Simple Analytical Expressions for the Magnetic Field of Current Coils

Joan R. Ciric, Senior Member, IEEE

Abstract—Current coils are decomposed in straight segments of rectangular cross section, whose sides are trapezoidal in general. For curved coil portions, an appropriate number of straight segments are considered in terms of the desired accuracy. Simple formulas are derived for the contribution of such a coil segment to the resultant magnetic field by modeling the given volume current density in terms of a distribution of fictitious magnetization inside the segment volume and corresponding surface currents and magnetic charges. The expressions obtained contain only elementary functions, and computed results illustrate their efficiency with respect to existing analytical and numerical integration methods.

I. INTRODUCTION

THREE-DIMENSIONAL magnetic fields from practical current coils can be computed by using filament or thinsheet approximation algorithms, or by various methods of numerical integration in the Biot-Savart formulas [1]-[3]. A more efficient computation procedure consists of using analytical expressions for the contributions from current tubes in the form of rectangular boxes and of sections of toroid of rectangular cross section [4] in which a given volume current distribution is decomposed. However, in some special situations the coil geometry does not allow such a decomposition to be performed exactly, for instance, in the case of the saddle-shaped coils for the magnetic systems in magnetohydrodynamic energy converters. It should be remarked that even when an exact decomposition in such current tubes is possible, the field due to the circular arc segments is obtained by computing Jacobian elliptic functions and elliptic integrals, along with a numerical evaluation of another type of integral which evades an analytical treatment [4].

In order to reduce the amount of computation required for a specified accuracy, in this paper we consider straight segments only, of the same rectangular cross section but with trapezoidal sides in general, in which a coil of arbitrary geometry can approximately be decomposed, as shown in Fig. 1(a). The curved coil sections are modeled by an appropriate number of such straight segments. The contribution of each segment carrying stationary or quasi-stationary current to the resultant coil magnetic field can be expressed exactly in terms of elementary functions on the basis of a modeling method developed by the author [5]. A numerical investigation of the magnetic field from toroidal coils, as well as a comparison with existing algorithms of the same rectangular cross section but with trapezoidal coil sections are modeled by an appropriate number of such straight segments only, of the same rectangular cross section but with trapezoidal sides in general, in which a coil of arbitrary geometry can approximately be decomposed, as shown in Fig. 1(a). The curved coil sections are modeled by an appropriate number of such straight segments.

Fig. 1. Model construction. (a) The six sides $k$ of a straight segment. (b) Cross section with fictitious magnetization and surface current and charge.

II. MODELING THE CURRENT COIL

From the point of view of the magnetic field produced, a volume distribution of current density $J$ can be replaced by equivalent distributions of fictitious magnetization $M_c$ and surface densities of current $J_s$ and magnetic charge $\rho_{tc}$, with no volume magnetic charge distribution if $M_c$ is chosen such that its divergence is zero everywhere [5]. The relationship between these quantities, for a homogeneous medium of permeability $\mu$, are given by

\[
\text{curl } M_c = J
\]

\[
J_{sc} = -\hat{n}_i \times (M_{cj} - M_{ci})
\]

\[
\rho_{tc} = -\mu \hat{n}_i \cdot (M_{cj} - M_{ci})
\]

where $\hat{n}_i$ is the normal unit vector from side $i$ to side $j$ of each surface of discontinuity of $M_c$. Using such a model yields the magnetic field intensity in an unbounded, homogeneous, and linear region expressed in terms of surface integrals only, as

\[
H(r) = M_c(r) + \frac{1}{4\pi} \left[ \frac{1}{\mu} \int \frac{\rho_{tc}(r') R}{R^3} \, dS' \right. \\
+ \left. \int \frac{J_{sc}(r') \times R}{R^3} \, dS' \right] 
\]

with $r$ and $r'$ defining the position vectors of the field point and the source point, respectively, and $R = r - r'$.

Consider the coil decomposed in straight segments only, with the six side surfaces of every segment denoted in the same manner with respect to the current direction, for instance, $k = 0, 1, 2, 3, 4, 5$ as illustrated in Fig. 1, such that two sides with the...
same \( k = 0, 1, 2, 3 \), from adjacent segments have a common edge.

For a uniformly distributed current, one can choose the fictitious magnetization \( \mathbf{M}_c \) in (1) to be different from zero only inside the coil, for each segment as shown in Fig. 1(b), perpendicular to sides 1 and 3 (from 1 to 3), and increasing linearly with the distance from the side \( k = 0 \), where \( \mathbf{M}_c = 0 \), to the side \( k = 2 \), where \( \mathbf{M}_c = \mathbf{J}_a \). Equation (1) is now satisfied and \( \text{div} \mathbf{M}_c = 0 \) everywhere. According to (2) and (3), the model will have for each segment a uniform surface current of density \( J_{sc} = \mathbf{J}_a \), flowing over side 2 in the direction of \( \mathbf{J} \), and negative and positive magnetic surface charges on sides 1 and 3, respectively, of a density which increases in magnitude linearly with the distance from \( k = 0 \), where \( \rho_{sc} = 0 \), to \( k = 2 \), where \( |\rho_{sc}| = \mu \mathbf{J}_a \). On sides 4 and 5 there are positive or negative charges varying linearly from \( k = 0 \) to \( k = 2 \), according to (3); the total surface currents given by (2) on sides 4 and 5, obtained by the superposition of the contributions from adjacent segments, are equal to zero.

Therefore, the contribution of each straight segment to the resultant field of the coil can be calculated as the sum of the fields due to the uniformly distributed surface current on side 2 and to the surface charges of linearly varying densities on sides 1, 3, 4, and 5.

### III. Formulas for the Magnetic Field

To determine the contribution of a coil segment to the resultant magnetic field on the basis of the model presented in Section II, consider an arbitrary trapezoidal side surface \( k \) with a local Cartesian coordinate system as shown in Fig. 2. The side has either a uniform surface current of density \( J_{sc} \) along the positive \( z \) axis or a surface charge of a density varying linearly from \( \rho_{sc} = 0 \) at \( y = 0 \) to \( \rho_{sc} = \rho_d \) at \( y = d \). Performing the corresponding integrations in (4) yields the Cartesian components of the field intensity at any point \((x, y, z)\), in local coordinates (see the Appendix)

\[
H_x^{(k)} = \frac{J_{sc}}{4\pi} \lambda
\]

\[
H_y^{(k)} = \frac{J_{sc}}{4\pi} \gamma
\]

\[
H_z^{(k)} = 0
\]

for a side \( k = 2 \) carrying a surface current of density \( J_{sc} = \mathbf{J}_a \), and

\[
H_x^{(k)} = \frac{\rho_d}{4\pi \mu_d} (x\lambda + y\gamma)
\]

\[
H_y^{(k)} = \frac{\rho_d}{4\pi \mu_d} \left[ -y\lambda + x\gamma - \frac{z_2}{d^2} (R_1 - R_2) \right] - \frac{z_3 - z_4}{d^2} \left( R_3 - R_4 \right)
\]

\[
H_z^{(k)} = \frac{\rho_d}{4\pi \mu} \left[ \frac{R_1 - R_2}{l_{12}^2} + \frac{R_3 - R_4}{l_{34}^2} + \frac{Q_{12} \lambda_{12}}{l_{12}^2} + \frac{Q_{34} \lambda_{34}}{l_{34}^2} \right]
\]

for any side with surface charge \( k = 1, 3, 4, 5 \), where \( R_1, R_2, R_3, \) and \( R_4 \) are the distances from the field point to the vertices of the trapezoidal side considered, \( l_{12} \) and \( l_{34} \) are the lengths

of the nonparallel sides of the trapezoid (as in Fig. 2), and

\[
P_{12} = -z_2 y + dz
\]

\[
P_{34} = -(z_3 - z_4)y + d(z - z_4)
\]

\[
Q_{12} = dy + z_2 z
\]

\[
Q_{34} = dy + (z_3 - z_4)(z - z_4)
\]

\[
\lambda_{12} = \frac{R_1 l_{12} - Q_{12}}{(R_2 + l_{12}) l_{12} - Q_{12}}
\]

\[
\lambda_{34} = \frac{(R_3 + l_{34}) l_{34} - Q_{34}}{R_4 l_{34} - Q_{34}}
\]

\[
\lambda = \frac{(R_1 + z)(R_3 + z - z_3) + z_2}{(R_2 + z - z_2)(R_4 + z - z_4)}
\]

\[
\gamma = \frac{-z_2 z_3 - z_2 R_1}{x R_1 d} - \frac{-z_2 z_3 - z_2 R_2}{x R_2 d}
\]

\[
+ \frac{-z_3 z_4 - Q_{34} - l_{34}^2}{x R_3 d}
\]

\[
- \frac{-z_3 z_4 - Q_{34} - l_{34}^2}{x R_4 d}
\]

\[
\gamma = \tan^{-1} (z - z_2) (Q_{12} - l_{12}^2) - z_2 R_2^2
\]

\[
\times R_2 d
\]

\[
- \tan^{-1} (z - z_3) (Q_{34} - l_{34}^2) - (z - z_4) R_4^2
\]

\[
\times R_4 d
\]

(7)

\( \gamma \) is just the solid angle under which the trapezoidal side surface is seen from the observation point \((x, y, z)\) and is positive for \( x > 0 \) and negative for \( x < 0 \). The trapezoidal sides of the coil segment considered have a width \( d \) and a charge density \( \rho_d \) given by:

\[
d = \begin{cases} a, & \text{for } k = 1, 3 \\ b, & \text{for } k = 2 \\ a/(n_{ik}^2 + n_{kj}^2)^{1/2}, & \text{for } k = 4, 5 \\ -\mu \mathbf{J}_a, & \text{for } k = 1 \\ +\mu \mathbf{J}_a, & \text{for } k = 3 \\ \mu \mathbf{J}_a n_{ik}, & \text{for } k = 4, 5 \end{cases}
\]

(8)

where \( n_{ik} \) and \( n_{kj} \) are the cosines of the angles made by the outwardly oriented (with respect to the segment considered) normal unit vector of side \( k \) with the fictitious magnetization \( \mathbf{M}_c \) and the current density \( \mathbf{J} \), respectively.

The resultant coil magnetic field at any point in an unbounded, homogeneous, and linear region is obtained by adding the local value of the fictitious magnetization vector \( \mathbf{M}_c \neq 0 \) at observation points inside the coil segment volume) and the corresponding expressions (5) and (6) for the sides \( k = 2 \) and \( k = 1, 3, 4, 5 \), respectively, for all of the straight segments com-

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Fig. 2. Local coordinates for a side \( k \) of coil segment.
posing the coil. When crossing the surface of a segment side, the solid angle \(\gamma\) jumps by \(4\pi\) in magnitude. The continuity of the resultant field components is insured by the fact that the discontinuity in \(\gamma\) is compensated fully by the discontinuity in the fictitious magnetization \(M_p\) (see (4)), which is zero outside the coils. It should be noted that the contributions (6) from a side \((k = 4, 5)\) shared by adjacent segments to the fields due to those segments are equal, i.e., they are to be calculated only once for such a segment side and taken twice in the resultant field.

For almost all the current coils in electromagnetic devices and systems the density \(J\) throughout a coil has a direction that is everywhere parallel to a given plane. This is an important particular case of a coil geometry. For such a coil, the fictitious magnetization \(M_p\) within all the segments can be chosen to have a unique direction, namely, perpendicular to the above mentioned plane (for instance, from side 1 to side 3 for all segments, as shown in Fig. 1). In this model only three side surfaces of each coil segment contribute to the resultant field, namely, \(k = 1, 3\) with the surface charge and \(k = 2\) with the surface current. The necessary computation is thus correspondingly reduced, since there is no contribution from sides 4 and 5.

IV. INTRINSIC EXPRESSIONS

The formulas presented in Section III for a local Cartesian system of coordinates can easily be expressed in a vector form, independently of any reference frame. Using the geometry in Fig. 2 yields the following expressions for the unit vectors of the local Cartesian axes and the local coordinates:

\[
\hat{x} = (l_{12} \times \hat{z})/d
\]

\[
\hat{y} = \hat{z} \times \hat{x}
\]

\[
\hat{z} = l_{14}/l_{34}
\]

\[
x = \hat{x} \cdot R_1
\]

\[
y = \hat{y} \cdot R_1
\]

\[
z = \hat{z} \cdot R_1
\]

\[
l_p = \hat{z} \cdot l_p
\]

with

\[
l_{pq} = R_p - R_q, \quad p, q = 1, 2, 3, 4
\]

\[
d = (l_{12}^2 - l_{34}^2)^{1/2}
\]

where \(R_1, R_2, R_3,\) and \(R_4\) are the position vectors of the field point with respect to the vertices of the trapezoid. \(P_{12}\) and \(P_{34}\) in (6) and (7) are equal to twice the projections of the area vectors \(R_1 \times R_2/2\) and \(R_4 \times R_3/2\) along the normal to the trapezoidal side

\[
P_{12} = \hat{x} \cdot (R_1 \times R_2)
\]

\[
P_{34} = -\hat{x} \cdot (R_3 \times R_4)
\]

and \(Q_{12}\) and \(Q_{34}\) are the scalar products

\[
Q_{12} = R_1 \cdot l_{12}
\]

\[
Q_{34} = -R_4 \cdot l_{34}.
\]

Using these expressions allows the field components to be calculated directly in terms of the global coordinates, common to the entire coil, without being necessary to apply the general transformation of coordinates to get the local coordinates. Moreover, the local unit vectors and coordinates for sides 1, 2, and 3 of a given coil segment are related in a simple manner, as seen from Fig. 1. Thus, (10) need be used only for one of these three side surfaces, from which the local unit vectors and coordinates for the other two can readily be obtained. As well, for identical straight segments modeling a coil section of constant curvature, the local coordinates \(z_p, p = 2, 3, 4,\) and \(d\) in (5)-(7) corresponding to homologous sides are identical, and they are to be determined only once. On the other hand, the field components in global coordinates can easily be calculated from the local expressions (5) and (6) in terms of the intrinsic forms of the unit vectors in (10).

V. SPECIAL CASES

For a side surface in the shape of a parallelogram (e.g., \(k = 4, 5\)) we have

\[
z_2 = z_2 = z_4, \quad l_{12} = l_{34}.
\]

If a side has a rectangular shape, then

\[
z_2 = 0
\]

\[
z_3 = z_4
\]

\[
l_{12} = l_{34} = d
\]

and (6) and (7) reduce to

\[
H^x = \frac{\rho_d}{4\pi\mu d} (x\lambda + y\gamma)
\]

\[
H^y = \frac{\rho_d}{4\pi\mu d} [-y\lambda + x\gamma + z(\lambda_{12} + \lambda_{34}) - z_4\lambda_{34}]
\]

\[
H^z = \frac{\rho_d}{4\pi\mu d} [R_1 - R_2 + R_3 - R_4 + y(\lambda_{12} + \lambda_{34})]
\]

\[
\lambda_{12} = \ln \frac{R_1 - y}{R_3 - y}
\]

\[
\lambda_{34} = \ln \frac{R_3 + d - y}{R_4 - y}
\]

\[
\lambda = \ln \frac{(R_1 + z)(R_3 + z - z_4)}{(R_2 + z)(R_4 + z - z_4)}
\]

\[
\gamma = \tan^{-1} \frac{yz}{xR_1} - \tan^{-1} \frac{(y - d)z}{xR_3} + \tan^{-1} \frac{(y - d)(z - z_4)}{xR_4}.
\]

For a straight conductor of finite length whose sides are all rectangular, results obtained with these formulas are the same as those calculated with the expressions deduced by triple integrations in the Biot–Savart formula (for instance, as given in [4]).

In the case where the side surface in Fig. 2 is made infinitely long in the \(z\) direction, as shown in the cross section in Fig. 3, one obtains

\[
H^x = \frac{\rho_d}{2\pi\mu d} \left( x \ln \frac{R_d}{R_0} + y\gamma \right)
\]

\[
H^y = \frac{\rho_d}{2\pi\mu d} \left( -y \ln \frac{R_d}{R_0} + x\gamma' - d \right)
\]
where \( R_0 (R_0 = |R_0|) \) and \( R_d (R_d = |R_d|) \) are the "distance" vectors to the observation point from the \( z \)-axis and from the other edge of the infinite strip, respectively, and \( \gamma' \) is the angle between \( R_0 \) and \( R_d \), with \( \gamma' > 0 \) for \( x > 0 \) and \( \gamma' < 0 \) for \( x < 0 \). The expressions in (18) are identical to those used to derive closed formulas for the two-dimensional magnetic field of polygonal cross section conductors [6].

VI. DISCUSSION

The simple analytical expressions presented in Section III give the exact values of the field components for coils constructed in the form of a chain of straight segments, as shown in Fig. 1(a). For usual coils, their curved sections are replaced by an appropriate number of straight segments in terms of the desired accuracy, as illustrated for the example in Fig. 4. The relationship between the number of these straight segments and accuracy has been analyzed by using a toroidal coil having a rectangular cross section [7]. We have found that in order to obtain the best overall accuracy in computing the field, for a given number of straight segments, each toroidal section should be replaced by a straight segment of the same rectangular cross section and the same average length, i.e., of the same volume. When segments with an angular opening of 30° are used, the accuracy in the center region is 1.24%, the larger errors being at points inside the coil volume and reaching at the edges 5.26%. These values were obtained for a radial thickness of a toroidal coil equal to twice its axial thickness and to 0.4 of its mean radius. The largest error decreases when this latter ratio increases, being 3.10% for a 1.6 ratio. If the angular opening of the segment is reduced to one half, the above relative errors are approximately four times smaller. Obviously, the accuracy in the computed field improves rapidly for the observation points that are farther away from the coil conductors.

To illustrate the efficiency of the simple formulas derived in this paper, a typical tokamak field coil of rectangular cross section [4] is decomposed in segments as shown for one half of it at the indicated scale in Fig. 4. To compute the field with the accuracy indicated above, a coil consisting of four arc segments has been decomposed in 13 straight segments with an angular opening of 30° or less. The accuracy in the resultant field, due to all the coils of the system, is better since the field of an individual coil is determined more accurately at more distant observation points. The computation time is reduced on an average by a factor of nine with respect to that when using the expressions in [4] for arc segments, which in turn as an average is four times shorter than that required when applying the method of numerical integration considered in [4] and twice shorter than that for the improved numerical technique reported in [3]. It should be remarked that the computational effort is practically proportional to the number of straight segments, but, as mentioned above, the accuracy improves much faster when the number of segments increases.

VII. CONCLUSION

In order to simplify the computation of their magnetic field, the current-carrying coils of an arbitrary geometry are decomposed only in straight segments of rectangular cross section and modeled in terms of a distribution of fictitious magnetization and surface distributions of current and charge. On the basis of this modeling procedure, simple formulas have been derived for the contribution of each coil segment to the resultant field. These formulas contain only elementary functions and have been presented in such a form that a computer algorithm can be constructed in a straightforward manner. A comparison of the amount of computation required in the case of toroidal coils and of a tokamak field coil illustrates clearly the advantage of these simple expressions with respect to those for arc segments, containing elliptic functions and integrals, or to existing methods of numerical integration. The number of straight segments necessary to model curved coil sections with the desired accuracy has been determined. When a superior accuracy is needed for the field in a specified region of the coil, it is sufficient to take more segments only about that region, since the impact of the rest of the coil is less important.

The surface source models described in Section II can also be employed directly for the solution of boundary-value problems in the presence of practical coils by using a single-valued scalar potential [5], without the need to separately compute the field produced by coils in an unbounded homogeneous space. By using the same modeling technique, the analysis presented can be extended to coils of an arbitrary polygonal cross section. The corresponding formulas will be given elsewhere. On the other hand, surface models for coils of arbitrary cross section permit the computation of the field by a numerical integration over the surface instead of the usual volume integrals with the corresponding time-saving effect. Such surface source models, with no volume distribution of charge, can be constructed even for nonuniform current distributions, for instance, whenever the current density depends on only one Cartesian coordinate over the coil cross section.

APPENDIX

The quantities in (5) and (6) represent the Cartesian components of the following vector expressions:
The magnetic field of current coils can be expressed analytically as:

\[
H^{(j)} = \frac{J_c}{4\pi} \int_0^d dy' \oint_{l(y')} \frac{\vec{z} \times \vec{R}}{R^3} dz'
\]

\[
H^{(s)} = \frac{\rho_d}{4\pi \mu_0} \int_0^d y' dy' \oint_{l(y')} \frac{\vec{R}}{R^3} dz'
\]

where

\[
\vec{R} = x\hat{\vec{x}} + (y - y')\hat{\vec{y}} + (z - z')\hat{\vec{z}}
\]

\[
R = \left[ x^2 + (y - y')^2 + (z - z')^2 \right]^{1/2}
\]

\[
l_1(y') = \frac{z_2 y'}{d}
\]

\[
l_2(y') = z_4 + (z_3 - z_4) y'/d.
\]

Thus

\[
H^{(j)}_y = \frac{J_c}{4\pi} (-y I_0 - I_1)
\]

\[
H^{(j)}_z = \frac{J_c}{4\pi} x I_0
\]

\[
H^{(s)}_y = 0
\]

\[
H^{(s)}_z = \frac{\rho_d}{4\pi \mu_0} x I_1
\]

\[
I_1 = \int_0^d y' dy'
\]

\[
I_2 = \int_0^d y' dz'
\]

\[
I = \int_0^d \frac{(z_3 - z_4)}{R^3} dz'
\]

The latter integral is

\[
I = \frac{1}{x^2 + (y - y')^2} \left[ \frac{z_2}{R} \right]_{z = l(y')}^{z = l(y')}
\]

By applying the change of variables

\[
y' = y - x \tan \nu
\]

and then

\[
\tan (\nu - C) = w
\]

\[
\tan C = \frac{x(z_3 - z_4)}{P_{34}}
\]

one obtains, after performing elementary integrations,

\[
I_0 = \frac{1}{x} y
\]

\[
I_1 = y I_0 + \lambda
\]

\[
I_2 = 2y I_1 - (x^2 + y^2) I_0
\]

\[
+ d \left[ \frac{z_2}{l_{12}} (R_1 - R_2) + \frac{z_2 - z_4}{l_{34}^2} (R_3 - R_4) \right]
\]

\[
- d^2 \left( \frac{P_{12} \lambda_{12}}{l_{12}^3} + \frac{P_{34} \lambda_{34}}{l_{34}^3} \right)
\]

where the notation is that used in (7). These results, as well as the elementary integration in the last equation in (A5), yield the expressions in (5) and (6).

REFERENCES


Ioan R. Ciric (SM’84) received the Dipl.-Ing and Dr.-Ing. degrees in electrical engineering from the Polytechnic Institute of Bucharest, Romania, in 1959 and 1970, respectively. Since 1980, he has been with the University of Manitoba, Winnipeg, Canada, where he is currently Professor of Electrical Engineering. His major interests are in the mathematical modeling of stationary and quasi-stationary fields, electromagnetic fields in the presence of moving solid conductors and levitation, field theory of special electrical machines, dc corona ionized fields, analytical and numerical methods for wave scattering problems, propagation along waveguides with discontinuities, and transients. He has authored and coauthored three electrical engineering textbooks, and has been a contributor to chapters on magnetic field modeling and electromagnetic scattering by complex objects for two books.

Dr. Ciric is a past Chairman of the Winnipeg Section of the IEEE, where he also served as Secretary, Treasurer, and Vice Chairman.