Formulas for the Magnetic Field of Polygonal Cross Section Current Coils

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Abstract — An extension to the case of general polygonal cross section of the analytical expressions obtained previously by the author for the magnetic field of rectangular cross section current coils is presented. The polygonal cross section coils are decomposed in straight segments whose sides along the current direction are trapezoidal in general. By using surface source models, the field due to the volume current distribution is expressed in terms of that due to sources distributed over the segment side surfaces, for which simple formulas are derived. All these formulas contain elementary functions only and numerical results even for the special case of rectangular cross section coils show a substantially higher computational efficiency with respect to existing methods.

I. INTRODUCTION

Magnetic fields from curved current conductors are usually computed by applying methods of fast numerical integration in the Biot-Savart formula [1], [2]. Recently, analytical expressions for the contributions to the resultant coil fields from current tubes in the form of sections of toroid of rectangular cross section [3], as well as of general polygonal cross section [4], were derived. They contain Jacobian elliptic functions and elliptic integrals, and for computing the field it is also necessary to evaluate numerically another type of integrals, which evade an analytical treatment. However, this procedure is superior to numerical integration techniques in accuracy and stability, and it requires a computation time that is reduced on an average by a factor of four for same accuracy [4].

Since in special situations the coil volume cannot be decomposed exactly in straight tubes and tubes in the form of sections of toroid, and in order to further reduce the amount of computation required for a specified accuracy, we have considered straight segments only, of same cross section, but with sides along the current direction of a trapezoidal shape in general, in which a coil of arbitrary geometry can approximately be decomposed. As shown for coils of rectangular cross section [5], the formulas derived yield, for a satisfactory accuracy, a reduction in the computation time by a factor of nine on an average, with respect to that when using the expressions for sections of toroid. In this paper we consider the general case of arbitrary polygonal cross section current coils. The contribution of each carrying-current straight segment of polygonal cross section to the resultant coil magnetic field can be expressed exactly in terms of elementary functions on the basis of a modeling method developed by the author [6]. Namely, the volume distribution of current density \( J \) is replaced by equivalent distributions of fictitious magnetization \( M_c \) and surface densities of current \( \rho_s \) and magnetic charge \( \rho_m \) over surfaces of discontinuity of \( M_c \), with no volume distribution of magnetic charge if \( M_c \) is chosen such that its divergence is zero everywhere. The magnetic field intensity in an unbounded, homogeneous and linear region of permeability \( \mu \) can be determined in terms of surface integrals only, as

\[
H(r) = \frac{1}{4\pi} \left[ \int \frac{J_{sc}(r') \times R}{R^3} \ dS' \right] + \frac{1}{\mu} \left[ \int \frac{\rho_{sc}(r') R}{R^3} \ dS' \right]
\]

where \( r \) and \( r' \) are the position vectors of the field point and the source point, respectively, with \( R = r - r' \).

II. SURFACE SOURCE MODELING

Assume the coil decomposed in straight segments only, with the \( n + 2 \) sides of every segment denoted in the same manner with respect to the current direction, as shown in Fig. 1, such that two sides with the same \( k, k = 1, 2, 3, ..., n \), from adjacent segments have a common edge. Assuming a uniformly distributed current throughout all the segments, which is a good approximation even for the straight segments simulating curved sections of the coil [5], the fictitious magnetization \( M_c \) is chosen to be different from zero only.
As a result of our modeling method, the contribution of each straight segment to the resultant magnetic field of the coil is calculated as the sum of the fields due to the surface currents and charges on the sides $k = 1, 2, 3, \ldots, n$, and due only to the surface charges on the sides $k = n+1$ and $k = n+2$.

### III. Formulas for the Field Components

Consider an arbitrary trapezoidal side surface $k$ and a local Cartesian coordinate system as shown in Fig. 2, with the $z$ axis chosen along the edge $(k-1, k)$. The side has a surface current of density $J_{sc}$ in the direction of the positive $z$ axis and a surface charge density $\rho_{sc}$, determined from (2) and (3), respectively, in the form:

$$J_{sc} = Jn''(D - n'y)$$

$$\rho_{sc} = \mu Jn'(D - n'y)$$

where $D$ is the distance $X$ to the edge $(k-1, k)$, and $n'$ and $n''$ are the cosines of the angles made by $\hat{n}$ with $M_c$ and $M_c \times J$, respectively. Performing the corresponding integrations in (1) yields the Cartesian components of the field intensity at any point $(x, y, z)$ in local coordinates due to a side $k$, $k = 1, 2, 3, \ldots, n$,

$$H_x = \frac{J}{4\pi} \left\{ \left[ n''D - n' (n'x + n''y) \right] \lambda 
+ \left[ n'D - n' (-n''x + n'y) \right] \gamma - n'n''\psi \right\}$$

$$H_y = \frac{J}{4\pi} \left\{ n'D - n' (n'x + n''y) \right\} \lambda$$

$$H_z = \frac{J}{4\pi} \left\{ \left[ n''D - n' (n'x + n''y) \right] \gamma
+ \left[ n'D - n' (-n''x + n'y) \right] \lambda - n'n''\psi \right\}$$

$$-n^2d^2 \left[ \frac{R_1 - R_2}{l_{12}} + \frac{R_3 - R_4}{l_{34}} + \frac{Q_{12}^2}{l_{12}^2} + \frac{Q_{34}^2}{l_{34}^2} \right]$$

with

$$\psi = \frac{d_{23}}{l_{12}^2} (R_1 - R_2) + \frac{d_{24}}{l_{34}^2} (R_3 - R_4)$$

$$-d^2 \left( \frac{P_{12}^2}{l_{12}^2} + \frac{P_{34}^2}{l_{34}^2} \right)$$
where \( R_1, R_2, R_3, \) and \( R_4 \) are the distances from the field point to the vertices of the trapezoidal side considered, \( l_{12} \) and \( l_{34} \) are the lengths of the nonparallel sides of the trapezoid (see Fig. 2), and [5]

\[
P_{12} = -z_2y + dz \quad , \quad \lambda_{12} = \ln \frac{R_1 l_{12} - Q_{12}}{(R_2 + l_{12}) l_{12} - Q_{12}} \quad , \quad \lambda_{34} = \ln \frac{(R_3 + l_{34}) l_{34} - Q_{34}}{R_4 l_{34} - Q_{34}}
\]

\[
Q_{12} = dy + z_2z \quad , \quad Q_{34} = dy + (z_3 - z_4)(z - z_4)
\]

\[
\gamma = \tan^{-1} \frac{z_2 R_1^2}{x R_1 d} - \tan^{-1} \frac{(z - z_2)(Q_{12} - l_{12}^2) - z_2 R_2^2}{x R_2 d} + \tan^{-1} \frac{(z - z_3)(Q_{34} - l_{34}^2) - (z_3 - z_4) R_3^2}{x R_3 d} - \tan^{-1} \frac{(z - z_4) Q_{34} - (z_3 - z_4) R_4^2}{x R_4 d}.
\]

\( \gamma \) is the solid angle under which the trapezoidal side considered is seen from the observation point, with \( \gamma > 0 \) for \( x > 0 \) and \( \gamma < 0 \) for \( x < 0 \).

The coil magnetic field at any point in an unbounded and homogeneous space is obtained by adding the local values of the fictitious magnetization components \( (M_c = 0) \) outside the coil volume) and the corresponding expressions (6) for the sides \( k = 1, 2, 3, \ldots, n \), as well as the field components due to the surface charges (3) on the polygonal sides \( k = n+1, n+2 \), of all of the straight segments constituting the coil. The latter can be calculated by decomposing the polygonal sides in trapezoidal and triangular sections and then by using the formulas (6) accordingly. The contributions from a side \( k = n+1, n+2 \) shared by adjacent segments to the fields due to those segments are equal, thus being sufficient to be computed only once and then taken twice for the resultant field. It should be noted that the resultant coil field is independent of the choice of the origin \( X = 0 \) for measuring the distance \( D \). As remarked in [5], for the important practical case when the current density \( J \) throughout a coil is everywhere parallel to a given plane, the fictitious magnetization \( M_c \) can be chosen to have a unique direction within all the segments of the coil, namely perpendicular to this plane. In such a model there is no contribution from the sides \( k = n+1 \) and \( k = n+2 \), the necessary amount of computation being correspondingly reduced.

The formulas (6)-(8) in the local Cartesian system of coordinates can be expressed in an intrinsic, vector form, independently of any reference frame, as indicated in [5]. This allows the field components to be calculated directly in terms of the global coordinates, common to the entire coil, thus reducing appreciably the computation time.

IV. CONCLUSION

For coils constructed as a succession of straight segments of polygonal cross section, the formulas presented in the previous Section yield the exact values of the field components. For usual coils, their curved sections are replaced by an appropriate number of straight segments in terms of the desired accuracy. The relationship between the number of straight segments and accuracy has been analyzed previously [5] by using toroidal coils of rectangular cross section.

The simple formulas derived on the basis of our modeling procedure contain only elementary functions and they are presented in a form allowing a straight-forward implementation in a computer algorithm. As an illustration of their efficiency, we mention that for an accuracy of about 1% in the center region of toroidal coils and tokamak coils of rectangular cross section [5], the amount of computation required when applying the formulas derived in this paper is approximately an order of magnitude less than that when using the expressions for toroidal segments, containing elliptic functions and integrals. This reduction of the computation time is even more substantial when comparing with results obtained by numerical integration methods [4].

The formulas presented can readily be particularized for coils of rectangular cross section and for infinitely long conductors of an arbitrary polygonal cross section [7].
REFERENCES


