Scattering by systems of spheroids in arbitrary configurations

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By expanding the incident, scattered, and transmitted electromagnetic fields in terms of appropriate vector spheroidal wave functions, an analytic solution is obtained to the problem of electromagnetic scattering by \( n \) dielectric spheroids of arbitrary orientation. The incident wave is considered to be a monochromatic uniform plane electromagnetic wave of arbitrary polarization and angle of incidence. The boundary conditions at the surface of a given spheroid are imposed by expressing the electromagnetic fields scattered by all the other \( n-1 \) spheroids in terms of the spheroidal coordinates attached to the spheroid considered, using the rotational-translational addition theorems for vector spheroidal wave functions. The solution of the associated set of algebraic equations yields the column matrix of the unknown scattered and transmitted field expansion coefficients expressed as the product of a system matrix and the column matrix of the known incident field expansion coefficients. The numerical evaluation of various matrix elements and spheroidal wave functions is presented in detail. Even though the formulation is general, the numerical results in the form of far-field scattering cross-sections are presented only for two spheroids of arbitrary orientation with various axial ratios, orientations, and center-to-center distances.

1. Introduction

The analysis of electromagnetic wave scattering by spheroids is important, as it is possible to simulate many real system objects by using spheroids of appropriate sizes. In this paper we present an exact solution for the general case of scattering by \( n \) dielectric spheroids of arbitrary orientation, which is obtained on the basis of the rotational-translational addition theorems for vector spheroidal wave functions derived by the authors [1,2], and also independently in ref. [3]. The already available analytic solutions for two spheroids of arbitrary orientation [4–7], for two spheroids in parallel configuration [8–10], for two spheres [11], and that for \( n \) perfectly conducting spheroids of arbitrary orientation, can be obtained from the corresponding solution for \( n \) dielectric spheroids, by particularization.

In section 2 we discuss the formulation and analysis of the problem and in section 3, the imposing of the boundary conditions and the derivation of the system matrix \([G_d]\). Next, as a special case, the derivation of the system matrix \([G_c]\) associated with the solution to the problem of scattering by \( n \) perfectly conducting spheroids of arbitrary orientation is presented in section 4. Given in sections 5 and 6 are the numerical computations involved in calculating the far-field scattering cross-sections, along with a brief description about the different plots obtained. Finally, the conclusions are presented in section 7.

2. Formulation and analysis of the general problem

Let us consider, in general, \( n \) prolate spheroids of arbitrary orientation, with their centers located at the origins \( O_r \) of the Cartesian coordinate systems \( O_r x_r, y_r, z_r \) \( (r = 1, 2, \ldots, n) \), respectively, and the \( r \) th coordinate system attached to the \( r \) th spheroid, as shown in fig. 1. The major axes of the spheroids are along the \( z \) axes of the respective Cartesian systems. Each of the origins \( O_r \) has spherical coordinates \( d_r \),...
The geometry of the $q$th and $r$th prolate spheroids and the associated Cartesian systems of arbitrary orientation.

Let a linearly polarized, monochromatic uniform plane electromagnetic wave with an electric field of unit amplitude be incident at an angle $\theta_i$ with respect to the $z$ axis of the system $Oxyz$, the plane of incidence being chosen as the $x$–$z$ plane ($\phi_i = 0$). The polarization angle $\gamma_i$ is the angle between the direction of the incident electric field intensity vector and the direction of the normal to the plane of incidence. For transverse electric (TE) polarization $\gamma_i$ is zero and for transverse magnetic (TM) polarization it is $\frac{1}{2}\pi$. It is assumed that the medium in which the spheroids are embedded is isotropic and nonconducting, and further that both the medium and the spheroids are nonmagnetic. A time dependence of $e^{i\omega t}$ is assumed throughout, but suppressed for convenience.

Since the direction of the incident wave vector $k$ with respect to the system $Oxyz$ is specified by the angular spherical coordinates $\theta_i, \phi_i = 0$, we have

$$k = -k (\sin \theta_i \hat{x} + \cos \theta_i \hat{z}). \quad (1)$$

If the direction of $k$ with respect to the system $O, x, y, z$, is specified by the angular spherical coordinates $\theta_i, \phi_i$, then

$$k = -k (\sin \theta_i \cos \phi_i \hat{x}_r + \sin \theta_i \sin \phi_i \hat{y}_r + \cos \theta_i \hat{z}_r). \quad (2)$$

The unit vectors $\hat{x}, \hat{y}, \hat{z}$ in $Oxyz$ can be expressed in terms of $\hat{x}_r, \hat{y}_r, \hat{z}_r$ in $O, x, y, z$, as

$$\hat{a} = c_{ax} \hat{x}_r + c_{ay} \hat{y}_r + c_{az} \hat{z}_r, \quad a = x, y, z. \quad (3)$$
with \( \alpha, \beta, \gamma \) being the Euler angles as defined in ref. [12]. Substituting \( \hat{x} \) and \( \hat{y} \) from eq. (3) in eq. (1) and identifying the corresponding coefficients of \( x, y, z \) with those in eq. (2) gives

\[
\begin{align*}
\sin \theta_i \cos \phi_i &= c_{xx} \sin \theta_i + c_{xy} \cos \theta_i, \\
\sin \theta_i \sin \phi_i &= c_{xy} \sin \theta_i + c_{yx} \cos \theta_i, \\
\cos \theta_i &= c_{zz} \sin \theta_i + c_{xz} \cos \theta_i,
\end{align*}
\]

from which \( \theta_i \) and \( \phi_i \) can be evaluated.

The incident electric field intensity \((r)E_i\) in the system \(O_{r}x_{r}y_{r}z_{r}\) can be written as

\[
(r)E_i = (r)E_{i}^{\text{TE}} \cos \gamma_k + (r)E_{i}^{\text{TM}} \sin \gamma_k,
\]

where \((r)E_{i}^{\text{TE}}\) and \((r)E_{i}^{\text{TM}}\) are the TE and TM components of \((r)E_i\), respectively. \((r)E_i\) can then be expanded in terms of spheroidal vector wave functions \((r)M\) associated with the system \(O_{r}x_{r}y_{r}z_{r}\) in the form

\[
(r)E_i = e^{-jk \cdot d} \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \left( p_{mn}^{(r)}M_{mn}^{(1)} + p_{mn}^{(r)}M_{mn}^{(-1)} + p_{mn}^{(r)}M_{mn}^{z(1)} \right),
\]

where

\[
\begin{align*}
p_{mn}^\pm &= \frac{2j^{n-1}}{kN_{mn}(h_r)} S_m(h_r, \cos \theta_i) e^{-jm\phi_i} \left[ \left( c_{xx} + j c_{xy} \right) \cos \theta_i - c_{xy} \sin \theta_i \right] \sin \gamma_k \quad \text{for } \theta_i \neq \frac{1}{2} \pi, \\
p_{mn}^z &= \frac{2j^{n-1}}{kN_{mn}(h_r)} S_m(h_r, \cos \theta_i) e^{-jm\phi_i} \left[ c_{yx} \sin \gamma_k + \left( c_{zz} \cos \theta_i \right) \left( -c_{xx} \cos \theta_i \right) \right] \sin \gamma_k \quad \text{for } \theta_i \neq \frac{1}{2} \pi,
\end{align*}
\]

\(S_{mn}(h_r, \cos \theta_i)\) and \(N_{mn}(h_r)\) are the spheroidal angle function and the normalization constant, respectively [13], and \(h_r = kF_r\), with \(F_r\) being the semi-interfocal distance of the \(r\)th spheroid. \(d_r\) is the position vector of \(O_r\) relative to \(O\). In eq. (7), the argument of each \((r)M\), which is \((h_r, \xi_r, \eta_r, \phi_r)\), with \(\xi_r, \eta_r, \phi_r\) denoting the spheroidal coordinates in the \(r\)th system, has been suppressed for convenience. If the terms in the series expansion of \((r)E_i\) are arranged in the \(\phi_r\) sequence \(e^{j0}, e^{\pm j\phi_r}, e^{\pm 2j\phi_r}, \ldots\), then we can write this expansion in a matrix form [4–7],

\[
(r)E_i = (r)M_i^{(1)\dagger} \cdot \hat{I},
\]
where \( \vec{M}_i^{(1)} \) and \( \vec{T} \) are column matrices whose elements are prolate spheroidal vector wave functions of the first kind, expressed in terms of the coordinates in the \( r \)th spheroidal system, and the corresponding known expansion coefficients, respectively. Similarly, the incident electric field intensity on any of the spheroids can be expanded in terms of the spheroidal coordinates attached to that particular spheroid.

Next, if we consider the electromagnetic field scattered by the \( q \)th spheroid which corresponds to a nonplane wave, then the scattered electric field intensity \( (q)E_s \) can be expanded in terms of a set of vector spheroidal wave functions associated with the system \( O_{q_x,y_q,z_q} \) in the form [8]

\[
(q)E_s = \sum_{m=0}^{\infty} \sum_{n=-m}^{m} \left( \beta_{mn}^{+} (q)M_{mn}^{(4)} + \beta_{m+1,n+1}^{+} (q)M_{m+1,n+1}^{(4)} \right) + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( \beta_{-m,n}^{-} (q)M_{-m,n}^{(4)} + \beta_{-(m+1),n+1}^{-} (q)M_{-(m+1),n+1}^{(4)} \right).
\]

If the terms in the expansion of \( (q)E_s \) are arranged in the \( \phi_q \) sequence \( e^{j0}, e^{\pm j\phi_q}, e^{\pm 2j\phi_q}, \ldots \), then we can write this scattered field expansion in a matrix form

\[
(q)E_s = (q)\vec{M}_s^{(4)T} \vec{\beta}.
\]

where \( (q)\vec{M}_s^{(4)} \) and \( q\vec{\beta} \) are column matrices whose elements are prolate spheroidal vector wave functions of the fourth kind, expressed in terms of the spheroidal coordinates associated with the system \( O_{q_x,y_q,z_q} \), and the corresponding unknown expansion coefficients, respectively. The structure and elements of these matrices are similar to the corresponding ones defined in refs. [8,10].

To be able to impose the boundary conditions at the surface of the \( r \)th spheroid, it is necessary to express the electromagnetic fields scattered by all the other \( n-1 \) spheroids as incoming fields to the \( r \)th spheroid. This is done by using the rotational–translational addition theorems for vector spheroidal wave functions. Let us consider first the electric field scattered by the \( q \)th spheroid \( (q)E_s \). To express this field as an incoming field to the \( r \)th spheroid, we have to express the vector spheroidal wave functions of the fourth kind associated with the system \( O_{q_x,y_q,z_q} \), in terms of vector spheroidal wave functions of the first kind associated with the system \( O_{r_x,y_r,z_r} \), using the appropriate rotational–translational addition theorems for vector spheroidal wave functions [1,2].

\[
(q)M_{mn}^{(4)}(h_q; h_r) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \left( qC_1^{(r)}M_{\mu\nu}^{(1)}(h_r; h_r) \right) + \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \left( qC_2^{(r)}M_{\mu\nu}^{(1)}(h_r; h_r) \right), \quad r_r \leq d_{qr},
\]

\[
(q)M_{mn}^{(4)}(h_q; h_r) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \left( qC_3^{(r)}M_{\mu\nu}^{(1)}(h_r; h_r) \right) + \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \left( qC_4^{(r)}M_{\mu\nu}^{(1)}(h_r; h_r) \right), \quad r_r \leq d_{qr},
\]

\[
(q)M_{mn}^{(4)}(h_q; h_r) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \left( qC_5^{(r)}M_{\mu\nu}^{(1)}(h_r; h_r) \right) + \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \left( qC_6^{(r)}M_{\mu\nu}^{(1)}(h_r; h_r) \right), \quad r_r \leq d_{qr},
\]

where \( r_q \) and \( r \) represent the coordinate triads \((\xi_q, \eta_q, \phi_q)\) and \((\xi_r, \eta_r, \phi_r)\), respectively. \( d_{qr} \) is the position
vector of \( O_r \) relative to \( O_q \) (see fig. 1) and \( \alpha_{qr}, \beta_{qr}, \gamma_{qr} \) are the Euler angles which describe the rotation of the system \( O_x,y,z, \) relative to \( O_{x_q,y_q,z_q} \).

\[
q^rC_1 = \frac{1}{2} \left[ (q^rC_{xx} + q^rC_{yy}) + j(q^rC_{xy} - q^rC_{yx}) \right], \quad q^rC_2 = \frac{1}{2} \left[ (q^rC_{xx} - q^rC_{yy}) + j(q^rC_{xy} + q^rC_{yx}) \right], \\
q^rC_3 = \frac{1}{2} \left( q^rC_{xx} + j q^rC_{xy} \right), \quad q^rC_4 = q^rC_{xz} - j q^rC_{yz}, \quad q^rC_5 = q^rC_{zz},
\]

(16)

with the asterisk denoting the complex conjugate. \( q^rC_{xx}, q^rC_{xy}, \ldots \) are obtained from \( c_{xx}, c_{xy}, \ldots \) (see eq. (4)), respectively, by replacing \( \alpha, \beta, \gamma \) by \( \alpha_{qr}, \beta_{qr}, \gamma_{qr} \), respectively. \( \alpha_{qr}, \beta_{qr}, \gamma_{qr} \) are the rotational-translational coefficients in the expansion of scalar spheroidal wave functions of the fourth kind associated with \( O_{x_q,y_q,z_q} \) in terms of the same functions of the first kind associated with \( O_x,y,z, \) for \( r,.,.; dq,., \) and are given in ref. [14]. Details regarding the computation of these coefficients are given in appendix A. By arranging the terms in the series expansions (13)–(15), in the \( \phi_r \) sequence \( e^{i0}, e^{\pm i<\phi}, e^{\pm 2i<\phi}, \ldots \), we can express the outgoing vector wave functions associated with \( O_{x_q,y_q,z_q} \), \( (q)M 5 < 4 > \), in terms of incoming vector wave functions associated with \( O_x,y,z, \) \( (qr)M(l) > \), in the form

\[
(q^rM_{s}^{(4)}) = [\Gamma_{qr}] (qr)^{(1)}M^{(1)},
\]

(17)

in which the structure and elements of the matrix \( [\Gamma_{qr}] \) are defined in ref. [15]. The transpose of \( (qr)^{(1)}M^{(1)} \) is

\[
(qr)^{(1)}M^{(1)T} = \left[ (qr)^{(1)T} (q^rM_0^{(1)T}) (qr)^{(1)^{(1)}T} (qr)^{(2)^{(1)}T} \ldots \right],
\]

(18)

where

\[
(qr)^{(1)T} = \left[ \begin{array}{cccc}
q^rM_{-1}^{(1)T} & q^rM_{-1}^{(1)T} & q^rM_{0}^{(1)T} & q^rM_{0}^{(1)T} \\
(q^rM_{-1}^{(1)T})^T & (q^rM_{-1}^{(1)T})^T & (q^rM_{0}^{(1)T})^T & (q^rM_{0}^{(1)T})^T \\
(q^rM_{-1}^{(1)T})^T & (q^rM_{-1}^{(1)T})^T & (q^rM_{0}^{(1)T})^T & (q^rM_{0}^{(1)T})^T \\
(q^rM_{-1}^{(1)T})^T & (q^rM_{-1}^{(1)T})^T & (q^rM_{0}^{(1)T})^T & (q^rM_{0}^{(1)T})^T \\
\end{array} \right],
\]

(19)

for \( \sigma \geq 1 \). Denoting the secondary incident field on the \( r \)th spheroid due to \( (q)E_s \) by \( (qr)E_s \), taking the transpose of both sides of eq. (17) and then substituting \( (qr)^{(1)}M^{(1)} \) in eq. (12) gives

\[
(qr)^{(1)}E_s = (qr)^{(1)T} [\Gamma_{qr}]^T q^r\bar{E}.
\]

(20)

Thus for \( q = 1, 2, \ldots, r - 1, r + 1, \ldots, n \), we get the secondary incident electric fields on the \( r \)th spheroid, due to the electric fields scattered by each of the spheroids 1, 2, \ldots, \( r - 1, r + 1, \ldots, n \).

Due to the incidence of all the secondary fields and the field corresponding to the incident plane wave, the spheroid \( r \) scatters an electric field \( (r)E_s \) which can also be expanded similar to \( (qr)^{(1)}E_s \) as

\[
(r)E_s = (r)^{(1)}M_s^{(1)T} r \bar{E}.
\]

(21)

The column matrices \( (r)^{(1)}M_s^{(1)} \) and \( r \bar{E} \) have the same form as those of \( (qr)^{(1)}M_s^{(1)} \) and \( q^r\bar{E} \), respectively, with the vector wave functions evaluated with respect to the \( r \)th spheroidal coordinate system.

The electromagnetic field transmitted inside the \( r \)th spheroid also corresponds to a nonplane wave whose electric field intensity \( (r)E_t \) can be expanded in terms of a set of vector spheroidal wave functions as [10]

\[
(r)E_t = (r)^{(1)T} r \bar{\alpha},
\]

(22)

where \( (r)^{(1)}M_t^{(1)} \) and \( r \bar{\alpha} \) are column matriceswhose elements are prolate spheroidal vector wave functions of the first kind, expressed in terms of spheroidal coordinates associated with \( O_x,y,z, \) taking into account the permittivity of the material inside the \( r \)th spheroid, and the corresponding unknown coefficients in the series expansion, respectively.
Using Maxwell's equation

\[ H = j k^{-1} (\epsilon / \mu)^{1/2} \nabla \times E, \tag{23} \]

where \( \epsilon \) and \( \mu \) are the permittivity and the permeability, respectively, the expansions of the different magnetic \((H)\) fields in terms of appropriate vector spheroidal wave functions can be obtained from those of the corresponding electric \((E)\) fields. This is done by replacing \( M \) by \( N \), where \( N = k^{-1} (\nabla \times M) \), and multiplying each expansion by the appropriate value of \( j(\epsilon / \mu)^{1/2} \). Thus we have

\[ (r)H_i = j(\epsilon / \mu_0)^{1/2} (r)N_i^{(1)T} \bar{T}, \tag{24} \]
\[ (qr)H_s = j(\epsilon / \mu_0)^{1/2} (qr)N_i^{(1)T} [\Gamma_{qr}]^T q \bar{\beta}, \tag{25} \]
\[ (r)H_s = j(\epsilon / \mu_0)^{1/2} (r)N_i^{(1)T} r \bar{\beta}, \tag{26} \]
\[ (r)H_t = j(\epsilon / \mu_0)^{1/2} (r)N_i^{(1)T} r \alpha, \tag{27} \]

in which \( \epsilon \) and \( \epsilon_r \) are the permittivities of the media outside and inside the \( r \)th spheroid, and \( \mu_0 \) is the permeability of free space. The elements of the matrices \( (r)N_i^{(1)T}, (qr)N_i^{(1)T}, (r)N_i^{(4)T}, \) and \( (r)N_i^{(1)T} \) can be obtained from the corresponding elements of the matrices \( (r)M_i^{(1)T}, (qr)M_i^{(1)T}, (r)M_i^{(4)T}, \) and \( (r)M_i^{(1)T} \), respectively, by replacing the vector spheroidal wave functions \( M \) by \( N \).

3. Boundary conditions

The boundary conditions require that on the surface of each dielectric spheroid \( \xi_r = \xi_{rs} \) \((r = 1, 2, \ldots, n)\), the tangential components of both \( E \) and \( H \) fields be continuous across the boundary. Thus considering the \( r \)th spheroid we can write

\[ \left( (r)M_i^{(1)T} \right) \bar{T} + \sum_{q=1}^{n} \left[ (1 - \delta_{qr}) (qr)M_i^{(1)T} [\Gamma_{qr}]^T + \delta_{qr} (q)M_i^{(4)T} \right] q \bar{\beta} \times \hat{\xi}_r |_{\xi_r = \xi_{rs}}, \tag{28} \]

\[ = \left( (r)M_i^{(1)T} \right) \bar{\alpha} \times \hat{\xi}_r |_{\xi_r = \xi_{rs}}, \]

\[ \left( (r)N_i^{(1)T} \right) \bar{T} + \sum_{q=1}^{n} \left[ (1 - \delta_{qr}) (qr)N_i^{(1)T} [\Gamma_{qr}]^T + \delta_{qr} (q)N_i^{(4)T} \right] q \bar{\beta} \times \hat{\xi}_r |_{\xi_r = \xi_{rs}}, \tag{29} \]

\[ = \left( \epsilon / \epsilon_r \right)^{1/2} \left( (r)N_i^{(1)T} \right) \bar{\alpha} \times \hat{\xi}_r |_{\xi_r = \xi_{rs}}, \]

where \( \delta_{qr} \) is the Kronecker delta function. For \( r = 1, 2, \ldots, n \), we obtain \( 2n \) such equations in total after imposing the boundary conditions on the surfaces of all the \( n \) spheroids. Taking the scalar product of both sides of eqs. (28) and (29) by

\[ \left\langle T_{\eta_r} \right \rangle S_m, h_{\eta_r} (h, \eta) e^{\pm j (m \pm 1) \phi_r}, \]

for \( r = 1, 2, \ldots, n, \ m = \ldots -2, -1, 0, 1, 2, \ldots, \kappa = 0, 1, 2, \ldots, \) integrating correspondingly over the surfaces of the \( n \) spheroids, and using the orthogonality properties of the trigonometric and spheroidal angle functions, yields after rearranging [15]

\[ [G_d] \hat{S}_d = [R_d] \hat{T}, \tag{30} \]
where

$$
\begin{bmatrix}
[ P_{M1} ] [ 0 ] \ldots [ 0 ] [ Q_{M1} ] [ R_{M21}] [ I_{21}]^T \ldots [ R_{Mn1}] [ I_{n1}]^T \\
[ P_{N1} ] [ 0 ] \ldots [ 0 ] [ Q_{N1} ] [ R_{N21}] [ I_{21}]^T \ldots [ R_{Nn1}] [ I_{n1}]^T \\
[ 0 ] [ P_{M2} ] \ldots [ 0 ] [ R_{M21}] [ I_{12}]^T [ Q_{M2} ] \ldots [ R_{Mn2}] [ I_{n2}]^T \\
[ 0 ] [ P_{N2} ] \ldots [ 0 ] [ R_{N12}] [ I_{12}]^T [ Q_{N2} ] \ldots [ R_{Nn2}] [ I_{n2}]^T \\
[ 0 ] [ 0 ] \ldots [ R_{Mn1}] [ I_{1n}]^T [ R_{Mn2}] [ I_{2n}]^T \ldots [ Q_{Mn}] \\
[ 0 ] [ 0 ] \ldots [ R_{N1n}] [ I_{1n}]^T [ R_{N2n}] [ I_{2n}]^T \ldots [ Q_{Nn}] \\
\end{bmatrix}
$$

$$
\left[ \begin{array}{c}
\frac{1}{\alpha} \\
\frac{2}{\alpha} \\
\vdots \\
\frac{n}{\alpha} \\
\frac{1}{\beta} \\
\frac{2}{\beta} \\
\vdots \\
\frac{n}{\beta} \\
\end{array} \right], \quad \left[ R_d \right] = \left[ \begin{array}{c}
[ R_{M1} ] [ 0 ] \ldots [ 0 ] \\
[ R_{N1} ] [ 0 ] \ldots [ 0 ] \\
[ 0 ] [ R_{M2} ] \ldots [ 0 ] \\
[ 0 ] [ R_{N2} ] \ldots [ 0 ] \\
[ 0 ] [ 0 ] \ldots [ R_{Mn} ] \\
[ 0 ] [ 0 ] \ldots [ R_{Nn} ] \\
\end{array} \right], \quad \vec{I} = \left[ \begin{array}{c}
\frac{1}{I} \\
\frac{2}{I} \\
\vdots \\
\frac{n}{I} \\
\end{array} \right].
$$

The coefficients $\ell_\eta$ and $\ell_\phi$ used in eq. (28) are given by $\ell_\eta = j2F_\eta(\xi_{rs}^2 - \eta_r^2)^{1/2}$, $\ell_\phi = 2F_\phi(\xi_{rs}^2 - \eta_r^2)$ and those used in eq. (29) by $\ell_\eta = 2F_\eta^2(\xi_{rs}^2 - \eta_r^2)^{1/2}/(\xi_{rs}^2 - 1)^{1/2}$, $\ell_\phi = j2F_\phi^2(\xi_{rs}^2 - \eta_r^2)/(\xi_{rs}^2 - 1)$. $\xi_{rs}$ is the value of $\xi_r$ on the surface of the $r$th spheroid. Definitions of all the matrices are given in appendix B.

Equation (30) can now be written in the form

$$
\vec{S}_d = [ G ] \vec{I},
$$

where

$$
[G] = [G_d]^{-1} [ R_d ]
$$

is the system matrix which is independent of the direction and polarization of the incident wave. The matrix form (33) gives the coefficients in the expansion of the electromagnetic fields scattered and transmitted by the $n$ arbitrarily oriented spheroids. Because of the above mentioned special feature of the system matrix, it is possible to calculate the unknown coefficients for a different direction and/or polarization of the incident wave, without repeatedly solving a new set of algebraic equations.

### 4. Case of perfectly conducting spheroids

The solution for the case of $n$ perfectly conducting spheroids can be derived from the one for $n$ dielectric spheroids, by letting the permittivity of each of the $n$ dielectric spheroids become very high (theoretically infinite). In this case since the spheroids cannot sustain any field inside them, the boundary conditions require that the tangential component of the resultant electric field be zero on the surface of each of the $n$ spheroids. Hence if we consider the $r$th spheroid we can write

$$
\left( \vec{M}_{i(1)} \right)^T \vec{I} + \sum_{q=1}^{n} \left( (1 - \delta_{qr}) \left[ (q \vec{r}) \vec{M}_{s(q)} \right]^T \left[ \vec{I}_{qr} \right] + \left[ (q \vec{r}) \vec{M}_{s(q)} \right]^T \vec{I}_{qr} \left( (q \vec{r}) \vec{M}_{s(q)} \right) \right) = 0.
$$
For \( r = 1, 2, \ldots, n \), we obtain \( n \) such equations after imposing the boundary conditions on the surfaces of all the \( n \) spheroids. Following a procedure identical to that described in the previous section, we can finally obtain a system of algebraic equations which could be written in matrix form as

\[
[G_c]\vec{S}_c = [R_c]\vec{I},
\]

where

\[
[G_c] = \begin{bmatrix}
[Q_{M1}] & [R_{M21}][T_{21}]^T & \cdots & [R_{M1n}][T_{1n}]^T \\
[R_{M12}][T_{12}]^T & [Q_{M2}] & \cdots & [R_{M2n}][T_{2n}]^T \\
\vdots & \vdots & \ddots & \vdots \\
[R_{M1n}][T_{1n}]^T & [R_{M2n}][T_{2n}]^T & \cdots & [Q_{Mn}]
\end{bmatrix},
\]

\[
\vec{S}_c = \begin{bmatrix}
\vec{1} \\
\vec{2} \\
\vdots \\
\vec{n}
\end{bmatrix}, \quad [R_c] = \begin{bmatrix}
[Q_{M1}] & [0] & \cdots & [0] \\
[0] & [R_{M2}] & \cdots & [0] \\
\vdots & \vdots & \ddots & \vdots \\
[0] & [0] & \cdots & [R_{mn}]
\end{bmatrix}, \quad \vec{I} = \begin{bmatrix}
\vec{1} \\
\vec{2} \\
\vdots \\
\vec{n}
\end{bmatrix}.
\]

Equation (36) can be rearranged and written as

\[
\vec{S}_c = [G']\vec{I}
\]

in which

\[
[G'] = [G_c]^{-1}[R_c].
\]

Similar to \([G]\), the system matrix \([G']\) in this case is also independent of the direction and polarization of the incident wave. However, the size of \([G']\) is half of that of \([G]\). The solution for the case of imperfectly conducting spheroids can be obtained by incorporating the surface impedance in the boundary conditions. For the case of a mixture of dielectric and perfectly conducting spheroids of arbitrary orientation, the solution can be obtained from that for the dielectric spheroids, by considering the permittivity of the perfectly conducting spheroids as being infinite. Both the size of the system matrix and the computational time increases with the number of spheroids. In this paper numerical computations associated with the above analytical formulation are illustrated for the case of a system of two spheroids of arbitrary orientation, with the incident wave being a monochromatic uniform plane electromagnetic wave of arbitrary polarization and angle of incidence.

5. Normalized scattering cross-sections

Consider two spheroids A and B with the Cartesian system \( Oxyz \) attached to the spheroid A, \( O'x'y'z' \) attached to the spheroid B, and a point of observation having spherical coordinates \( r, \theta, \phi \) and \( r', \theta', \phi' \) with respect to the two systems \( Oxyz \) and \( O'x'y'z' \), respectively, as shown in fig. 2. The spheroidal coordinates associated with the two systems are given by \( \xi, \eta, \phi \) and \( \xi', \eta', \phi' \), respectively. Using the
asymptotic expressions of different vector spheroidal wave functions, the electric field intensity in the far zone can be written as

\[
E_s = E_s^A + E_s^B
\]

\[
= \frac{e^{-jkR}}{kr} \left[ F_{\theta A}(\theta, \phi) \hat{\theta} + F_{\phi A}(\theta, \phi) \hat{\phi} + F_{\theta B}(\theta', \phi') \hat{\theta}' + F_{\phi B}(\theta', \phi') \hat{\phi}' \right]
\]

\[
= \frac{e^{-jkR}}{kr} \left[ F_{\theta A}(\theta, \phi) \hat{\theta} + F_{\phi A}(\theta, \phi) \hat{\phi} + F_{\theta B}(\theta', \phi') \{ g_1 \hat{\theta} + g_2 \hat{\phi} \} + F_{\phi B}(\theta', \phi') \{ g_3 \hat{\theta} + g_4 \hat{\phi} \} \right]
\]

\[
= \frac{e^{-jkR}}{kr} \left[ F_{\theta}(\theta, \phi) \hat{\theta} + F_{\phi}(\theta, \phi) \hat{\phi} \right],
\]

where

\[
F_{\theta}(\theta, \phi) = F_{\theta A}(\theta, \phi) + F_{\theta B}(\theta, \phi), \quad F_{\phi}(\theta, \phi) = F_{\phi A}(\theta, \phi) + F_{\phi B}(\theta, \phi),
\]

\[
\begin{bmatrix} g_1 & g_2 \end{bmatrix}^T = [\Omega] \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} \cos \theta' \cos \phi' & \cos \theta' \sin \phi' & -\sin \theta' \end{bmatrix}^T,
\]

\[
\begin{bmatrix} g_3 & g_4 \end{bmatrix}^T = [\Omega] \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} -\sin \phi' \cos \phi' & 0 \end{bmatrix}^T,
\]
with

$$
F_{\theta A}(\theta, \phi) = - \sum_{m=0}^{\infty} \sum_{n=-m}^{m} j^{n+1} \frac{S_{mn}}{2} \left\{ (\alpha_{mn}^+ - \alpha_{mn}^-) \cos(m+1)\phi + j(\alpha_{mn}^+ + \alpha_{mn}^-) \sin(m+1)\phi \right\}
- \sum_{n=1}^{\infty} j^{n+1} \frac{S_{1n}}{2} \alpha_{-1n}^-,
$$
(45)

$$
F_{\phi A} = \sum_{m=0}^{\infty} \sum_{n=-m}^{m} j^{n} \left[ \cos \theta \frac{S_{mn}}{2} \left\{ (\alpha_{mn}^+ + \alpha_{mn}^-) \cos(m+1)\phi + j(\alpha_{mn}^+ - \alpha_{mn}^-) \sin(m+1)\phi \right\} + j \sin \theta \frac{S_{m+1,n+1}}{2} (\alpha_{m+1,n+1}^\prime + \alpha_{m+1,n+1}^-) \cos(m+1)\phi 
+ j(\alpha_{m+1,n+1}^\prime - \alpha_{m+1,n+1}^-) \sin(m+1)\phi \right] \right\} + \cos \theta \sum_{n=1}^{\infty} j^{n} \frac{S_{1n}}{2} \alpha_{-1n}^- - \sin \theta \sum_{n=1}^{\infty} j^{n} S_{0n} \alpha_{0n}^\prime,
$$
(46)

and

$$
[\Omega] = \begin{bmatrix}
\cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta
\end{bmatrix}, \quad [C] = \begin{bmatrix}
c_{xx}^\prime & c_{xy}^\prime & c_{xz}^\prime \\
c_{yx}^\prime & c_{yy}^\prime & c_{yz}^\prime \\
c_{zx}^\prime & c_{zy}^\prime & c_{zz}^\prime
\end{bmatrix},
$$
(47)

in which

$$
\alpha_{mn}^\pm = k \alpha_{mn}^\pm, \quad \alpha_{mn}^\pm = k \alpha_{mn}^\pm, \quad S_{mn} = S_{mn}(h, \cos \theta).
$$
(48)

\(\alpha_{mn}^\pm\) and \(\alpha_{mn}^\pm\) are the coefficients in the series expansion of the electric field scattered by spheroid A in terms of vector spheroidal wave functions in the unprimed system, which are evaluated beforehand by solving the set of algebraic equations

$$
\begin{bmatrix}
[Q_{MA}] \\
[R_{MAB}]^T
\end{bmatrix} \begin{bmatrix}
[\bar{\alpha}]
\end{bmatrix} = \begin{bmatrix}
[\bar{\beta}]
\end{bmatrix} = \begin{bmatrix}
[R_{MA}] \\
[0]
\end{bmatrix} \begin{bmatrix}
[\bar{I}_A]
\end{bmatrix}
$$
(49)

for the perfectly conducting case, which has the form of eq. (36) for \(n = 2\), with \(\bar{\beta} \) and \(\bar{\gamma} \) denoted here \(\bar{\alpha} \) and \(\bar{\beta} \), respectively, and

$$
\begin{bmatrix}
P_{MA} \\
[0]
\end{bmatrix} \begin{bmatrix}
[Q_{MA}] \\
[R_{MBA}]^T
\end{bmatrix} \begin{bmatrix}
[\bar{\gamma}]
\end{bmatrix} = \begin{bmatrix}
[\bar{\delta}]
\end{bmatrix} = \begin{bmatrix}
P_{NA} \\
[0]
\end{bmatrix} \begin{bmatrix}
[Q_{NA}] \\
[R_{NBA}]^T
\end{bmatrix} \begin{bmatrix}
[\bar{\alpha}]
\end{bmatrix} = \begin{bmatrix}
P_{MB} \\
[0]
\end{bmatrix} \begin{bmatrix}
[Q_{MB}] \\
[R_{MB}]^T
\end{bmatrix} \begin{bmatrix}
[\bar{\beta}]
\end{bmatrix} = \begin{bmatrix}
P_{NB} \\
[0]
\end{bmatrix} \begin{bmatrix}
[Q_{NB}] \\
[R_{NB}]^T
\end{bmatrix} \begin{bmatrix}
[\bar{\gamma}]
\end{bmatrix}
$$
(50)

for the dielectric case, which has the form of eq. (30) for \(n = 2\), with \(\bar{\alpha} \), \(\bar{\beta} \), \(\bar{\gamma} \), \(\bar{\delta} \), \(\bar{\alpha} \), and \(\bar{\beta} \), respectively. These coefficients are obtained by imposing the appropriate boundary conditions at the surface of each of the spheroids. \(c_{ax}^\prime \), \(c_{ay}^\prime \), and \(c_{az}^\prime \) are obtained from \(c_{ax} \), \(c_{ay} \), and \(c_{az} \), respectively, for \(a = x, y, z\), by replacing \(\alpha_{r} \), \(\beta_{r} \), \(\gamma_{r} \), by \(\alpha_{0} \), \(\beta_{0} \), \(\gamma_{0} \), respectively, which specify the rotation of the spheroid B relative to spheroid A. The explicit expressions of \(F_{\phi B}(\theta', \phi')\), and \(F_{\phi B}(\theta', \phi')\) in primed coordinates are obtained from those of \(F_{\phi A}(\theta, \phi)\) and \(F_{\phi A}(\theta, \phi)\), respectively, by replacing \(\alpha\) by the corresponding \(\beta\) in eqs. (49) and (50), and multiplying each expression by an overall phase factor \(e^{ik_s d}\), with \(k_s\) given by

$$
k_s = k (\sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}).
$$
(51)
The expressions of $F_{\theta'}(\theta, \phi)$ and $F'_{\phi'}(\theta, \phi)$ are obtained from those of $[g_1 F_{\theta'}(\theta', \phi') + g_3 F_{\phi'}(\theta', \phi')]$ and $[g_2 F_{\theta'}(\theta', \phi') + g_4 F_{\phi'}(\theta', \phi')]$, respectively, by substituting all the functions in primed variables $\theta'$, $\phi'$ in terms of the unprimed variables $\theta, \phi$.

The bistatic radar cross section is defined as

$$\sigma(\theta, \phi) = \lim_{r \to \infty} 4\pi r^2 \frac{|E_s \cdot \hat{\tau}|^2}{|E_i|^2},$$

with the unit vector $\hat{\tau}$ denoting the direction of polarization of the receiver at the point of observation. When $\hat{\tau}$ has the same direction as $E_s$, the normalized bistatic cross-section is

$$\frac{\pi \sigma(\theta, \phi)}{\lambda^2} = |F_{\theta}(\theta, \phi)|^2 + |F_{\phi}(\theta, \phi)|^2.$$

The normalized bistatic cross-sections in the $E$- and $H$-planes are obtained by substituting $\phi = \pi/2$ and $\phi = 0$, respectively, in eq. (53).

The normalized backscattering cross-section is obtained from eq. (53) for $\theta = \theta_i$ and $\phi = \phi_i = 0$,

$$\frac{\pi \sigma(\theta_i)}{\lambda^2} = |F_{\theta}(\theta_i, 0)|^2 + |F_{\phi}(\theta_i, 0)|^2.$$

### 6. Numerical results for two spheroids of arbitrary orientation

Results of numerical computation are presented in the form of plots of normalized bistatic and backscattering cross-sections in the far field for scattering by two spheroids having various displacements of their centers and different relative orientations. As the formulation and the computation for the case of scattering by two perfectly conducting spheroids is much simpler than for the case of scattering by two dielectric spheroids, the plots for the perfectly conducting spheroids will be presented first, followed by those for the dielectric spheroids.

#### 6.1. Perfectly conducting spheroids

Since the series expansions of the different electromagnetic fields in terms of vector spheroidal wave functions are infinite in extent, all the matrices introduced in sections 2, 3, and 4 are of infinite dimensions. Thus, in order to obtain numerical results it is necessary to truncate these matrices according to the required accuracy. Detailed numerical experiments were performed on the eqs. (13)–(15) describing the rotational–translational addition theorems for vector spheroidal wave functions. These experiments show that for major axes of the spheroids in the range of magnitude considered, it is sufficient to consider $-2 \leq \mu \leq 2$ and $\nu = |\mu|$, $|\mu| + 1, ..., |\mu| + 5$ on the right hand sides of the equations in order to obtain a two significant digit accuracy when compared with the values of the corresponding left hand sides, for different values of $m$ and $n$ [15]. All the vector spheroidal wave functions and the rotational–translational coefficients have been calculated with a five significant digit accuracy. When using these functions and coefficients in our calculations, it has been found that to obtain a two significant digit accuracy in the computed bistatic and backscattering cross-sections, it is sufficient to consider only the $\phi$-harmonics $e^{j0}$, $e^{\pm j\phi}$, and $e^{\pm 2j\phi}$. All the results given in this section have thus been obtained with $m$ corresponding to the above $\phi$-harmonics, $n = |m|, ..., |m| + 3$, $\kappa = 0, 1, 2, 3$, in truncating the matrices $[Q_{MA}]$, $[Q_{MB}]$, $[R_{MA}]$, and $[R_{MB}]$, and with $n = |m|, ..., |m| + 5$, $\kappa = 0, 1, 2, 3$, in truncating the matrices $[T_{AB}]$, $[T_{BA}]$, $[R_{MAB}]$, and $[R_{MBA}]$ in eq. (49).
The formulation presented in section 2 is general. However, it is interesting to note that for a particular system of only two spheroidal objects, the relative position of one with respect to the other can always be obtained by choosing the $x$ and $y$ axes appropriately, and then by performing only one rotation through the Euler angle $\beta_0$, i.e. with $\alpha_0 = 0^\circ$, $\gamma_0 = 0^\circ$, followed by the corresponding translation. Thus, the results presented in this section are with $\alpha_0$ and $\gamma_0$ equal to zero. To demonstrate the generality of the theory and the validity of the software being used, the results presented in the next section for dielectric spheroids are obtained with $\alpha_0$ and $\gamma_0$ different from zero. However, the reduction in the total amount of computation time required when using $\alpha_0 = 0^\circ$, $\gamma_0 = 0^\circ$ is only about 5% with respect to the case when $\alpha_0$ and $\gamma_0$ are different from zero.

Figure 3 shows the normalized bistatic cross-section as a function of the scattering angle, for two identical sets of prolate spheroids of axial ratios 2 and 10, semi-major axes $\lambda/4$, with the spheroid centers displaced along the $z$ axis of spheroid A. The orientation of spheroid B with respect to A is specified by the Euler angles $\alpha_0 = 0^\circ$, $\beta_0 = 45^\circ$, $\gamma_0 = 0^\circ$. The incident field propagates along the negative $z$ axis, as shown in the figure. In fig. 3a the center-to-center distance is $\lambda/2$ and in fig. 3b, it is $\lambda$. When the axial ratio changes from 2 to 10, a significant decrease in the magnitude of the bistatic cross-section is visible in both $E$- and $H$-plane patterns which is partly due to the reduction of the area available for scattering. When the distance between the centers of the spheroids is increased from $\lambda/2$ to $\lambda$ we observe that the scattering cross-sections are subject to more oscillations in general due to the interference pattern of the two spheroids. In fig. 4 the plots of normalized backscattering cross-section versus angle of incidence are given for TE and TM polarizations of the incident wave. The spheroids are identical to those in fig. 3, and so are the orientation and center-to-center distances between the spheroids. It is interesting to note that the behavior of the backscattering cross-sections is almost the same for both polarizations, when the axial ratio of the spheroids is 2.

### 6.2 Two dielectric spheroids

In this section we present the results for scattering by two dielectric spheroids of arbitrary orientation. When computing the numerical results for this case, again we have found that in order to obtain a two significant digit accuracy in the computed bistatic and backscattering cross-sections, it is sufficient to consider only the $\phi$-harmonics $e^{i\phi}$, $e^{i\phi}$, and $e^{i\phi}$. Thus the values of $m$ being used in the truncation of the associated matrices in eq. (50) remain the same as in the perfectly conducting case, but the values of $n$ and $\kappa$ are now given by $n = |m|$, $|m| + 1$, …, $|m| + 5$, and $\kappa = 0, 1, \ldots, 5$.

Figure 5 shows the normalized bistatic cross section for TE polarization of the incident wave versus the scattering angle for two identical prolate spheroids of axial ratio 5, dielectric constant $\epsilon_r = 3.0$, with the spheroid centers displaced along the $z$ axis of spheroid A. The orientation of spheroid B with respect to A is specified by the Euler angles $\alpha_0 = 30^\circ$, $\beta_0 = 45^\circ$, $\gamma_0 = 60^\circ$. The incident field propagates along the negative $z$ axis. It should be noted that the geometries of the systems of spheroids considered are similar to those shown for the perfectly conducting case. These are therefore not shown again with the figures. In fig. 5a the center-to-center distance is $\lambda/2$ and in fig. 5b, it is $\lambda$. Here we observe that the magnitude of the forward scattering cross-section ($\theta = \pi$) is higher than that of the backscattering cross-section ($\theta = 0$) for both plots. This is partly due to the contribution to the forward scattered field from the field transmitted inside the spheroid.

The variation of the normalized backscattering cross-section with the angle of incidence is shown in fig. 6, for the general case of two nonidentical spheroids, separated center-to-center by a distance $\lambda/2$ in the direction specified by the spherical coordinates $\theta_0 = 60^\circ$, $\phi_0 = 20^\circ$. The orientation of the spheroid B with respect to A is given by the Euler angles $\alpha_0 = 30^\circ$, $\beta_0 = 45^\circ$, $\gamma_0 = 60^\circ$. The dielectric constant for both spheroids is 3.0. In this case we see that the behavior of the curves for both TE and TM polarizations is almost the same, with minima around $\theta_i = 125^\circ$. 
Fig. 3. Normalized bistatic cross-section for TE polarization of the incident wave, as a function of the scattering angle for two identical prolate spheroids and for two axial ratios, with $a_A = a_B = \frac{1}{2} \lambda$, Euler angles $\alpha_0 = 0^\circ$, $\beta_0 = 45^\circ$, $\gamma_0 = 0^\circ$, and displaced along the $z$ axis: (a) $d = \frac{1}{2} \lambda$; (b) $d = \lambda$.

It should be noted that the results for a system of perfectly conducting spheroids can be obtained as a special case from the corresponding results for a system of dielectric spheroids for $\varepsilon_r \to \infty$. When the plots of backscattering cross-section for the dielectric case are compared with the corresponding ones for the perfectly conducting case, we observe that the magnitudes of the backscattering cross-sections in the dielectric case are lower. This is due to the fact that a part of the incident field is now being transmitted inside the spheroid, without being scattered.

To show further the applicability of the general software to limiting cases of eccentricity, the backscattering cross section has been calculated for two spheroids of axial ratio 1.001 with arbitrary Euler
angles and a given separation between the centers. The results have been compared with those obtained for two spheres having the same center-to-center distance, and are in good agreement, with the maximum relative difference being 3.9%. Also the backscattering cross-sections calculated for the same two spheroids with two different sets of Euler angles and a given separation are found to be almost the same.

7. Conclusions

An analytic solution to the problem of electromagnetic scattering by $n$ dielectric spheroids of arbitrary orientation has been obtained, on the basis of rotational–translational addition theorems for vector
Fig. 5. Normalized bistatic cross-section for TE polarization of the incident wave with $\theta_i = 0$ versus scattering angle, for two identical spheroids with $a_A = a_B = \frac{1}{2} \lambda$, Euler angles $\alpha_0 = 30^\circ$, $\beta_0 = 45^\circ$, $\gamma_0 = 60^\circ$, $\epsilon_r = 3.0$, and displaced along the z axis: (a) $d = \frac{1}{2} \lambda$; (b) $d = \lambda$.

Fig. 6. Normalized backscattering cross-section as a function of the angle of incidence, for two prolate spheroids of different axial ratios, with $a_A = a_B = \frac{1}{2} \lambda$, Euler angles $\alpha_0 = 30^\circ$, $\beta_0 = 45^\circ$, $\gamma_0 = 60^\circ$, dielectric constant 3.0, and centers displaced along the direction $\theta_0 = 60^\circ$, $\phi_0 = 20^\circ$ by $d = \frac{1}{2} \lambda$. 
spheroidal wave functions. The exact boundary conditions are imposed by expanding the resultant field seen from a system of coordinates attached to each spheroid in terms of appropriate vector spheroidal eigenfunctions. The unknown expansion coefficients of the scattered and transmitted electromagnetic fields are obtained by using a matrix formulation, in which the column matrix of the total transmitted and scattered field expansion coefficients is equal to the product of a matrix, which is generally known as the system matrix, and the column matrix of the known incident field expansion coefficients. As in the case of scattering by two spheroids with parallel major axes [8,10], the system matrix has the special feature of being independent of the direction and polarization of the incident wave. This makes it possible to evaluate the unknown transmitted and scattered field expansion coefficients for various angles of incidence and for both TE and TM polarizations of the incident wave, using the same system matrix, which is a great advantage in numerical calculations. Results of a prescribed accuracy, corresponding to a whole range of angles of incidence, are therefore calculated with a high computational efficiency.

The solution for the case of \( n \) perfectly conducting spheroids of arbitrary orientation is derived from that for \( n \) dielectric spheroids, by letting the dielectric constant (or the refractive index) of the material of each dielectric spheroid become very high (theoretically infinite). The solutions for the special case of scattering by two dielectric spheroids and by two perfectly conducting spheroids of arbitrary orientation are obtained directly from the general formulation for scattering by \( n \) spheroids of arbitrary orientation. The overall accuracy and the corresponding number of terms depend on the semi-major axis lengths and the separation between the centers of the two spheroids; larger semi-major axis lengths and smaller separations require more terms for a given accuracy. Due to the complexity of the geometries considered, there are no results available in the literature obtained by using other methods.

These solutions are useful in analyzing models which have similar configurations for important engineering problems such as scattering of radar signals from hydrometeors, visible light absorption by heterogeneous particles, and also in biomedical engineering. Results obtained by the exact method developed here with a controllable accuracy are also important for evaluating the accuracy of other approximate methods and validating numerical codes which can be used for the analysis of electromagnetic scattering by similar configuration systems.

Appendix A

Here we give details about the numerical calculation of the rotational-translational coefficients \( \langle q_r Q_{mn}^{\mu \nu}(\alpha_{qr}, \beta_{qr}, \gamma_{qr}; d_{qr}) \rangle \) that appear in eqs. (13)-(15), which are given in ref. [14]:

\[
(4) Q_{mn}^{\mu \nu}(\alpha_{qr}, \beta_{qr}, \gamma_{qr}; d_{qr}) = \sum_{u=0,1}^{\infty} d_{u}^{mn}(h_q) \sum_{\mu=-(|m|+u)}^{m+u} R_{\mu,m+u}^{\nu,m+u}(\alpha_{qr}, \beta_{qr}, \gamma_{qr}) \times \sum_{w=0,1}^{\infty} d_{w}^{\nu}(h_r) \frac{N_{\mu}(|m|+w)}{N_{\mu w}^{\nu}(h_r)} d_{w}^{\nu}(h_r),
\]

(A.1)

with \( d_{u}^{mn}(h_q) \) and \( d_{w}^{\nu}(h_r) \) being the spheroidal expansion coefficients. The rest of the notation is as follows:

\[
N_{\mu} = \frac{2}{(2l+1)} \frac{(l+\mu)!}{(l-\mu)!},
\]

(A.2)

\[
R_{\mu,m+u}^{\nu,m+u}(\alpha, \beta, \gamma) = (-1)^{\mu-m} \left[ \frac{N_{\mu}^{m}}{N_{\mu}} \right]^{1/2} e^{j\mu}\nu d_{\mu,m+u}^{(1)}(\beta) e^{j\mu a},
\]

(A.3)

\[
d_{\mu,m+u}^{(1)}(\beta) = \left[ \frac{(l+\mu+1)(l-\mu)}{(l+m+1)(l-m)!} \right]^{1/2} (\cos \frac{1}{2}\beta)^{\mu+m} (\sin \frac{1}{2}\beta)^{\mu-m} P_{l}^{(\mu-m,\mu+m)}(\cos \beta),
\]

(A.4)
with $P_{l-\mu}^{(\mu-m+\mu)}(\cos \beta)$ being the Jacobi polynomial of argument $\cos \beta$;

$$
^{(4)}a^{\mu}_{\mu}(d) = (-1)^{\mu} \sum_{p=p_0, p_0+1}^{l+s} j^{l+p-s}(2l + 1) a(\mu, s | -\mu, l | p) \Psi^{(4)}_{\mu-\mu, p}(d), \tag{A.5}
$$

in which $a(\mu, s | -\mu, l | p)$ are the linearization expansion coefficients and

$$
\Psi^{(4)}_{\mu-\mu, p}(d) = z^{(4)}_p(kd) P^\mu_{p+\mu}(\cos \theta_d) e^{i(\mu-\mu)\phi}, \tag{A.6}
$$

where $z^{(4)}_p$ is the spherical Hankel function of the second kind and $P^\mu_{p+\mu}$ is the associated Legendre function of the first kind. The spheroidal expansion coefficients $d^{mn}_{\mu}(h)$ are calculated using a recurrence formula, which is a linear homogeneous difference equation of the second order [13].

In both eqs. (A.1) and (A.5) the prime over the summation indicates that the summation is either only over even values or only over odd values of the summing variable. In eq. (A.5), since the last term of the summation is fixed at $l+s$, the first term is chosen to be either $p_0 = \max(|l-s|, |\mu-\mu|)$ or $p_0 + 1$, depending on the parity of $l+s$.

The linearization expansion coefficients $a(\mu, s | -\mu, l | p)$ result as a consequence of expressing the product of two associated Legendre functions in terms of a summation of associated Legendre functions. These are calculated by using a sequence of recurrence relations which are in ref. [16], together with a general recursion scheme and some numerical values for these coefficients.

When calculating the rotational-translational coefficients we have found that it is sufficient to consider the last terms in the series of $u$ and $w$ as $n - |m| + 12$ and $n - |\mu| + 12$, respectively, to obtain an accuracy of five significant digits, with the accuracy of the largest spheroidal expansion coefficients being twelve digits. The Jacobi polynomials are evaluated by expressing them in the form of factorials, as given in ref. [12].

Appendix B

The elements of the matrices $[P_{1r}]$, $[Q_{1r}]$, $[R_{1r}]$, and $[R_{1qr}]$, for $q = 1, 2, \ldots n$ and $r = 1, 2, \ldots n$, can be grouped in submatrices, such that all these matrices are quasi-diagonal in the sense that only the diagonal submatrices are different from zero. All null off-diagonal submatrices have the same size as the corresponding diagonal submatrices. The diagonal submatrices of $[P_{1r}]$, $[Q_{1r}]$, $[R_{1r}]$, and $[R_{1qr}]$ can be written as $[4,5,8,10]$:

$$
[P_{1r}]_0 = \begin{bmatrix}
[r] Y^{(4)}_{-1} & [r] Y^{(4)}_{0} \\
[r] Y^{(4)}_{0} & [r] Y^{(4)}_{0} \\
\end{bmatrix},
$$

$$
[P_{1r}]_m = \begin{bmatrix}
[r] Y^{(4)}_{m-1} & [r] Y^{(4)}_{m} \\
[r] Y^{(4)}_{m} & [r] Y^{(4)}_{m} \\
0 & [r] Y^{(4)}_{m} \\
\end{bmatrix}, \quad m \geq 1, \tag{B.2}
$$

$$
[Q_{1r}]_0 = \begin{bmatrix}
[r] Y^{(4)}_{-1} & [r] Y^{(4)}_{0} \\
[r] Y^{(4)}_{0} & [r] Y^{(4)}_{0} \\
\end{bmatrix},
$$

$$
[Q_{1r}]_m = \begin{bmatrix}
[r] Y^{(4)}_{m-1} & [r] Y^{(4)}_{m} \\
[r] Y^{(4)}_{m} & [r] Y^{(4)}_{m} \\
0 & [r] Y^{(4)}_{m} \\
\end{bmatrix}, \quad m \geq 1, \tag{B.3}
$$

$$
[R_{1r}]_0 = \begin{bmatrix}
[r] Y^{(4)}_{-1} & [r] Y^{(4)}_{0} \\
[r] Y^{(4)}_{0} & [r] Y^{(4)}_{0} \\
\end{bmatrix},
$$

$$
[R_{1r}]_m = \begin{bmatrix}
[r] Y^{(4)}_{m-1} & [r] Y^{(4)}_{m} \\
[r] Y^{(4)}_{m} & [r] Y^{(4)}_{m} \\
0 & [r] Y^{(4)}_{m} \\
\end{bmatrix}, \quad m \geq 1, \tag{B.4}
$$

$$
[R_{1qr}]_0 = \begin{bmatrix}
[r] Y^{(4)}_{-1} & [r] Y^{(4)}_{0} \\
[r] Y^{(4)}_{0} & [r] Y^{(4)}_{0} \\
\end{bmatrix},
$$

$$
[R_{1qr}]_m = \begin{bmatrix}
[r] Y^{(4)}_{m-1} & [r] Y^{(4)}_{m} \\
[r] Y^{(4)}_{m} & [r] Y^{(4)}_{m} \\
0 & [r] Y^{(4)}_{m} \\
\end{bmatrix}, \quad m \geq 1, \tag{B.5}
$$
\[ [Q_{Jr}]_0 = - \begin{bmatrix} [r] X_+^{(4)} & [r] X_0^{(4)} \\ [\eta] X_{m-1}^{+} & [\eta] X_{m-1}^{0} \end{bmatrix}, \] 
\[ [Q_{Jr}]_m = - \begin{bmatrix} [r] X_{m-1}^{+} & [r] X_{m-1}^{0} \\ [\phi] X_{m-1} & [\phi] X_{m-1} \end{bmatrix} \begin{bmatrix} [r] X_+^{(4)} & [r] X_0^{(4)} \\ [\eta] X_{m-1}^{+} & [\eta] X_{m-1}^{0} \end{bmatrix}, \quad m \geq 1, \] 
\[ [R_{Jr}]_0 = \begin{bmatrix} [r] X_+^{(1)} & [r] X_0^{(1)} \\ [\eta] X_{m-1}^{+} & [\eta] X_{m-1}^{0} \end{bmatrix}, \] 
\[ [R_{Jr}]_m = - \begin{bmatrix} [r] X_{m-1}^{+} & [r] X_{m-1}^{0} \\ [\phi] X_{m-1} & [\phi] X_{m-1} \end{bmatrix} \begin{bmatrix} [r] X_+^{(1)} & [r] X_0^{(1)} \\ [\eta] X_{m-1}^{+} & [\eta] X_{m-1}^{0} \end{bmatrix}, \quad m \geq 1 \] 
\[ [R_{Jqr}]_0 = - \begin{bmatrix} [r] X_+^{(1)} & [r] X_0^{(1)} \\ [\eta] X_{-1}^{+} & [\eta] X_{-1}^{0} \end{bmatrix}, \] 
\[ [R_{Jqr}]_m = - \begin{bmatrix} [r] X_{m-1}^{+} & [r] X_{m-1}^{0} \\ [\phi] X_{m-1} & [\phi] X_{m-1} \end{bmatrix} \begin{bmatrix} [r] X_+^{(1)} & [r] X_0^{(1)} \\ [\eta] X_{m-1}^{+} & [\eta] X_{m-1}^{0} \end{bmatrix}, \quad m \geq 1. \] 

The submatrices \([X_m]\) have the form
\[ [X_m] = \begin{bmatrix} X_{m,0,|m|} & X_{m,0,|m|+1} & X_{m,0,|m|+2} & \cdots & \cdots \\ X_{m,1,|m|} & X_{m,1,|m|+1} & X_{m,1,|m|+2} & \cdots & \cdots \\ X_{m,2,|m|} & X_{m,2,|m|+1} & X_{m,2,|m|+2} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots \end{bmatrix} \]
where the elements are given by

\[
(r) \Delta_{m+n,\kappa,n}^{\pm(i)}(r) = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{-1}^{1} \left[ \frac{\partial}{\partial \phi} \right]_{i} \left( \int_{-1}^{1} \left( \frac{\partial}{\partial \phi} \right)_{j} J_{m+n}(h, \eta, \kappa) e^{-j(m+v\pm \phi)} \frac{d\eta}{d\phi}, \right.
\]

\[
B(10)
\]

\[
(r) \Delta_{m+1,\kappa,n}^{\pm(i)}(r) = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{-1}^{1} \left[ \frac{\partial}{\partial \phi} \right]_{i} \left( \int_{-1}^{1} \left( \frac{\partial}{\partial \phi} \right)_{j} J_{m+1}(h, \eta, \kappa) e^{-j(m+v\pm \phi)} \frac{d\eta}{d\phi}, \right.
\]

\[
B(11)
\]

with \( J \) being either \( M \) or \( N \), i.e. the respective component of \( M \) or \( N \). The submatrices \([Y_m]\) have the same form as those of \([X_m]\), with the elements given by

\[
(r) \Delta_{m+n,\kappa,n}^{\pm(i)} = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{-1}^{1} \left[ \frac{\partial}{\partial \phi} \right]_{i} \left( \int_{-1}^{1} \left( \frac{\partial}{\partial \phi} \right)_{j} J_{m+n}(h, \eta, \kappa) e^{-j(m+v\pm \phi)} \frac{d\eta}{d\phi}, \right.
\]

\[
B(12)
\]

\[
(r) \Delta_{m+1,\kappa,n}^{\pm(i)} = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{-1}^{1} \left[ \frac{\partial}{\partial \phi} \right]_{i} \left( \int_{-1}^{1} \left( \frac{\partial}{\partial \phi} \right)_{j} J_{m+1}(h, \eta, \kappa) e^{-j(m+v\pm \phi)} \frac{d\eta}{d\phi}, \right.
\]

\[
B(13)
\]

in which \( J \) is either \( M \) or \( N \). The explicit expressions of \( X \) for \( J = M \) are given in ref. [17]. The explicit expressions of \( X_{m,n}^{\pm(i)}(v = 0) \), \( X_{m,2,n}^{\pm(i)}(v = 2) \), and \( X_{m+1,1,n}^{\pm(i)}(v = 1) \) for \( J = N \) are given below (with \( R_m^i \) being the radial spheroidal function of the \( i \)th kind):

\[
(r) \Delta_{m,n}^{\pm(i)} = \left[ \left( \xi_s^2 - 1 \right) \frac{\partial^2}{\partial \xi_s^2} R_m^i(h, \xi_s) |_{\xi_s = \xi_s} + \xi_s \frac{\partial}{\partial \xi_s} R_m^i(h, \xi_s) |_{\xi_s = \xi_s} \right] \left( \frac{\xi_s^2 - 1}{\xi_s^2} I_{1m \kappa} + I_{14m \kappa} \right)
\]

\[
+ \xi_s \frac{\partial}{\partial \xi_s} R_m^i(h, \xi_s) |_{\xi_s = \xi_s} \left( \frac{\xi_s^2 - 1}{\xi_s^2} I_{4m \kappa} + I_{15m \kappa} + 2I_{14m \kappa} \right)
\]

\[
- R_m^i(h, \xi_s) \left[ I_{16m \kappa} - \frac{\xi_s^2}{(\xi_s^2 - 1) I_{15m \kappa}} \right]
\]

\[
B(14)
\]

\[
(r) \Delta_{m+2,n}^{\pm(i)} = \left[ \left( \xi_s^2 - 1 \right) \frac{\partial^2}{\partial \xi_s^2} R_{m+2,n}^i(h, \xi_s) |_{\xi_s = \xi_s} + \xi_s \frac{\partial}{\partial \xi_s} R_{m+2,n}^i(h, \xi_s) |_{\xi_s = \xi_s} \right] \times \left( \frac{\xi_s^2 - 1}{\xi_s^2} I_{8m \kappa} + I_{19m \kappa} \right)
\]

\[
+ \xi_s \frac{\partial}{\partial \xi_s} R_{m+2,n}^i(h, \xi_s) |_{\xi_s = \xi_s} \left( \frac{\xi_s^2 - 1}{\xi_s^2} I_{9m \kappa} + I_{20m \kappa} + 2I_{19m \kappa} \right)
\]

\[
- R_m^i(h, \xi_s) \left[ I_{21m \kappa} - \frac{\xi_s^2}{(\xi_s^2 - 1) I_{20m \kappa}} \right]
\]

\[
B(15)
\]
\[
\begin{align*}
(\mu) X^{(\mu)(i)}_{m+1, n} &= 2 \left[ (\xi_s^2 - 1)^{3/2} \frac{d}{d\xi_r} R_{m+1, n}^{(i)}(h_r, \xi_s) \big|_{\xi_s = \xi} \left( I_{6m+1, n} - I_{5m+1, n} \right) \\
&\quad + \left( \xi_s^2 - 1 \right)^{1/2} \frac{d}{d\xi_r} R_{m+1, n}^{(i)}(h_r, \xi_s) \big|_{\xi_s = \xi} \left( I_{24m+1, n} - I_{25m+1, n} \right) \\
&\quad - \xi_s \left( \xi_s^2 - 1 \right)^{1/2} \frac{d^2}{d\xi_r^2} R_{m+1, n}^{(i)}(h_r, \xi_s) \big|_{\xi_s = \xi} \left( \left( \xi_s^2 - 1 \right)^2 I_{26m+1, n} + 2(\xi_s^2 - 1) I_{25m+1, n} \right) \\
&\quad + \frac{(m + 1)^2 \xi_s}{(\xi_s^2 - 1)^{3/2}} \frac{d}{d\xi_r} R_{m+1, n}^{(i)}(h_r, \xi_s) \big|_{\xi_s = \xi} \left( \left( \xi_s^2 - 1 \right)^2 I_{26m+1, n} + 2(\xi_s^2 - 1) I_{25m+1, n} \right) \\
&\quad + \frac{2\xi_s}{(\xi_s^2 - 1)^{1/2}} R_{m+1, n}^{(i)}(h_r, \xi_s) I_{24m+1, n} - \frac{2\xi_s^2}{(\xi_s^2 - 1)^{1/2}} \frac{d}{d\xi_r} R_{m+1, n}^{(i)}(h_r, \xi_s) \big|_{\xi_s = \xi} I_{25m+1, n} \right], \\
(\mu) X^{(\mu)(i)}_{m+1, n} &= \left( \xi_s^2 - 1 \right) R_{mn}^{(i)}(h_r, \xi_s)(mI_{28m+1, n} \pm I_{27m+1, n}) \\
&\quad \pm \left[ (\xi_s^2 - 1)^2 \frac{d^2}{d\xi_r^2} R_{mn}^{(i)}(h_r, \xi_s) \big|_{\xi_s = \xi} \left( -1 \pm m \right) \left( \pm m \right) R_{mn}^{(i)}(h_r, \xi_s) \right] I_{1m+1, n}, \\
(\mu) X^{(\mu)(i)}_{m+2, n} &= \left( \xi_s^2 - 1 \right) R_{m+2, n}^{(i)}(h_r, \xi_s)((m + 2)(I_{31m+1, n} + I_{32m+1, n}) \pm (I_{33m+1, n} - I_{31m+1, n})) \\
&\quad \pm \left[ (\xi_s^2 - 1)^2 \frac{d^2}{d\xi_r^2} R_{m+2, n}^{(i)}(h_r, \xi_s) \big|_{\xi_s = \xi} \left( -1 \pm (m + 2) \right) \right] I_{7m+1, n}, \\
(\mu) X^{(\mu)(i)}_{m+1, n} &= -2(m + 1)(\xi_s^2 - 1)^{1/2} \left[ \xi_s R_{m+1, n}^{(i)}(h_r, \xi_s) I_{29m+1, n} \\
&\quad + \left( \xi_s^2 - 1 \right) \frac{d}{d\xi_r} R_{m+1, n}^{(i)}(h_r, \xi_s) \big|_{\xi_s = \xi} I_{29m+1, n} \right], \\
(\mu) X^{(\mu)(i)}_{0, n} &= -2 \left[ (\xi_s^2 - 1)^{1/2} \frac{d}{d\xi_r} R_{0n}^{(i)}(h_r, \xi_s) \big|_{\xi_s = \xi} \left( \xi_s^2 - 1 \right) I_{11m+1, n} + I_{12m+1, n} \right) \\
&\quad - \xi_s (\xi_s^2 - 1)^{1/2} \frac{d^2}{d\xi_r^2} R_{0n}^{(i)}(h_r, \xi_s) \big|_{\xi_s = \xi} \left( \xi_s^2 - 1 \right) I_{10m+1, n} + I_{13m+1, n} \right) \\
&\quad - (\xi_s^2 - 1)^{1/2} \frac{d}{d\xi_r} R_{0n}^{(i)}(h_r, \xi_s) \big|_{\xi_s = \xi} \left( \xi_s^2 - 1 \right) I_{10m+1, n} + I_{13m+1, n} \right) \\
&\quad + \frac{2\xi_s}{(\xi_s^2 - 1)} R_{0n}^{(i)}(h_r, \xi_s) \left( I_{12m+1, n} - \xi_s (\xi_s^2 - 1)^{1/2} I_{13m+1, n} \right), \\
(\mu) X^{(\mu)(i)}_{0, n} &= 0.
\end{align*}
\]
Explicit expressions of \( Y_{m,k,n}^{\pm}(v = 0) \), \( Y_{m+2,k,n}^{\pm}(v = 2) \), and \( Y_{m+1,k,n}^{\pm}(v) \) for \( J = M \) and \( J = N \) have the same structure as those of the corresponding \( X \), but with the functions \( J \) evaluated with respect to \( h'_r \), which is the value of \( h \) inside the \( r \)th spheroid. \( \xi_s \) is the value of \( \xi \) on the surface of the spheroid considered. \( I_{m,k,n} - I_{11,k,n} \) are given in ref. [17] and \( I_{12,k,n} - I_{33,m,k,n} \) in appendix C. It should be noted that for computational purposes, the following relations are used [8,13]:

\[
S_{mn}(h, \eta) = K_{mn} S_{|m|\eta}(h, \eta), \quad R_{mn}^{(i)}(h, \xi) = R_{|m|\eta}^{(i)}(h, \xi), \quad i = 1, 2, 3, 4,
\]

\[
d_m^{mn}(h) = (-1)^{(m-|m|)/2} \frac{(|m| - m + q)!}{q!} K_{mn} d_{|m|\eta}^{mn}(h), \quad N_{mn}(h) = K_{mn} N_{|m|\eta}(h), \tag{B.22}
\]

where

\[
K_{mn} = (-1)^{(|m| - m)/2} \frac{(n + m)!}{(n + |m|)!}. \tag{B.23}
\]

Appendix C

The integrals \( I_{m,Nn} - I_{33,m,Nn} \) in appendix B, that result when applying the orthogonality properties of the spheroidal angle functions are evaluated using the recurrence relations for the associated Legendre functions and the integrals [17,18]

\[
\int_{-1}^{1} P_{\mu}^m(\eta) P_{\nu}^m(\eta) \, d\eta = \frac{2}{(2\mu + 1)} \frac{(\mu + m)!}{(\mu - m)!} \delta_{\nu \mu}, \tag{C.1}
\]

\[
\int_{-1}^{1} P_{\mu}^{m+2}(\eta) P_{\nu}^m(\eta) \, d\eta = \begin{cases} 
0, & \nu > \mu, \\
-\frac{2}{(2\nu + 1)} \frac{(\nu + m)!}{(\nu - m - 2)!}, & \nu = \mu, \\
2(\nu + m)! \left[ 1 + (-1)^{\nu + \mu} \right], & \nu < \mu,
\end{cases} \tag{C.2}
\]

where \( \delta_{\nu \mu} \) is the Kronecker delta function. The integrals \( I_{1m,Nn} - I_{9m,Nn}, I_{10,Nn}, \) and \( I_{11,Nn} \) are evaluated in ref. [17]. We derived expressions for \( I_{12,Nn}, I_{13,Nn}, \) and \( I_{14m,Nn} - I_{33,m,Nn} \) in the form:

\[
I_{12,Nn} = \int_{-1}^{1} \eta(1 - \eta^2)^{3/2} \frac{d}{d\eta} S_{0n} S_{1,n+N} \, d\eta
\]

\[
= 2 \sum_{q=0,1}^\infty \frac{(q + 1)(q + 2)}{(2q + 3)} \left( -\frac{(q - 2)(q - 1)q d_{q-2}^{0n}}{(2q - 3)(2q - 1)(2q + 1)} + \frac{q(q + 1)^2 d_q^{0n}}{(2q - 1)(2q + 1)(2q + 5)} \right)
\]

\[
+ \frac{(q + 2)(q + 3) d_{q+2}^{0n}}{(2q + 1)(2q + 5)(2q + 7)} - \frac{(q + 3)(q + 4)(q + 5) d_{q+4}^{0n}}{(2q + 5)(2q + 7)(2q + 9)} \right) d_{q}^{1,n+N}, \quad (n + N) \text{ even},
\]

\[
= 0, \quad (n + N) \text{ odd}, \tag{C.3}
\]
\[ I_{13.Nn} = \int_{-1}^{1} (1 - \eta^2)^{3/2} S_{0n} S_{1.1+N} \, d\eta \]
\[ = 2 \sum_{q=0,1}^{\infty} \frac{(q + 1)(q + 2)}{(2q + 3)} \left[ -\frac{(q - 1)q d_{q-2}^{0n}}{(2q - 3)(2q - 1)(2q + 1)} + \frac{(3q^2 + 5q - 4)d_{q}^{0n}}{(2q - 1)(2q + 1)(2q + 5)} \right. \]
\[ - \frac{(3q^2 + 13q + 8)d_{q+2}^{0n}}{(2q + 1)(2q + 5)(2q + 7)} - \frac{(q + 3)(q + 4)d_{q+4}^{0n}}{(2q + 5)(2q + 7)(2q + 9)} \left(d_{q}^{1.1+N}, \ (n + N \text{ even}, \right) \]
\[ = 0, \ (n + N \text{ odd}, \quad (C.4) \]
\[ I_{14m.Nn} = \int_{-1}^{1} (1 - \eta^2) \eta S_{m,n} S_{m,m+N} \, d\eta \]
\[ = 2 \sum_{q=0,1}^{\infty} \left[ -\frac{(q - 2)(q - 1)q}{(2m + 2q - 3)(2m + 2q - 1)} \left( \frac{d_{q-3}^{m,m+N}}{(2m + 2q - 5)} - \frac{d_{q-1}^{m,m+N}}{(2m + 2q - 1)} \right) \right. \]
\[ + \frac{(2m + 1)(2m + q + 1)q}{(2m + 2q - 1)(2m + 2q + 3)} \left( \frac{d_{q-1}^{m,m+N}}{(2m + 2q - 1)} - \frac{d_{q+1}^{m,m+N}}{(2m + 2q + 3)} \right) \]
\[ + \frac{(2m + q + 1)(2m + q + 2)(2m + q + 3)}{(2m + 2q + 3)(2m + 2q + 5)} \left( \frac{d_{q+1}^{m,m+N}}{(2m + 2q + 3)} - \frac{d_{q+3}^{m,m+N}}{(2m + 2q + 7)} \right) \]
\[ \times \frac{(2m + q)!}{(2m + 2q + 1)q!} d_{q}^{mn}, \ (n + N \text{ odd}, \quad (C.5) \]
\[ I_{15m.Nn} = \int_{-1}^{1} (1 - \eta^2)^2 \frac{d}{d\eta} S_{m,n} S_{m,m+N} \, d\eta \]
\[ = 2 \sum_{q=0,1}^{\infty} \frac{(2m + q)!d_{q}^{mn}}{(2m + 2q + 1)q!} \left[ \frac{(m + q + 1)(q - 2)(q - 1)q}{(2m + 2q - 3)(2m + 2q - 1)} \left( \frac{d_{q-3}^{m,m+N}}{(2m + 2q - 5)} - \frac{d_{q-1}^{m,m+N}}{(2m + 2q - 1)} \right) \right. \]
\[ + \frac{[2(m + q)^2 + 5m + 2q]}{(2m + 2q - 1)(2m + 2q + 3)} \left( \frac{d_{q-1}^{m,m+N}}{(2m + 2q - 1)} - \frac{d_{q+1}^{m,m+N}}{(2m + 2q + 3)} \right) \]
\[ - \frac{(m + q)(2m + q + 1)}{(2m + 2q + 1)(2m + 2q + 3)} \frac{(2m + q + 2)(2m + q + 3)}{(2m + 2q + 3)(2m + 2q + 5)} \]
\[ \times \left( \frac{d_{q+1}^{m,m+N}}{(2m + 2q + 3)} - \frac{d_{q+3}^{m,m+N}}{(2m + 2q + 7)} \right) \right] \left( n + N \text{ odd}, \quad (C.6) \]
\[ = 0, \ (n + N \text{ even}, \quad (C.6) \]
\[ I_{16m.Nn} = \int_{-1}^{1} (1 - \eta^2) \eta^2 \frac{d}{d\eta} S_{m,n} S_{m,m+N} \, d\eta \]
\[ = I_{4m.Nn} - I_{15m.Nn}, \quad (C.7) \]
\[ I_{17m, N_n} = \int_{-1}^{1} \frac{m \eta}{1 - \eta^2} S_{mn} S_{m, m+N} \, d\eta \]

\[ = \sum_{q=0}^{\infty} \frac{(2m + q)!}{q!} d_q^{mn} \sum_{r=q+1}^{\infty} d_r^{m, m+N} + \sum_{q=1, 0}^{\infty} \frac{(2m + q)!}{q!} d_q^{m, m+N} \sum_{r=q+1}^{\infty} d_r^{mn}, \quad (n + N) \text{ even}, \quad (C.8) \]

\[ I_{18m, N_n} = \int_{-1}^{1} \frac{d}{d\eta} S_{mn} S_{m, m+N} \, d\eta \]

\[ = 2 \sum_{q=0}^{\infty} d_q^{0N} \sum_{r=q}^{\infty} d_r^{0N}, \quad (n + N) \text{ odd}, \quad m = 0, \quad (C.9) \]

\[ I_{19m, N_n} = \int_{-1}^{1} (1 - \eta^2) \eta S_{m+2n, n} S_{m, m+N} \, d\eta \]

\[ = 2 \sum_{q=0}^{\infty} \left[ \frac{(2m + q + 5)}{(2m + 2q + 5)(2m + q + 7)} \left( \frac{d_{q+1}^{m, m+N}}{(2m + 2q + 3)} - \frac{2d_{q+3}^{m, m+N}}{(2m + 2q + 9)} \right) \right. \]

\[ + \frac{1}{(2m + 2q + 7)} \left( \frac{q d_{q+3}^{m, m+N}}{(2m + 2q + 3)(2m + 2q + 5)} + \frac{(2m + q + 5)d_{q+5}^{m, m+N}}{(2m + 2q + 9)(2m + 2q + 11)} \right) \]

\[ + \frac{q}{(2m + 2q + 1)} \left( \frac{d_{q+1}^{m, m+N}}{(2m + 2q - 1)(2m + 2q + 3)} - \frac{2d_{q+3}^{m, m+N}}{(2m + 2q + 3)(2m + 2q + 5)} \right) \]

\[ \left. \times \frac{(2m + q + 4)!}{(2m + 2q + 5)q!} d_q^{m+2n}, \quad (n + N) \text{ odd}, \quad (C.10) \right. \]

\[ I_{20m, N_n} = \int_{-1}^{1} (1 - \eta^2)^2 \frac{d}{d\eta} S_{m+2n, n} S_{m, m+N} \, d\eta \]

\[ = 2 \sum_{q=0}^{\infty} \left[ \frac{(m + q + 3)q}{(2m + 2q + 1)(2m + 2q + 3)} \left( \frac{d_{q-1}^{m, m+N}}{(2m + 2q - 1)} - \frac{2d_{q+1}^{m, m+N}}{(2m + 2q + 5)} \right) \right. \]

\[ - \frac{(m + q + 2)(2m + q + 5)}{(2m + 2q + 5)(2m + 2q + 7)} \left( \frac{d_{q+1}^{m, m+N}}{(2m + 2q + 3)} - \frac{2d_{q+3}^{m, m+N}}{(2m + 2q + 9)} \right) \]

\[ + \left( \frac{(m + q + 3)q d_{q+3}^{m, m+N}}{(2m + 2q + 3)(2m + 2q + 5)} - \frac{(m + q + 2)(2m + q + 5)d_{q+5}^{m, m+N}}{(2m + 2q + 9)(2m + 2q + 11)} \right) \frac{1}{(2m + 2q + 7)} \]

\[ \left. \times \frac{(2m + q + 4)!}{(2m + 2q + 5)q!} d_q^{m+2n}, \quad (n + N) \text{ odd}, \quad (C.11) \right. \]
\[
I_{21m,Nn} = \int_{-1}^{1} (1 - \eta^2)^{3/2} \frac{d}{d\eta} S_{m+1,n} S_{m,m+N} d\eta \\
= I_{9m,Nn} - I_{20m,Nn},
\]

\[
I_{22m,Nn} = \int_{-1}^{1} \frac{\eta}{(1 - \eta^2)} S_{m+2,n} S_{m,m+N} d\eta \\
= 2 \sum_{q=0,1}^\infty \frac{(2m + q)!}{q!} d_q^{m,m+N} \sum_{r=0}^\infty \left[ rd_{r-1}^{m+2,n} + (2m + 2r + 3) \sum_{s=r+1}^\infty d_s^{m+2,n} \right], \quad (n + N) \text{ odd}, \\
= 0, \quad (n + N) \text{ even},
\]

\[
I_{23m,Nn} = \int_{-1}^{1} \frac{d}{d\eta} S_{m+2,n} S_{m,m+N} d\eta \\
= 2 \sum_{q=0,1}^\infty \frac{d_q^{m,m+N}}{(2m + q)!} \sum_{r=0}^\infty \left[ -(m + r + 1)d_{r-1}^{m+2,n} \right] \\
+ (m + 2)(2m + 2r + 3) \sum_{s=r+1}^\infty d_s^{m+2,n}, \quad (n + N) \text{ odd}, \\
= 0, \quad (n + N) \text{ even},
\]

\[
I_{24m,Nn} = \int_{-1}^{1} (1 - \eta^2)^{3/2} \frac{d}{d\eta} S_{m+1,n} S_{m,m+N} d\eta \\
= 2 \sum_{q=0,1}^\infty \frac{(2m + q + 2)!d_q^{m+1,n}}{(2m + 2q + 3)q!} \left\{ \frac{(m + q + 2)q}{(2m + 2q + 1)} \left( \frac{(q - 1)d_{q-2}^{m,m+N}}{(2m + 2q - 3)(2m + 2q - 1)} \right) \right. \\
+ \frac{(2m + q + 2)d_{q+2}^{m,m+N}}{(2m + 2q - 1)(2m + 2q + 3)} - \frac{(2m + 2q + 5)}{(2m + 2q + 3)(2m + 2q + 5)} \right\} \\
- \frac{(m + q + 1)(2m + q + 3)}{(2m + 2q + 5)} \left( \frac{(q + 1)d_{q+1}^{m,m+N}}{(2m + 2q + 1)(2m + 2q + 3)} \right) \\
+ \frac{(2m + 1)d_{q+2}^{m,m+N}}{(2m + 2q + 3)(2m + 2q + 7)} - \frac{(2m + q + 4)d_{q+4}^{m,m+N}}{(2m + 2q + 7)(2m + 2q + 9)} \right\}, \quad (n + N) \text{ even}, \\
= 0, \quad (n + N) \text{ odd},
\]

\[
I_{25m,Nn} = \int_{-1}^{1} (1 - \eta^2)^{3/2} S_{m+1,n} S_{m,m+N} d\eta \\
= 2 \sum_{q=0,1}^\infty \frac{(2m + q + 2)!d_q^{m+1,n}}{(2m + 2q + 3)q!} \left\{ \frac{(2m + q + 3)(2m + q + 4)}{(2m + 2q + 5)} \left( \frac{d_q^{m,m+N}}{(2m + 2q + 1)(2m + 2q + 3)} \right) \right. \\
- \frac{2d_{q+2}^{m,m+N}}{(2m + 2q + 3)(2m + 2q + 7)} + \frac{d_{q+4}^{m,m+N}}{(2m + 2q + 7)(2m + 2q + 9)} \right\} \\
- \frac{(q - 1)q}{(2m + 2q + 1)} \left( \frac{d_{q-2}^{m,m+N}}{(2m + 2q - 3)(2m + 2q - 1)} - \frac{2d_{q+2}^{m,m+N}}{(2m + 2q - 1)(2m + 2q + 3)} \right) \\
+ \frac{d_{q+2}^{m,m+N}}{(2m + 2q + 3)(2m + 2q + 5)} \right\}, \quad (n + N) \text{ even}, \\
= 0, \quad (n + N) \text{ odd},
\]
\[ I_{26mNn} = \int_{-1}^{1} \frac{1}{(1 - \eta^2)^{1/2}} S_{m+1,n} S_{m,m+N} \, d\eta \]
\[ = 2 \sum_{q=0,1}^{\infty} \frac{(2m + q)!}{q!} d_{q}^{m,m+N} \sum_{r=q}^{\infty} d_{r}^{m+1,n}, \quad (n + N) \text{ even}, \]
\[ = 0, \quad (n + N) \text{ odd}, \quad \text{(C.17)} \]

\[ I_{27mNn} = \int_{-1}^{1} \frac{1}{(1 - \eta^2)^{1/2}} \frac{d}{d\eta} \left( \frac{\eta}{(1 - \eta^2)^{1/2}} S_{mn} \right) S_{m,m+N} \, d\eta \]
\[ = -2 \sum_{q=0,1}^{\infty} \frac{(q + 1)^2}{(2q + 3)} d_{q+1}^{0n} d_{q+1}^{0N} + 2 \sum_{r=0,1}^{\infty} d_{r}^{0N} \sum_{s=r+1}^{\infty} d_{s+1}^{0n}, \quad (n + N) \text{ even}, \quad m = 0, \]
\[ = -2 \sum_{q=0,1}^{\infty} \frac{(2m + q)!}{q!} \left( \frac{q(m + q) + (q + 1)(m + q + 1)}{(2m + 2q + 1)} \right) d_{q}^{m,m+N} \]
\[ - (m + 1) d_{q}^{m,m+N} \sum_{r=q+2}^{\infty} d_{r}^{m,m+N} -(m - 1) d_{q}^{m,m+N} \sum_{r=q+2}^{\infty} d_{r}^{m,m+N} \right), \quad (n + N) \text{ even}, \quad m \neq 0, \]
\[ = 0, \quad (n + N) \text{ odd}, \quad \text{(C.18)} \]

\[ I_{28mNn} = \int_{-1}^{1} m(1 - \eta^2)^{1/2} \frac{d}{d\eta} \left( \frac{\eta}{(1 - \eta^2)^{1/2}} S_{mn} \right) S_{m,m+N} \, d\eta \]
\[ = \sum_{q=0,1}^{\infty} \frac{(2m + q)!}{q!} \left( \frac{(m + 2q + 1)}{(2m + 2q + 1)} \right) d_{q}^{m,m+N} \]
\[ + (m + 1) d_{q}^{m,m+N} \sum_{r=q+2}^{\infty} d_{r}^{m,m+N} -(m - 1) d_{q}^{m,m+N} \sum_{r=q+2}^{\infty} d_{r}^{m,m+N} \right), \quad (n + N) \text{ even}, \]
\[ = 0, \quad (n + N) \text{ odd}, \quad \text{(C.19)} \]

\[ I_{29mNn} = \int_{-1}^{1} (1 - \eta^2)^{1/2} \frac{d}{d\eta} S_{m+1,n} S_{m,m+N} \, d\eta \]
\[ = -2 \sum_{q=0,1}^{\infty} \frac{(m + q + 1)(2m + q + 1)!}{(2m + 2q + 3)q!} d_{q}^{m+1,n} d_{q+1}^{m,m+N} \]
\[ + 2(m + 1) \sum_{q=1,0}^{\infty} \frac{(2m + q)!}{q!} d_{q}^{m,m+N} \sum_{r=q+1}^{\infty} d_{r}^{m+1,n}, \quad (n + N) \text{ odd}, \]
\[ = 0, \quad (n + N) \text{ even}, \quad \text{(C.20)} \]

\[ I_{30mNn} = \int_{-1}^{1} \eta^2 S_{m+2,n} S_{m,m+N} \, d\eta \]
\[ = 2 \sum_{q=0,1}^{\infty} \frac{(2m + q)!}{(2m + 2q + 1)q!} d_{q}^{m,m+N} \]
\[ \times \left[ \frac{(2m + q + 1)(2m + q + 2)(2m + q + 3)}{(2m + 2q + 3)(2m + 2q + 5)} \left( q + 1 \right) d_{q}^{m+2,n} + (2m + 2q + 5) \sum_{r=q+2}^{\infty} d_{r}^{m+2,n} \right) \]
\[\begin{align*}
&+ \frac{(2m+1)(2m+q+1)q}{(2m+2q-1)(2m+2q+3)} \left( (q-1)d_{q-2}^{m+2,n} + (2m+2q+1) \sum_{r=q}^{\infty} d_{r}^{m+2,n} \right) \\
- \frac{(q-2)(q-1)q}{(2m+2q-3)(2m+2q-1)} \left( (q-3)d_{q-4}^{m+2,n} + (2m+2q-3) \sum_{r=q-2}^{\infty} d_{r}^{m+2,n} \right),
\end{align*}\]

\[
(n + N) \text{ even,} = 0, \quad (n + N) \text{ odd,} \quad (C.21)
\]

\[
I_{31m, Nn} = \int_{-1}^{1} \frac{d}{d\eta} S_{m+2,n} S_{m, m+N} \, d\eta
\]

\[
= 2 \sum_{q=0,1}^{\infty} \frac{(2m+q)!d_{q}^{m, m+N}}{q!}
\]

\[
\times \left[ \frac{q}{(2m+2+q+1)} \left( -(q-1)(q+m)d_{q-2}^{m+2,n} + (m+2)(2m+2q+1) \sum_{r=q}^{\infty} d_{r}^{m+2,n} \right) \\
+ \sum_{r=q+1}^{\infty} \left( -r(m+r+1)d_{r-1}^{m+2,n} + (m+2)(2m+2r+3) \sum_{s=r+1}^{\infty} d_{s}^{m+2,n} \right) \right], \quad (n + N) \text{ even,}
\]

\[
= 0, \quad (n + N) \text{ odd},
\quad (C.22)
\]

\[
I_{32m, Nn} = \int_{-1}^{1} \frac{1}{(1-\eta^2)} S_{m+2,n} S_{m, m+N} \, d\eta
\]

\[
= -2 \sum_{q=0,1}^{\infty} \frac{(2m+q-2)!}{q!} \left[ (2m+q-1)(2m+q)(2m+2q+3) \sum_{r=q}^{\infty} d_{r}^{m+2,n} \\
\times \sum_{r=q+2}^{\infty} d_{r}^{m, m+N} - 2m(2m+2q-1) \sum_{r=q}^{\infty} d_{r}^{m, m+N} \\
\times \left( \sum_{r=q}^{\infty} (2m+2r+3) \sum_{s=r}^{\infty} d_{s}^{m+2,n} \right) \right], \quad (n + N) \text{ even,}
\]

\[
= 0, \quad (n + N) \text{ odd},
\quad (C.23)
\]

\[
I_{33m, Nn} = \int_{-1}^{1} (1-\eta^2) \frac{d^2}{d\eta^2} S_{m+2,n} S_{m, m+N} \, d\eta
\]

\[
= 2I_{31m, Nn} + (m + 2)^2 I_{32m, Nn} - \lambda_{m+2,n} I_{m, Nn} + h^2 I_{30m, Nn}, \quad (C.24)
\]

where \(\lambda_{m+2,n}\) are the eigenvalues corresponding to the angle and radial spheroidal functions. In the series expressions of these integrals there are single, double, or triple summations. The summations in the form \(\sum d_{m}^{m,n}\) are very rapidly convergent, three or four terms being sufficient to yield an accuracy of ten digits. For the second summations in the double and triple series, five or six terms are sufficient to give an accuracy of seven digits. The last summations in the triple series require nine or ten terms to yield an accuracy of five digits.
References