Fields, losses and forces due to moving current distributions coaxial with a cylindrical conductor

Ioan R. Ciric

Department of Electrical and Computer Engineering, The University of Manitoba, Winnipeg, Manitoba, Canada

Keywords Electromagnetics, Magnetic fields, Motion

Abstract An exact mathematical solution has been obtained for the quasistationary electromagnetic field of a circular current loop coaxial with a conducting circular cylinder in uniform relative motion with respect to each other. The vector magnetic potential corresponding to a filamentary loop carrying a variable with time current is determined by applying the method of separation of variables. Expressions for the fields outside and inside the cylindrical conductor, the eddy-current distribution, the power loss and the interaction force are derived directly from the vector potential. The fields due to practical current coils are obtained by integration from the results for the filamentary loop. An approximate simple formula is presented for a loop carrying direct current at high velocities. The analysis performed is relevant to the design and operation of magnetic devices with metallic cores in motion, linear electrical machines and electromagnetic launching systems.

Introduction

Electromagnetic fields in the presence of moving solid conductors, as those in linear electrical machines, launching systems and numerous other electromagnetic devices, have usually been treated by applying various approximate methods or numerical techniques (Nasar and Boldea, 1976; Poloujadoff, 1980; Hsieh and Driga, 1993). Exact analytic solutions have only been obtained for a few idealized configurations (Ciric, 1979, 1983; Ciric and Mathur, 1984, 1985, 1986). Such benchmark solutions are desired for evaluating the accuracy of various numerical formulations and solutions for electromagnetic systems with moving induced solid conductors or/and moving current distributions.

In this paper, we consider a very long circular cylinder of radius $a$, conductivity $\sigma$, and permeability $\mu$, in the presence of a coaxial circular filamentary loop of radius $r_0$ carrying a sinusoidal with time current, $i = I \cos \omega t$, the loop and the cylindrical conductor being in relative axial motion with a constant velocity $v$, as shown in Figure 1. The medium outside the cylinder has a permeability $\mu_0$. For linear media the fields due to a general time-varying current carried by the loop can be analyzed by superposing the fields due to its time-harmonic current components.

At velocity values encountered in engineering applications one can apply the Maxwell-Hertz equations for moving media, by using a reference system
attached to either the cylinder, in which the loop has a velocity \( \mathbf{v} \), or to the loop, in which the cylinder has a velocity \(-\mathbf{v}\), the field quantities being defined in both cases in the local reference system, attached to the moving media (Ciric, 1979). The magnetic induction and the electric field intensity are expressed in terms of a vector magnetic potential as:

\[
\mathbf{B} = \nabla \times \mathbf{A},
\]

(1)

\[
\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \times (\nabla \times \mathbf{A}),
\]

(2)

where \( \mathbf{u} \) is the local velocity in the reference system considered. Due to the axisymmetry of the structure in Figure 1, \( \mathbf{A} \) is chosen to have only an azimuthal component depending on time and on the circular cylindrical coordinates \( r \) and \( z \), \( A(r,z,t) \). Equations (1) and (2) yield the radial and the axial components of the magnetic induction, and the only component, i.e. the azimuthal component, of the electric field intensity, in the form:

\[
B_r = -\frac{\partial A}{\partial z}, \quad B_z = \frac{\partial A}{\partial r} + \frac{A}{r},
\]

(3)

\[
E = -\frac{\partial A}{\partial t} - \mathbf{u} \cdot \frac{\partial \mathbf{A}}{\partial z}.
\]

(4)

**Field analysis**

In a coordinate system attached to the moving loop (see Figure 1) the vector magnetic potential and all the other field quantities are sinusoidal with time, of same frequency as the current carried by the loop. We use a phasor representation with the factor \( \exp(j\omega t) \) suppressed.
Vector magnetic potential solution

In the loop coordinate system the conducting cylinder has a velocity \( u = -v \) and the vector potential satisfies the equations

\[
\frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} - \frac{A}{r^2} + \frac{\partial^2 A}{\partial z^2} + v \mu \sigma \frac{\partial A}{\partial z} - j \omega \mu \sigma A = 0, \quad r < a, \tag{5}
\]

\[
\frac{\partial^2 A_e}{\partial r^2} + \frac{1}{r} \frac{\partial A_e}{\partial r} - \frac{A_e}{r^2} + \frac{\partial^2 A_e}{\partial z^2} = -\mu_o I \delta(r - r_o) \delta(z - z_o), \quad r > a, \tag{6}
\]

where \( \delta \) is the Dirac delta function. The continuity conditions at \( r = a \) are:

\[
\left. \frac{A}{r} \right|_{r=a} = \left. \frac{A_e}{r} \right|_{r=a}, \tag{7}
\]

\[
\left( \frac{\partial A}{\partial r} + \frac{A}{r} \right) \bigg|_{r=a} = \frac{\mu}{\mu_o} \left( \frac{\partial A_e}{\partial r} + \frac{A_e}{r} \right) \bigg|_{r=a}. \tag{8}
\]

Applying the method of separation of variables and imposing the regularity conditions at \( r = 0 \) and at infinity yields

\[
A = \frac{\mu_o I r_0}{2\pi a} \int_0^\infty \frac{K_1(\lambda r_o)}{\lambda} \left[ C_+^\ast I_1(k_\lambda^+ r) e^{j\lambda(z-z_o)} + C_-^\ast I_1(k_\lambda^- r) e^{-j\lambda(z-z_o)} \right] d\lambda, \quad r \leq a \tag{9}
\]

\[
A_e = \frac{\mu_o I r_0}{2\pi} \int_0^\infty K_1(\lambda r_o) K_1(\lambda) \left[ D_+^+ e^{j\lambda(z-z_o)} + D_-^+ e^{-j\lambda(z-z_o)} \right] d\lambda \tag{10}
\]

\[
+ \frac{\mu_o I r_0}{\pi} \int_0^\infty I_1(\lambda r_o) K_1(\lambda) \cos \lambda(z-z_o) d\lambda, \quad r \geq a
\]

the asterisk indicating the complex conjugate and the last term in \( A_e \) being the vector potential due to the current loop alone in free space (Panofsky and Phillips, 1962), with

\[
r_- \equiv \min(r, r_o), \quad r_+ \equiv \max(r, r_o). \tag{11}
\]

\( I_1 \) and \( K_1 \) are the modified Bessel function of order 1 and the first and second kind respectively, and

\[
k_\lambda^\pm \equiv \sqrt{\lambda^2 + j(\pm \omega - \lambda \nu) \mu \sigma}. \tag{12}
\]
The constants of integration $C^+_\lambda$ and $D^+_\lambda$ are determined from the boundary conditions (7), (8) and have the following expressions:

$$C^+_\lambda = \frac{1}{|I_1(k_\lambda^+ a)K_0(\lambda a) + \beta^+_1 I_0(k_\lambda^+ a)K_1(\lambda a)|},$$  \hspace{1cm} (13)

$$D^+_\lambda = C^+_\lambda \left[ I_1(k_\lambda^+ a)I_0(\lambda a) - \beta^+_1 I_0(k_\lambda^+ a)I_1(\lambda a) \right],$$  \hspace{1cm} (14)

with

$$\beta^+_\lambda \equiv \mu_0 k_\lambda^+ / (\mu \lambda).$$  \hspace{1cm} (15)

The complex density of current induced in the cylindrical conductor is calculated from (see (4) and (9))

$$J = \sigma \left( -j\omega A + v \frac{\partial A}{\partial z} \right).$$  \hspace{1cm} (16)

**Special cases**

For $v = 0$ one obtains the solution for a current loop and a conducting cylinder at relative rest, with

$$k_\lambda^+ = k_\lambda^- = \sqrt{\lambda^2 + j\omega \mu \sigma}, \quad \beta^+_\lambda = \beta^-_\lambda = \mu_0 \sqrt{\lambda^2 + j\omega \mu \sigma} / (\mu \lambda),$$  \hspace{1cm} (17)

$$C^+_\lambda = C^-_\lambda, \quad D^+_\lambda = D^-_\lambda.$$  \hspace{1cm} (18)

The results known in the literature (Fluerasu and Galan, 1966) are readily recovered from (9), (10) with (13), (14) and (17), (18).

For $\omega = 0$ we have the special case of a moving circular loop carrying a direct current in the presence of a coaxial conducting cylinder. Now,

$$k_\lambda^+ = k_\lambda^- \equiv k_\lambda = \sqrt{\lambda^2 - j\lambda \nu \sigma}, \quad \beta^+_\lambda = \beta^-_\lambda \equiv \beta_\lambda = \mu_0 k_\lambda / (\mu \lambda)$$  \hspace{1cm} (19)

and

$$C^+_\lambda = C^-_\lambda \equiv C_\lambda = \frac{1}{|I_1(k_\lambda a)K_0(\lambda a) + \beta_\lambda I_0(k_\lambda a)K_1(\lambda a)|},$$  \hspace{1cm} (20)

$$D^+_\lambda = D^-_\lambda \equiv D_\lambda = C_\lambda \left[ I_1(k_\lambda a)I_0(\lambda a) - \beta_\lambda I_0(k_\lambda a)I_1(\lambda a) \right].$$  \hspace{1cm} (21)

From (9) and (10) we derive the vector potential, which in this case is real and constant with time:
where $I_d$ is the direct current carried by the loop.

The case of a carrying-current straight wire parallel to a conducting semispace (Ciric, 1979) is obtained for $a, r_o, r \to \infty, r_o/a \to 1, r/a \to 1$, but with $r_o - a$ and $r - a$ kept finite, by employing the asymptotic expansions of the Bessel functions.

*Fields due to moving coils*

The result for a filamentary loop can be used to derive the vector potential and the currents induced in the cylinder due to given moving coaxial distributions of current.

In the case of a *thin coil* of radius $r_o$ and length $2h$ centered at the origin $O$ (see Figure 1), the vector potential is obtained by integrating in (9) and (10) with respect to $z_o$ from $-h$ to $h$. For instance, (9) yields

\[
A = \frac{\mu_0 N l r_o}{2 \pi a h} \int_0^{\infty} \frac{K_1(\lambda r_o) \sin \lambda h}{\lambda^2} \left[ C_+ I_1(k_1^+ r) e^{i \lambda z} + C_- I_1(k_1^- r) e^{-i \lambda z} \right] d\lambda, \quad r \leq a
\]

(24)

where $N$ is the number of turns of the coil.

For a coaxial *solenoid* centered at the origin $O$, of length $2h$ and radii $r_1$ and $r_2$, $r_1 < r_2$, the vector potential inside the conducting cylinder, for example, is

\[
A = \frac{\mu_0 N l I}{2 \pi a h (r_2 - r_1)} \int_0^{\infty} \frac{K(\lambda) \sin \lambda h}{\lambda^2} \left[ C_+ I_1(k_1^+ r) e^{i \lambda z} + C_- I_1(k_1^- r) e^{-i \lambda z} \right] d\lambda, \quad r \leq a
\]

(25)
where $N$ is the total number of turns of the solenoid and

$$K(\lambda) \equiv \int_{r_1}^{r_2} r_o K_1(\lambda r_o) dr_o. \quad (26)$$

### Power loss and force

The time-average power loss within the cylindrical conductor is determined as the real part of the complex Poynting vector,

$$P + jQ = -\frac{\pi a}{\mu} \int_{-\infty}^{\infty} (E B^*_2) |_{r=a} dz$$

$$= -\frac{\pi a}{\mu} \int_{-\infty}^{\infty} \left[ -j \omega A + v \frac{\partial A}{\partial r} \right] \left( \frac{\partial A^*}{\partial r} + \frac{A^*}{r} \right) |_{r=a} dz, \quad (27)$$

where $Q$ is the reactive power and $A$ is the vector potential inside the conductor. For a single moving current loop, from (9) and (13), one obtains:

$$P = \frac{\mu_0 I^2 r_0^2}{2a} \int_0^\infty \frac{[K_1(\lambda r_0)]^2}{\lambda} \omega \text{Im} \left( |C_\lambda^+|^2 R_\lambda^+ - |C_\lambda^-|^2 R_\lambda^- \right) \text{Re} \text{Im} \left( |C_\lambda^+|^2 R_\lambda^+ + |C_\lambda^-|^2 R_\lambda^- \right) d\lambda \quad (28)$$

$$Q = \frac{\mu_0 I^2 r_0^2}{2a} \int_0^\infty \frac{[K_1(\lambda r_0)]^2}{\lambda} \omega \text{Re} \left( |C_\lambda^+|^2 R_\lambda^+ - |C_\lambda^-|^2 R_\lambda^- \right)$$

$$- \lambda \nu \text{Re} \left( |C_\lambda^+|^2 - |C_\lambda^-|^2 \right) d\lambda,$$

with

$$R_\lambda^\pm = \beta_\lambda^\pm I_o (k_\lambda^\pm a) I_h (k_\lambda^\pm a). \quad (30)$$

To calculate the force on the loop we use the elementary expression

$$d\mathbf{F} = id\mathbf{l} \times \mathbf{B}_{\text{ext}} \bigg|_{r=r_o}, \quad (31)$$

where $\mathbf{B}_{\text{ext}}$ is the magnetic induction produced only by the currents induced within the cylinder and $d\mathbf{l}$ is the vector length element of the loop wire. The time-average force along the positive $z$-axis is
\[ F = \pi I_r \rho \frac{\partial A_{\text{ext}}}{\partial z} |_{r=r_0}, \] 
(32)

with (see (10))

\[ A_{\text{ext}} = \frac{\mu_0 I_0}{2\pi} \int_0^\infty \left[ K_1(\lambda r_0) \right]^2 \left[ D_\lambda^+ e^{j\lambda(z-z_0)} + D_\lambda^- e^{-j\lambda(z-z_0)} \right] d\lambda. \] 
(33)

Thus,

\[ F = -\frac{\mu_0 I^2 r_0^2}{2} \int_0^\infty \lambda |K_1(\lambda r_0)|^2 \text{Im}(D_\lambda^+ + D_\lambda^-) d\lambda. \] 
(34)

For the special case of a loop carrying a direct current \( I_d (\omega = 0) \) there is no reactive power and we have (with (19)-(21))

\[ P = -Fv = \frac{2\mu_0 I_d^2 r_0^2}{a} \int_0^\infty |K_1(\lambda r_0)|^2 |C_\lambda|^2 \text{Im}(R_\lambda) d\lambda, \] 
(35)

where

\[ R_\lambda \equiv \beta_\lambda I_0(k_\lambda a)I_1(k_\lambda a). \] 
(36)

The electromagnetic force exerted on the loop is along the negative z-axis, i.e. it is a breaking force, and here the mechanical power \(-Fv\) necessary to move the loop is equal to the power loss in the conducting cylinder.

**Approximation for high velocities**

An accurate approximate elementary formula, which shows directly the effects of the geometric and kinematic parameters involved, can be derived for the case of a circular loop carrying a direct current and moving at high values of velocity with respect to a coaxial conducting circular cylinder. This formula also allows a ready evaluation of the power loss or force.

We remark that, for sufficiently high values of \( \lambda \), the integrands in the expressions derived in the previous sections become sufficiently small, and their contribution to the integral values can be neglected. On the other hand, where the value of the velocity \( v \) is high enough, such that the integrals in \( \lambda \) are essentially determined by values of \( \lambda \ll v\mu\sigma \), then one can approximate \( k_\lambda \) in (19) by \( k_\lambda \approx -j\lambda v\mu \sigma \). The corresponding modified Bessel functions in (35)-(36), for instance, can be evaluated as (Jahnke et al., 1960)
where \( J_0 \) and \( J_1 \) are the Bessel functions of the first kind and orders 0 and 1 respectively. In what follows, we assume a nonmagnetic conducting cylinder, \( \mu = \mu_0 \). Now, \(|\beta_\lambda| \approx \sqrt{\nu \mu_0 \sigma / \lambda} \gg 1 \) and we can also approximate \( C_\lambda \) in (20) by

\[
C_\lambda \approx 1/\left[ \beta_\lambda J_0 (k_\lambda a) K_1 (\lambda a) \right].
\]  

Substituting (37)-(39) in (35) yields

\[
P = -Fv \approx \frac{\sqrt{2I_0^2 \sqrt{1 - m}}}{\sigma_\delta_0 m} \int_0^{\infty} \left[ \frac{K_1 (\nu)}{K_1 (mn)} \right] \psi(w) \sqrt{w} dw,
\]  

where

\[
\nu \equiv \lambda r_0, \quad m \equiv a/r_0, \quad w \equiv (1 - m)(a/\delta_0)^2 \nu,
\]

\[
\psi(w) \equiv \text{Im} \left( \frac{J_1 (\sqrt{jw})}{J_0 (\sqrt{jw})} \right) - \text{Re} \left( \frac{J_1 (\sqrt{jw})}{J_0 (\sqrt{jw})} \right)
\]

and \( \delta_0 \) is a measure of the motional depth of penetration (Ciric, 1979)

\[
\delta_0 \equiv \sqrt{\frac{r_0 - a}{\nu \mu_0 \sigma}}.
\]

In order to evaluate the integral in (40), we notice that for \( \delta_0 \ll r_0 - a \), i.e. \( w \gg 1 \), the function \( \psi(w) \approx 1 \) (Jahnke et al., 1960) and we also use the approximation

\[
[K_1 (\nu)/K_1 (mn)]^2 \approx \beta_1 e^{- \beta_1 (1 - m) \nu} + \left( m^2 - \beta_1 \right) e^{- \beta_2 (1 - m) \nu},
\]

where \( b_1, \beta_1, \beta_2 \) depend on \( m \). For \( m = 0.8 \), for example, we have \( b_1 = 0.7611, \beta_1 = 1.9635 \) and \( \beta_2 = 8.1 \), which insures in (44) an accuracy better then 1.15 percent for \( \nu \leq 15 \). The values of the integrand in (40) are plotted in Figure 2 for \( a/\delta_0 = 10 \) and for \( m = 0.8 \) and \( m = 0.4 \). For higher \( a/\delta_0 \), the differences between the approximate and the exact values of the integrand in the expression of the power loss become smaller and smaller, which makes this approximation more and more accurate.
Using the definite integral (Dwight, 1961)
\[ \int_{0}^{\infty} e^{-q\nu} \sqrt{\nu} d\nu = \sqrt{\pi q^{-3/2}} / 2, \quad \text{Re}(q) > 0, \quad (45) \]
we obtain finally
\[ P = -Fv \approx \sqrt{\frac{\pi}{2}} \frac{I_0^2}{2 \sigma b_o} g(m), \quad (46) \]
with
\[ g(m) \equiv \frac{b_1 \beta_1^{3/2} + (m^2 - b_1) \beta_2^{3/2}}{m(1 - m)}. \quad (47) \]

**Conclusions**
Rigorous analytic expressions have been derived for the vector magnetic potential of current loops coaxial with a long solid conductor in relative translational motion. The magnetic field, as well as the induced electric field and current density, can be determined directly from the vector potential. Results corresponding to carrying-current coaxial coils have been obtained by integration from the results for a single loop. Joule losses and the reactive power in the induced solid conductor were calculated by means of the complex Poynting vector, and the force on the moving current loop was determined from the differential Ampère's force.

An elementary approximate expression has been obtained for the power loss and force in the case of a direct current loop moving at high velocities. As expected from previous investigations (Ciric, 1979, 1983), at high velocity
values, i.e. at very small motional depths of penetration, practically the power loss is directly proportional, while the breaking force is inversely proportional, to the square root of velocity.

Results presented are useful for the performance analysis of linear electrical machines and electromagnetic devices with conducting cylindrical cores in relative motion with respect to coaxial current-carrying coils. The exact mathematical expressions of the field quantities, power losses and forces for the benchmark structure considered in this paper are to be used for validating and evaluating results obtained by approximate numerical techniques developed for analyzing actual engineering systems.

References


