MODELLING FIELDS FROM VOLUME DISTRIBUTIONS OF CURRENT IN THE PRESENCE OF SOLID CONDUCTORS

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A new procedure is presented for computing quasistationary magnetic fields in the presence of induced solid conductors due to carrying-current coils, based on the surface magnetic charge model of the coils. A benchmark example illustrates the application of the proposed method, as well as its efficiency and versatility.

1. INTRODUCTION

Quasistationary magnetic fields and eddy currents induced in linear solid conductors due to various distributions of electric current can be determined by using equivalent distributions of magnetic dipole sources [1]–[4]. A simple model for the field of a magnetic dipole in the presence of a solid conductor was constructed recently, based on the equivalence between a dipole and a string of dipoles along an arbitrary path connecting the poles of the dipole [5]. As well, volume distributions of electric current encountered in electromagnetic devices have been modelled by only using equivalent distributions of fictitious surface magnetic charge and fictitious magnetization, in order to formulate magnetic field problems in terms of a single-valued scalar potential, instead of a vector potential [6], [7].

We show here how to employ the surface charge models of practical current coils to simplify drastically the analysis of their field in the presence of induced solid conductors. The modelling technique is illustrated for the case of a conducting spherical shell in the field of coils carrying alternating current.

2. SURFACE CHARGE MODEL OF CURRENT COILS

Given distributions of stationary or quasistationary electric current can be treated by using both the Ampèrian (current) and the Coulombian (charge) models for magnetized media, as shown below. Consider a volume distribution of current of known density \( J \) in free space. Applying first the Ampèrian model, we assume
that this distribution would represent an Ampèreian current distribution corresponding to a fictitious magnetization $M_c$, such that

$$\nabla \times M_c = J.$$  \hspace{1cm} (1)

According to the Ampèreian model, the effect of the magnetization $M_c$ is the same as that of the given distribution of $J$ and of a distribution of surface current over the surfaces of discontinuity $S_{12}^{(t)}$ of the tangential components of $M_c$ of density

$$J_{sm} = n_{12} \times (M_{c2} - M_{c1}),$$  \hspace{1cm} (2)

where $n_{12}$ is the normal unit vector from the side 1 to the side 2 of each $S_{12}^{(t)}$. Thus, for determining the magnetic induction $B_J$, the given distribution of $J$ can be replaced by the fictitious distributions of magnetization $M_c$ and of surface current of density $J_{sc} = -J_{sm}$ on all the surfaces $S_{12}^{(f)}$. At any point in a nonmagnetic region of permeability $\mu_0$ we have [6]

$$B_J = \mu_0 H_J = B_{M_c} + B_{J_{sc}},$$  \hspace{1cm} (3)

where $B_{M_c}$ is due to $M_c$ and $B_{J_{sc}}$ to $J_{sc}$.

On the other hand, according to the Coulombian model, the magnetic field intensity $H_{Mc}$ produced by $M_c$ in free space is identical to that produced by the following distributions of volume charge density $\rho_c$ and of surface charge density $\rho_{sc}$ on all the surfaces of discontinuity $S_{12}^{(n)}$ of the normal components of $M_c$

$$\rho_c = -\mu_0 \nabla M_c,$$  \hspace{1cm} (4)

$$\rho_c = -\mu_0 n_{12} (M_{c2} - M_{c1}),$$  \hspace{1cm} (5)

where $n_{12}$ is the normal unit vector from the side 1 to the side 2 of each $S_{12}^{(n)}$. Thus,

$$B_{M_c} = \mu_0 (H_{Mc} + M_c),$$  \hspace{1cm} (6)

and from (3)

$$H_J = M_c + H_{Mc} + H_{J_{sc}},$$  \hspace{1cm} (7)

with $H_{J_{sc}} = B_{J_{sc}} / \mu_0$.

Therefore, in a region of permeability $\mu_0$, the magnetic field intensity $H_J$ due to a volume distribution of current of density $J$ can be determined at any point by the sum of $M_c$ at that point and the magnetic field intensities due to the charge
distributions $\rho_c$ and $\rho_{sc}$, and due to the surface current distribution $J_{sc}$. It should be noted that, since we do not impose any \textit{a priori} conditions for its divergence, the field $M_c$ is not uniquely defined by (1) alone and, as a consequence, simple expressions for $M_c$ can be found extremely easily for practical current distributions [6], [7]. If $M_c$ is chosen such that $J_{sc}$ is zero everywhere, then the given volume distribution of current is modelled in terms of $M_c$ and only charge distributions. Obviously, if the chosen $M_c$ also has its divergence equal to zero everywhere, then the model is constructed with $M_c$ and only a surface charge distribution. Now the field computation is substantially reduced since we replace the original volume vector distribution of $J$ by an equivalent surface scalar distribution of $\rho_{sc}$.

As an illustrative example, consider a carrying-current toroidal coil with the equivalent current density $J$ assumed to be constant over its rectangular cross section. One of the possible models for this coil can be constructed as shown in Fig. 1, by choosing

$$
M_c = \begin{cases} 
  z' J (b' - r') & \text{inside the coil}, \\
  z' J (b' - a') & \text{for } z' \in (-c, c), r' \in [0, a'], \\
  0 & \text{elsewhere}, 
\end{cases}
$$

(8)

which yields only a surface charge distribution, with

$$
\rho_{sc} = \begin{cases} 
  \pm \mu_0 J (b' - r') & \text{for } z' = \pm c, \quad r' \in [a', b'], \\
  \pm \mu_0 J (b' - a') & \text{for } z' = \pm c, \quad r' \in [0, a']. 
\end{cases}
$$

(9)
Similar models can be constructed for other coils or for straight conductors of rectangular cross section [6]. In the next section, we use this type of a model for current coils to analyze the quasistationary field produced in the presence of a solid conductor.

3. CALCULATION OF QUASISTATIONARY FIELDS DUE TO CURRENT COILS

Consider a system of arbitrarily positioned carrying-current coils with rectangular cross sections in the presence of linear solid conductors. To compute the field, we first construct the surface charge model of each coil, as shown in the previous section. Then, we calculate the field contribution from a point magnetic charge corresponding to an element of surface charge in a particular position, by employing the field expressions for an equivalent string of dipoles [5] connected between the point charge and a point conveniently chosen either at finite distance or at infinity. Contributions from elements of surface charge arbitrarily located are obtained by using simple transformations related to the rotation of the axes of coordinates. The resultant field is computed by superposing the contributions from all the surface charge elements in the coil models.

This general procedure is illustrated below for the case of a conducting spherical shell where simple analytic expressions are derived for the contribution from a point magnetic charge.

3.1. "POINT MAGNETIC CHARGE" IN THE PRESENCE OF A CONDUCTING SPHERICAL SHELL

Assume a point magnetic charge \( Q_m \) on the \( z \)-axis of a system of coordinates with its origin at the centre of a spherical shell of radii \( a \) and \( b \), of conductivity \( \sigma \) and permeability \( \mu \). For current coils placed inside the shell, we take the point charge \( Q_m \) at an arbitrary distance \( r_d < a \) from the origin, as shown in Fig. 2, and we couple it with a point charge \(-Q_m\) placed at the origin. This pair of point magnetic charges corresponds physically to a string of magnetic dipoles between the two charges [5]. In the case of current coils located outside the shell, we consider the point charge \( Q_m \) on the \( z \)-axis at \( z > b \) and its pair \(-Q_m\) removed at infinity. In what follows, the field quantities are considered to be sinusoidal with time of same frequency and their representation in complex is used throughout.

To calculate the magnetic field due to such a pair of magnetic charges we can use in the regions \( r < a \) and \( r > b \) single-valued scalar magnetic potentials [6] which are functions only of the spherical coordinates \( r \) and \( \theta \).
Fig. 2 – Conducting spherical shell with a point magnetic charge inside it.

\[ V_c = V_c(r, \Theta), \quad (10) \]

while in the region \( a < r < b \) we use a vector magnetic potential which, due to axisymmetry, can be chosen to have only a \( \varphi \) – component and depending on \( r \) and \( \Theta \),

\[ A = e_\varphi A(r, \Theta). \quad (11) \]

The corresponding magnetic inductions are

\[ B = \mu_0 \mathbf{H} = -\mu_0 \nabla V_c = -\mu_0 \left[ e_r \frac{\partial V_c}{\partial r} + e_\varphi \frac{1}{r} \frac{\partial V_c}{\partial \Theta} \right], \quad r < a \text{ and } r > b \quad (12) \]
\[ B = \mu H = \nabla \times A = e_r \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A \sin \theta) - e_\theta \frac{1}{r} \frac{\partial}{\partial r} (Ar), \quad a < r < b. \tag{13} \]

For the case when the point charge \( Q_m \) is in \( r < a \), the scalar potential due to \( Q_m \) and \(-Q_m\) considered to be alone in free space is

\[ V_c = \frac{Q_m}{4\pi\mu_0} \left[ \frac{1}{\sqrt{r^2 - e_z r_d}} - \frac{1}{r} \right]. \tag{14} \]

The potentials in the three regions are calculated by using the expansion of \( 1/\sqrt{r - e_z r_d} \) in series of Legendre polynomials \( P_n(\cos \theta) \) and applying the method of separation of variables to the respective partial differential equations, with the corresponding boundary conditions at \( r = a \) and \( r = b \). If \( \mu = \mu_0 \), the vector potential in (11), for instance, is derived in the form

\[ A = \frac{Q_m}{4\pi a^2} \sum_{n=1}^{\infty} A_n \left[ k_{n-1}(\gamma b)j_n(\gamma r) + i_{n-1}(\gamma b)k_n(\gamma r) \right] P_n^1(\cos \theta), \quad a < r < b, \tag{15} \]

where \( \gamma = \sqrt{j \omega \mu_0 \sigma} \), \( j = \sqrt{-1} \), \( P_n^1 \) are the associated Legendre functions of the first kind [8], [9], \( i_n \) and \( k_n \) are the modified spherical Bessel functions of the first and second kind, respectively, defined in terms of the modified Bessel functions \( I \) and \( K \) [10] by

\[ i_n(\gamma r) = \frac{\pi}{\sqrt{2} \gamma r} I_{n+1/2}(\gamma r), \tag{16} \]

\[ k_n(\gamma r) = \frac{\pi}{\sqrt{2} \gamma r} K_{n+1/2}(\gamma r), \tag{17} \]

and

\[ A_n = \frac{2n + 1}{n} \left( \frac{r_d}{a} \right)^{n} \left[ i_{n+1}(\gamma a)k_{n-1}(\gamma b) - k_{n+1}(\gamma a)i_{n-1}(\gamma b) \right]. \tag{18} \]

The current density induced in the spherical shell has only a \( \varphi \) – component,

\[ J = -j \omega A. \tag{19} \]

To compute the field when the inducing coils are outside the spherical shell, one derives in a similar manner potential expressions corresponding to a point magnetic charge on the z-axis in the region \( r > b \).
3.2. FIELD OF THE EQUIVALENT SURFACE DISTRIBUTION OF MAGNETIC CHARGE

The resultant field due to the surface distributions of charge in the coil models is computed in a global system of Cartesian coordinates \( OXYZ \), in which a point charge representing the element of surface charge has specified spherical coordinates \( r_p, \Theta_p, \Phi_p \). In Section 3.1 we used a special system of Cartesian coordinates \( Oxyz \) that has its \( z \)-axis through the point charge. The system \( Oxyz \) can be obtained from \( OXYZ \) by two successive rotations, the first one with an angle \( \Phi_p \) about the \( Z \)-axis from \( X \) to \( Y \) and the second one with an angle \( \Theta_p \) about the new \( y \)-axis from \( Z \) to the new \( x \)-axis [11]. The coordinates \( x, y, z \) are related to \( X, Y, Z \) by

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} =
\begin{bmatrix}
  \cos \Theta_p \cos \Phi_p & \cos \Theta_p \sin \Phi_p & -\sin \Theta_p \\
  -\sin \Phi_p & \cos \Phi_p & 0 \\
  \sin \Theta_p \cos \Phi_p & \sin \Theta_p \sin \Phi_p & \cos \Theta_p
\end{bmatrix}
\begin{bmatrix}
  X \\
  Y \\
  Z
\end{bmatrix}, \tag{20}
\]

The field quantities at any point of coordinates \( X, Y, Z \) in the global system are calculated as follows. First, the coordinates \( x, y, z \) of the same point in the transformed system are determined from (20), with \( \Theta_p \) and \( \Phi_p \) corresponding to the point charge \( Q_n \) representing the charge \( \rho_{n \delta} \Delta S \) on the surface element \( \Delta S \) in the distribution (9) for the coil considered. Then, the spherical coordinates \( r, \theta \) corresponding to \( x, y, z \) are used in (10) – (18) to calculate the field components, for instance \( H_r \) and \( H_\theta \). From these components, one obtains the respective Cartesian components \( H_x, H_y, H_z \) in \( Oxyz \) and, finally, the Cartesian components \( H_x, H_y, H_z \) in the global system with

\[
\begin{bmatrix}
  H_x \\
  H_y \\
  H_z
\end{bmatrix}^T = R^{-1}(\Theta_p, \Phi_p) \begin{bmatrix}
  H_x \\
  H_y \\
  H_z
\end{bmatrix}^T, \tag{21}
\]

where \( R^{-1} \) is the inverse of the square matrix in (20). It should be noted that the computation time taken by these rotation transformations is practically negligible with respect to that required to evaluate the single series in the field expressions in the local system \( Oxyz \).

At any point where \( M_c = 0 \) the resultant field is obtained by superposing the contributions of all the elements \( \rho_{n \delta} \Delta S \) of the surface distributions in the charge models of the coils (see Fig. 1). As shown in (7), at the points where \( M_c \neq 0 \) in the coil models the resultant field is determined by adding \( M_c \) to the magnetic field intensity generated by the charge distributions.

The shielding factors are obtained from field values and the power loss dissipated within the shell can be computed by employing the Poynting vector [12].
Approximate formulas for usual values of geometric dimensions, conductivity and frequency, when $|\gamma b| >> |\gamma a| >> 1$, can be derived using the asymptotic expressions of the modified spherical Bessel functions, which are simple exponentials. Approximations for spherical shells with a thickness $b - a$ about the depth of penetration and for very thin shells can be worked out as shown in [12].

4. CONCLUSIONS

To show its efficiency, the proposed modelling procedure has been implemented for the case of conducting spherical shells. All the field quantities have been expressed in terms of single Legendre series in $\cos \theta$ which are much more convergent than the double series of associated Legendre functions $P_n^m(\cos \theta)$ in the expressions published recently in the literature [13]. Moreover, the fields due to given volume distributions of current are analyzed by using equivalent surface distributions of scalar sources. This reduces substantially the overall computation effort required.

Field solutions based on analytic expressions can also be developed by applying the modelling method presented to cylindrical and planar conducting structures.

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REFERENCES


