Circuit models for symmetric systems of linear equations

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SUMMARY

Electric circuit models are constructed for general symmetric systems of linear algebraic equations. The modelling procedure is based on the node-voltage analysis of linear circuits under steady-state conditions. These electric circuit models can, in principle, be physically realized and used for the solution of the systems of equations by simply measuring the electric voltage at the circuit nodes. Copyright © 2000 John Wiley & Sons, Ltd.

1. INTRODUCTION

General linear electric circuits can be solved by applying general methods based on Kirchhoff's theorems, such as the loop-current method and the node-voltage method [1,2]. A direct application of Kirchhoff's theorems yields a sparse matrix system of linear equations, with a number of equations which is equal to the number of branches in the circuit, the unknowns being the branch currents. In the loop-current method the fictitious mesh currents satisfy a symmetric linear system of equations, with one equation written for each independent loop of the circuit mesh. For purely resistive circuits, for instance, the resistance common to two different independent mesh loops is taken in the respective equations with a positive or a negative sign, in terms of the relative direction of the mesh currents in the two loops. The system of equations has a sparse matrix for usual topologies of complex circuits due to the fact that there are independent loops which do not have common branches. This makes it impractical to construct circuit models for dense symmetric systems of equations on the basis of the loop-current method of analysis.

On the other hand, the node-voltage method yields $n$ independent equations satisfied by the nodal potentials for a circuit with $n + 1$ nodes, where one of the nodes is taken as zero-potential reference. Each of these equations represents the Kirchhoff current theorem satisfied by the currents carried by the branches connected to a given independent node, with each branch current being expressed in terms of the voltages at the two respective branch nodes. The system of equations is symmetric and in the case of a circuit in the form of a complete polygon with $n + 1$
Figure 1. Topology of an electric circuit with \( n + 1 \) nodes: (a) complete polygon; (b) arbitrary branch.

For the circuit in Figure 1, the nodal potentials \( x_1, x_2, \ldots, x_n \), defined with respect to the zero potential of the node \( n + 1 \) chosen as reference, satisfy the following system of \( n \) equations:

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
  &\vdots \\
  a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{align*}
\]

(1)

where \( a_{ik} = a_{ki} \). In the case of a purely resistive circuit \( a_{ik} \), \( i \neq k \), is the conductance of the branch connected between the nodes \( i \) and \( k \), \( a_{ii} \) is the negative sum of the conductances of all the branches connected to the node \( i \)

\[
a_{ii} = -\sum_{k=1, k \neq i}^{n+1} a_{ik}
\]

(2)
and \( b_i \) is the sum of the short-circuit currents leaving the node \( i \) of all the branches connected to that node; for an arbitrary branch \( ik \) which contains, for instance, a voltage source of electromotive force \( E_{ik} \), as shown in Figure 1(b), this short-circuit current is equal to \( E_{ik}a_{ik} \). In the case of linear circuits under sinusoidal steady-state conditions, one uses the phasor representation and then, whenever there is no magnetic coupling between branches, \( a_{ik} \)'s are the corresponding complex admittances.

2. MODELLING PROCEDURE

Consider a symmetric linear system of \( n \) algebraic equations in \( n \) unknowns (1), with all the entries assumed to be real numbers. A direct model of the system of equations would be that in the Figure 1, with the branch conductances given by the entries \( a_{ik} \), \( i \neq k \), and with the conductances \( a_{i,n+1} \), \( i = 1, 2, \ldots, n \), of the branches connected to the node of reference determined from the condition that \( a_{ii} \) in (1) be the negative sum of the conductances of all the branches connected to the node \( i \)

\[
a_{i,n+1} = -a_{ii} - \sum_{k=1, k \neq i}^{n} a_{ik}
\]  

(3)

In each of the branches connected directly between the nodes \( i \) and \( n + 1 \), with its conductance \( a_{i,n+1} \neq 0 \), we insert a voltage source of an electromotive force oriented towards the node \( n + 1 \) and equal to

\[
E_{i,n+1} = \frac{b_i}{a_{i,n+1}}
\]  

(4)

Whenever \( a_{i,n+1} = 0 \) or is close to zero, we insert a voltage source in one of the branches connected between the node \( i \) and a selected node \( k \) that has its conductance \( a_{ik} \neq 0 \), with an electromotive force oriented towards the node \( k \) and equal to

\[
E_{ik} = \frac{b_i}{a_{ik}}
\]  

(5)

Alternatively, instead of voltage sources, we can use constant current sources and connect at each node \( i \) a source of constant current \( b_i \) leaving the node.

If all \( a_{ik} \geq 0, i \neq k, i, k = 1, 2, \ldots, n \), and if all \( a_{ii} < 0 \) such that (see (3)) even \( a_{i,n+1} \geq 0 \), then the circuit model can be simply constructed with ordinary passive resistors and voltage sources or current sources. In general, when some of the values \( a_{ik} < 0, i \neq k \), then one would have to use devices that present linear negative conductances, which are practically realizable. If some of \( a_{ik} = 0, i \neq k \), then the corresponding branches are missing in the current model.

The presence of electric sources in the models is undesirable, since calibrated sources, with constant parameters, are very expensive. Fortunately, it is possible to model the systems of equations without using any internal source.
3. CIRCUIT MODELS WITH A SINGLE EXTERNAL VOLTAGE SOURCE

In order to construct a model with no voltage source connected in the circuit branches or current source connected at the nodes, we write the given symmetric system of equations (1) in the equivalent form

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + a_{1,n+1}x_{n+1} &= 0 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + a_{2,n+1}x_{n+1} &= 0 \\
    &\vdots \\
    a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + a_{n,n+1}x_{n+1} &= 0 \\
    x_{n+1} &= 1
\end{align*}
\]

where \( a_{i,n+1} = -b_i, i = 1, 2, \ldots, n \). With these entries \( a_{i,n+1} \) considered to represent conductances, as all the other coupling conductances \( a_{ik}, i \neq k \), the circuit model of the system (6) is as shown in Figure 2, with \( n + 2 \) nodes and with a unit voltage applied to the node \( n + 1 \) with respect to the reference potential \( x_{n+2} = 0 \) of the node \( n + 2 \). The conductances \( a_{i,n+2} \),
Each of the first $n$ equations of (6) represents the Kirchhoff current theorem as applied to the respective node of the circuit in Figure 2.

In the process of modelling system (6), the values of $a_{i,n+1}, i = 1, 2, \ldots, n$, can be scaled by the same factor to obtain convenient conductance values, for instance of the same order of magnitude as those of other conductances in the circuit. Whenever this scaling is employed, the values of the unknowns $x_1, x_2, \ldots, x_n$ are to be automatically scaled by the same factor. Once the model in Figure 2 is finalized, a unit voltage is applied between the nodes $n+1$ and $n+2$, corresponding to the last equation in the equivalent system (6), and the solution of the original system (1) is simply obtained by measuring the potentials $x_1, x_2, \ldots, x_n$ of the respective nodes and dividing these potential values by the scaling factor.

If, again, all $a_{ik} = 0, i \neq k$, then the model in Figure 2 is constructed with only ordinary passive resistors. In general, when some entries $a_{ik} < 0, i \neq k$, then we would have to use linear circuit elements with negative conductances.

Remark. One can see directly from the model circuit in Figure 2 that if all $a_{ik} \geq 0, i \neq k$, including all $a_{i,n+2}$, then $0 \leq x_i \leq 1, i = 1, 2, \ldots, n$.

Note. When system (6) has all $a_{ik} \geq 0, i \neq k$, and all $a_{ii} < 0$ such that all $a_{i,n+2} \geq 0$ (in (7)), then the circuit model in Figure 2 can also be constructed employing, instead of resistors, static capacitors with capacitance values adjusted to be proportional to $a_{ik}, i \neq k$. Having all the capacitors initially discharged, by applying a unit direct-current potential difference between the nodes $n+1$ and $n+2$, the measured electrostatic potentials of the nodes $1, 2, \ldots, n$ give the solution $x_1, x_2, \ldots, x_n$, respectively, of the system of algebraic equations (1), after taking into account the factor of proportionality applied to the capacitance values and the scaling factor applied when passing from $b_i$ in (1) to the actual values of $a_{i,n+1}$ used to model (6). In this capacitor model, each of the first $n$ equations of (6) expresses the fact that, after applying the voltage between the nodes $n+1$ and $n+2$, in the new static final situation, the sum of the charges of all the capacitor plates connected at each node $i, i = 1, 2, \ldots, n$, is equal to zero. Practically, appropriate resistors are connected in series with the capacitors in the circuit model in order to ensure smooth transients between the initial and final states of the circuit.

4. MODELS FOR MODIFIED SYSTEMS WITH NON-NEGATIVE OFF-DIAGONAL ENTRIES

In general, the symmetric systems (1) to be modelled could have a large number of negative entries $a_{ik}, i \neq k$, and positive $b_i$'s. Practically, one will be interested to construct circuit models with as few negative conductances as possible, for as long as the respective resistor devices are more expensive than those with positive conductances. In order to reduce the number of necessary circuit elements with negative conductances to less than or a maximum of $n+1$, we proceed as follows.
First, we search for the maximum positive $b_i$ and for the negative $a_{ik}, i \neq k$, that has a maximum absolute value. Then, the following equation is added to each equation of (1):

$$c_1(x_1 + x_2 + \cdots + x_n + x_{n+1}) = -c_2$$

where

$$c_1 \geq \max(-a_{ik}), \quad i \neq k$$
$$c_2 \geq \max(b_i), \quad i, k = 1, 2, \ldots, n$$

If all $b_i \leq 0$, then $c_2$ can be chosen to be equal to zero. System (1) becomes

$$a'_{11}x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n + a'_{1,n+1}x_{n+1} = b'_1$$
$$a'_{21}x_1 + a'_{22}x_2 + \cdots + a'_{2n}x_n + a'_{2,n+1}x_{n+1} = b'_2$$
$$\vdots$$
$$a'_{n1}x_1 + a'_{n2}x_2 + \cdots + a'_{nn}x_n + a'_{n,n+1}x_{n+1} = b'_n$$
$$a'_{n+1,1}x_1 + a'_{n+1,2}x_2 + \cdots + a'_{n+1,n}x_n + a'_{n+1,n+1}x_{n+1} = b'_{n+1}$$

with the unknown $x_{n+1}$ and Equation (8) added, where

$$a'_{ik} = a_{ik} + c_1, \quad i, k = 1, 2, \ldots, n$$
$$a'_{i,n+1} = a'_{n+1,i} = c_1, \quad i = 1, 2, \ldots, n + 1$$
$$b'_i = b_i - c_2, \quad i = 1, 2, \ldots, n$$
$$b'_{n+1} = -c_2$$

This symmetric system has all the entries $a'_{ik} \geq 0, i \neq k$, all $b'_i \leq 0$, and we write it, as before (see (6)), in the form with one more supplementary 'unknown' and one more equation, as

$$a'_{11}x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n + a'_{1,n+1}x_{n+1} + a'_{1,n+2}x_{n+2} = 0$$
$$a'_{21}x_1 + a'_{22}x_2 + \cdots + a'_{2n}x_n + a'_{2,n+1}x_{n+1} + a'_{2,n+2}x_{n+2} = 0$$
$$\vdots$$
$$a'_{n1}x_1 + a'_{n2}x_2 + \cdots + a'_{nn}x_n + a'_{n,n+1}x_{n+1} + a'_{n,n+2}x_{n+2} = 0$$
$$a'_{n+1,1}x_1 + a'_{n+1,2}x_2 + \cdots + a'_{n+1,n}x_n + a'_{n+1,n+1}x_{n+1} + a'_{n+1,n+2}x_{n+2} = 0$$
$$x_{n+2} = 1$$

where \( a'_{i,n+2} = -b'_i, \ i = 1, 2, \ldots, n + 1 \). This modified system is modelled as in Section 3 (see Figure 2), with a potential reference node \( n + 3, x_{n+3} = 0 \), and with a unit voltage applied to the node \( n + 2, x_{n+2} = 1 \); no other voltage or current source is needed. In this model all the conductances are positive or zero, except for some in the branches connected to the node of reference. Such a branch has a conductance (see (7))

\[
a'_{i,n+3} = -a'_{ii} - \sum_{k=1, k \neq i}^{n+2} a'_{ik}, \quad i = 1, 2, \ldots, n
\]

which is positive or zero only if

\[
a'_{ii} \leq \sum_{k=1, k \neq i}^{n+2} a'_{ik}
\]

i.e. for the original system of equations

\[
a'_{ii} \leq -\sum_{k=1, k \neq i}^{n} a_{ik} + b_i - (n + 1)c_1 - c_2
\]

Notice that, since \( c_1 \geq 0 \) and \( c_2 \geq 0 \), the conductance between the nodes \( n + 1 \) and \( n + 3 \) is always negative if one or both of \( c_1 \) and \( c_2 \) are different from zero

\[
a'_{n+1,n+3} = -(n + 1)c_1 - c_2 \leq 0
\]

As in Section 3, a scaling of all the right-hand sides \( b'_i \)'s in (10) is performed (along with all \( x'_i \)'s) in order to adjust the values of the corresponding conductances in the model.

5. PURELY REACTIVE CIRCUIT MODELS

Circuit models for general symmetric systems of linear equations with real number entries can be constructed with purely reactive circuit elements instead of elements with negative conductance which may be more difficult to be manufactured within the existent technology. Consider system (1) written in the form (6), with real positive or negative entries. Multiplying all the coefficients \( a_{ik} \) by \( \sqrt{-1} \) yields the phasor (complex representation) form of the node-voltage equations for a linear circuit with the topology in Figure 2, whose branches consists of purely reactive elements, inductive if the respective coefficients \( a_{ik}, \ i \neq k \), are positive, or capacitive if the coefficients are negative, with a purely sinusoidal with time unit voltage (rms or peak value) applied between the node \( n + 1 \) and the node of reference \( n + 2 \), and no source element present in its branches. The voltages (in rms or peak values) measured between the nodes 1, 2, \ldots, \( n \) and the reference node \( n + 2 \) give the absolute values of the unknowns \( x_1, x_2, \ldots, x_n \); the actual value of an unknown is positive if the respective measured voltage is in phase with the applied voltage and is negative in the opposite case.
The choice of the operating frequency and the scaling of the entries in the given system of equations depend on the inductive and the capacitive circuit elements available. Accurately adjustable inductive or capacitive circuit elements can be manufactured by using segments of lossless or very low loss transmission lines [3], for instance. Conductors connecting various elements in the circuit model must have a resistance which is negligible with respect to the magnitude of the reactance of the elements at the operating frequency, and the inductive and capacitive couplings are to be designed such that the electromagnetic interference is practically eliminated. Such a 'purely' reactive circuit model remains in a stable state of operation as long as the matrix of the system of equations modelled is sufficiently far from being singular.

When it is more convenient to use, for instance, a reduced number of inductors in the circuit model, as compared with the number of capacitors, one can use a procedure similar to that in Section 4 to make negative or zero the entries $a'_{ik}, i \neq k$, in (12), such that the only positive values remain those among the quantities $a'_{i,n+3}$ (13) corresponding to reactances of branches connected to the node of reference.

6. CONCLUSIONS

A practical realization of circuit models as presented in this paper, with reliable values of the circuit elements, is, in principle, possible. Future developments in the area of integrated circuits, associated with sufficiently accurate and speedy techniques of adjusting and controlling the numerical values for the circuit elements, will make the construction of the proposed models feasible from an engineering point of view. Once these models are constructed, the solution of arbitrarily large symmetric systems of linear algebraic equations is obtained practically instantaneously, being given by electric voltages measured at the circuit nodes. Such ‘analogue’ computation devices would be extremely beneficial for a most rapid solution especially when some randomly positioned entries in the system matrix are changed many times, as in problems related to various imaging devices or to optimization techniques. Of course, the nodal voltages and the corresponding branch currents are limited in practice, such that the numerical values associated with the circuit elements remain within the accuracy limits required. This will limit the range of applicability of the proposed models to systems of equations which are not nearly singular.

The modelling procedure presented can be, in principle, readily extended to systems of linear equations with complex coefficients, when the branch conductances are replaced by the corresponding complex admittances. Circuit models with a reduced number of branches that have admittances with negative real part can be derived by applying a procedure similar to that described in Section 4.

REFERENCES

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