
backscattering cross section. Figure 2c consists of five spheres of equal radii \( ka = 0.5 \). We observe that the high peaks occur at specific electrical distances, i.e., \( kd = 3.14, 6.29, \) and 9.44, which is approximately every \( kd = \pi \). Again, very good agreement between the exact and approximate solutions is obtained even when the spheres are in contact. Figure 2d consist of five spheres of unequal radii, where the larger sphere has an electrical radius \( ka_1 = 0.5 \) and the other radii decrease from the largest towards the smallest sphere by an increment of 0.1. It is interesting to see that the small ripples in Fig. 2c can disappear by changing the electrical radii of the spheres in Fig. 2d while the high peaks remain at the same locations.

Figure 3a shows the normalized bistatic cross section versus the scattering angle \( \theta \) for an equispaced linear array of three spheres and end-fire incidence. The radius and separation between the successive spheres are \( ka = 0.5 \) and \( kd = 4.0 \). The agreement between the exact and approximate solutions is satisfactory except for a small variation in the forward scattering, and it can be seen that the backscattering and forward-scattering cross sections are equal in the \( E \) and \( H \) planes, as expected. In addition, a reduction of 7.1% in the backscattering is obtained relative to a similar array of conducting spheres (7). Figure 3b shows the same array with the second sphere being perfectly conducting. It can be seen from the latter case that
the ripple disappears over \(60^\circ \leq \theta \leq 120^\circ\) by replacing the dielectric sphere with a conducting sphere. Figure 3c consists of an array of five dielectric spheres. There is no significant change in the shape of the bistatic cross section relative to Fig. 3a, except an increase in the number of ripples. Figure 3d shows an array of eight spheres, the patterns in the two planes are virtually very close for \(\varepsilon_r < 35\), and the magnitude of the backscattering drops from 0.06 in Fig. 3c to 0.02 in this case.

In Figs. 4a and 4b we have plotted the normalized backscattering cross section versus the angle of incidence \(\alpha\) for an array of five and eight identical spheres with \(ka = 0.5\) for different values of \(kd\). We observe a significant increase in the oscillations of the curves by increasing \(kd\).

Figures 5a and 5b show the backscattering cross section plotted as a function of the dielectric constant \(1 \leq \varepsilon_r \leq 30\) for various values of \(kd\). The magnitude of the backscattering is zero at \(\varepsilon_r = 1.0\) and 30.0, and maximum at \(\varepsilon_r = 13.0\) for the arrays considered. This maximum is due to the interference between the field scattered by the illuminated region and the field transmitted back from the inside of the spheres. The presence of similar maxima and minima can be seen directly in the analogous, elementary case of dielectric slabs (8).

The normalized forward-scattering cross section is plotted
in Figs. 6a and 6b as a function of \( kd \) (1 < \( kd \) < 9) for various numbers of spheres, namely \( N = 3, 5, \) and 8, with \( ka = 0.5 \). The magnitude of the forward scattering is enhanced by increasing the number of spheres and also by changing the dielectric constant (\( \varepsilon_r \)) from 3 to 5. However, the forward scattering does not change significantly by varying the electrical separation and converges rapidly for large \( kd \).

Table 1 gives results for the special cases of end-fire and broadside backscattering cross sections for a linear array of \( N \) identical spheres with \( ka = 0.5, \varepsilon_r = 3.0 \).

<table>
<thead>
<tr>
<th>( kd )</th>
<th>( N )</th>
<th>( \alpha = 0^\circ ) (Endfire)</th>
<th>( \alpha = 90^\circ ) (Broadside)</th>
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<tr>
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<td>0.0369</td>
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</table>

5. Conclusions

The problem of electromagnetic-wave scattering by an arbitrary configuration of \( N \) dielectric spheres of arbitrary size, permittivity, and angle of incidence has been solved analytically by using the spherical vector wave functions and the translational addition theorem. The approximate method has been extended and compared numerically with the analytical solution for various electrical distances and numbers of spheres. The agreement between the exact and approximate methods for the normalized backscattering and bistatic cross sections is nearly excellent in all cases.

It should be pointed out that an increase in the number of spheres for a specific array leads to a reduction in the magnitude of the normalized backscattering cross section as shown in Figs. 3c and 3d, and of course an increase in the magnitude of normalized forward-scattering cross section as shown in Figs. 6a and 6b.

Of principal importance is the choice of the dielectric constant of the spheres that determines the overall backscattering and forward-scattering behaviour. On the other hand, the magnitude of the backscattering cross section is significantly modified by varying \( \varepsilon_r \) from 1 to 30 for different \( kd \) in Figs. 5a and 5b.

Finally, the authors are currently investigating the scattering by \( N \) lossy dielectric spheres, as well as dielectric-coated conducting spheres.

Acknowledgement

The authors wish to acknowledge the financial assistance of the Natural Sciences and Engineering Research Council of Canada which made this research possible.

Appendix

The translation addition theorem coefficients in [10] and [11] are given by

\[ A_{\mu \nu}(d_{pq}, \theta_{pq}, \phi_{pq}) = (-1)^{\mu} \sum_{p} a(m, n| - \mu, \nu|p)a(n, \nu, p)h_{p}^{(1)}(k_{dpq})p_{p}^{m-\mu} \cos(\theta_{pq})e^{i(m-\mu)\phi_{pq}} \]

\[ B_{\mu \nu}(d_{pq}, \theta_{pq}, \phi_{pq}) = (-1)^{\mu+1} \sum_{p} a(m, n| - \mu, \nu|p, p-1)b(n, \nu, p) \]

\[ \times h_{p}^{(1)}(k_{dpq})p_{p}^{m-\mu} \cos(\theta_{pq})e^{i(m-\mu)\phi_{pq}} \]

where

\[ a(m, n| \mu, \nu|p) = (-1)^{m+\nu}(2p + 1) \left[ \frac{(n+m)!(\nu+\mu)!(p-m-\mu)!}{(n-m)!(\nu-\mu)!(p+m+\mu)!} \right]^{1/2} \begin{pmatrix} n & \nu & p \\ m & \mu & -(m+\mu) \end{pmatrix} \begin{pmatrix} n & \nu & p \\ m & \mu & -(m+\mu) \end{pmatrix} \]

\[ a(m, n| \mu, \nu|p, q) = (-1)^{m+\nu}(2p + 1) \left[ \frac{(n+m)!(\nu+\mu)!(p-m-\mu)!}{(n-m)!(\nu-\mu)!(p+m+\mu)!} \right]^{1/2} \begin{pmatrix} n & \nu & p \\ m & \mu & -(m+\mu) \end{pmatrix} \begin{pmatrix} n & \nu & q \\ m & \mu & -(m+\mu) \end{pmatrix} \]

\[ a(n, \nu, p) = \frac{j^{\nu-n+p}}{2\nu(\nu+1)} [2\nu(\nu+1)(\nu+1)+(\nu+1)(n+\nu-p)(n+p-\nu+1) \]

\[ -\nu(n+\nu+p+2)(\nu+p-n+1) \]

\[ b(n, \nu, p) = -\frac{(2\nu+1)}{2\nu(\nu+1)} j^{\nu+p-n-1}[n+\nu+p+1](\nu+p-n)(n+p-\nu)(n+\nu-p+1)]^{1/2} \]

where

\[ \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right) \]

is the Wigner 3 - j symbol. The integer \( p \) in the summations takes the values \( n+\nu, n+\nu-2, \ldots, |n-\nu| \).