Use of Wavelets for an Efficient Solution of Electromagnetic Scattering by Conducting Bodies of Revolution

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Abstract—A fast wavelet transform (FWT) is applied to obtain highly sparse moment-method matrix equations from an integral equation formulation of the electromagnetic wave scattering by conducting bodies of revolution. These equations are solved by the generalized minimal residual (GMRES) method and numerical experiments are presented to illustrate the efficiency of the proposed solution technique.

Index Terms—Wavelet transforms, wave scattering.

I. INTRODUCTION

SCATTERING of electromagnetic waves by a body of revolution (BOR) can be described by constructing a specific vector integro-differential equation which takes into account the axial symmetry of the body [1]–[4]. Considerable effort has been made in recent years to enhance the computational efficiency for electrically large structures. A fast Fourier transform was used in [3] to improve the matrix fill time. Entire-domain functions were also employed in order to reduce the size of the moment-method matrices arising in the analysis of axially symmetric reflector antennas [4].

It has been shown that the use of wavelets in the method of moments (MoM) constitutes an effective means to obtain sparse matrix equations yielding sufficiently accurate solutions for electromagnetic field problems [5]–[10]. Fredholm integral equations for two-dimensional electromagnetic scattering problems were efficiently solved by using wavelets in [5]–[7]. The application of wavelets for the solution of coupled integral equations formulated for the eddy current problem of a two-conductor system was investigated in [8]. Thin-wire antenna problems were analyzed by utilizing wavelets in [9], [10]. In [7], the application of semi-orthogonal and orthogonal wavelets was studied and it has been found that orthogonal wavelets are optimal in terms of the matrix condition number.

In this paper, we present a fast solution technique for the moment-method equations describing the electromagnetic scattering by a conducting BOR based on a FWT and employing the GMRES algorithm. Numerical results are generated to demonstrate the efficiency of the solution procedure.

II. PROBLEM FORMULATION

Consider an arbitrary conducting BOR in free space, as shown in Fig. 1, illuminated by a plane electromagnetic wave. The scattered field can be determined from the induced surface current density \( \hat{J}_s \). This is related to the incident electric field intensity \( \mathbf{E}_{\text{inc}} \) via the so-called electric field integral equation [1]

\[
\hat{n} \times \mathbf{E}_{\text{inc}} = \mathbf{L}(\hat{J}_s) \tag{1}
\]

with the integro-differential operator

\[
\mathbf{L}(\hat{J}_s) = \hat{n} \times j\beta \eta \left[ \iint_S \hat{J}_s(\mathbf{r}') \mathbf{G}(\mathbf{r}, \mathbf{r}') \, ds' + \frac{1}{j\beta} \nabla_S \cdot \left[ \mathbf{J}_s(\mathbf{r}') \right] \mathbf{G}(\mathbf{r}, \mathbf{r}') \, ds' \right] \tag{2}
\]
where the freespace Green’s function is
\[ G(\mathbf{r}, \mathbf{r}') = e^{-j|\mathbf{r} - \mathbf{r}'|} / (4\pi|\mathbf{r} - \mathbf{r}'|) \] (3)
\( \hat{n} \) is the outward unit vector normal to the surface \( S \) of the conductor, \( \beta \) is the wave number, \( \eta \) the intrinsic impedance of free space, and \( j \equiv \sqrt{-1} \).

Equation (1) is solved numerically through the application of the MoM. To take advantage of the axisymmetry of the system, the unknown current density is first expanded in terms of Fourier modes as
\[ \mathbf{J}_s(t, \phi) = \sum_m [\mathbf{J}_m^t + \mathbf{J}_m^\phi(t)] e^{jm\phi} \] (4)
where \( \mathbf{J}_m^t \) and \( \mathbf{J}_m^\phi \) represent the components along the tangential unit vectors \( \hat{t} \) and \( \hat{\phi} \), respectively, and \( t \) is the length variable along the generator curve \( C \) of the BOR. Expanding the current density in terms of basis functions \( f_k \) as
\[ \mathbf{J}_m(t) = \hat{p} \sum_{k=1}^N [P^p_{mk} f_k(t)], \quad p = t, \phi \] (5)
and using the weighting functions
\[ W_{pq} = \hat{p} w_p(t)e^{-jm\phi} \] (6)
the application of the MoM yields a system of linear equations
\[ [Z][I] = [V] \] (7)
where
\[ [V]^p_m = \langle W^p_{p}M, E_0^p \rangle \]
\[ [Z^q_m]_{jk} = \langle W^q_{p}M, L(F^q_{mk}) \rangle, \quad p, q = t, \phi \] (8)
and \( F^q_{mk} = \hat{q} f_k(t)e^{jm\phi} \). The impedance matrix elements can be expressed as
\[ [Z^t_m]_{jk} = \beta \gamma \int_{C'} \int_{C'} \left( \Gamma^t_{mk} u_k f_k - \frac{g_m u_k f_k}{\beta^2} \right) dt' dt \]
\[ [Z^\phi_m]_{jk} = \beta \gamma \int_{C'} \int_{C'} \left( \Gamma^\phi_{mk} u_k f_k + \frac{g_m u_k f_k}{\beta^2 \rho} \right) dt' dt \]
\[ [Z^t_m]_{jk} = \beta \gamma \int_{C'} \int_{C'} \left( \Gamma^t_{mk} u_k f_k - \frac{g_m u_k f_k}{\beta^2 \rho} \right) dt' dt \]
\[ [Z^\phi_m]_{jk} = \beta \gamma \int_{C'} \int_{C'} \left( \Gamma^\phi_{mk} u_k f_k - \frac{g_m u_k f_k}{\beta^2 \rho^2} \right) dt' dt \] (9)

with
\[ \Gamma^t_m = \Gamma^\phi_m \sin \gamma \sin \gamma + g_m \cos \gamma \cos \gamma \]
\[ \Gamma^\phi_m = -\sin \gamma' g_{m+1} - g_{m-1} \]
\[ \Gamma^t_m = \sin \gamma (g_{m+1} - g_{m-1}) / 2 \]
\[ \Gamma^\phi_m = (g_{m+1} + g_{m-1}) / 2 \]. (10)
g_m represents the \( m \)th Fourier mode of (3) and \( \gamma \) denotes the angle between the unit vectors \( \hat{t} \) and \( \hat{\phi} \), being considered to be positive if \( \hat{t} \) points away from the \( z \)-axis and negative if \( \hat{t} \) points toward the \( z \)-axis.

Once the unknown current density expansion coefficients are found by solving (7) for the given incident field \( E_0 \), the corresponding scattered far field \( E_{sc} \) can be evaluated using the reciprocity theorem as
\[ E_{sc}^\phi = -\xi \sum_m \left( [P^t_m]^T [t^l_m] + [P^\phi_m]^T [\phi^l_m] \right) e^{jm\phi} \]
\[ E_{sc}^t = -\xi \sum_m \left( [P^t_m]^T [t^\phi_m] + [P^\phi_m]^T [\phi^t_m] \right) e^{jm\phi} \] (11)
where \( T \) indicates the transpose, \( \xi = j / (\eta \epsilon) \), and the vectors \( [P^t_m], [P^\phi_m] \), \( p = t, \phi \) and \( s = \theta, \phi \), are computed as shown in [1], [2].

III. Wavelet Transform and Solution Technique
Equation (7) can be transformed into
\[ [Z'][[I']] = [V'] \] (12)
where
\[ [Z'] = [W][Z][W]^T \]
\[ [V'] = [W][V] \]
\[ [I'] = [W]^T[I] \] (13)
with \( [W] \) being the wavelet transform matrix constructed from relevant wavelets through the FWT algorithm.

Although both orthogonal and semi-orthogonal wavelets have been frequently used for the solution of electromagnetic integral equations, yielding highly sparse moment-method matrices, we have shown in [7] that the condition number of \( [W] \) is 141 in the case of semi-orthogonal wavelets with 8 vanishing moments, resulting in a considerably increased condition number of the transformed matrices, while it is 1 in the case of the Daubechies wavelets, which is optimal. The analysis of the two-dimensional scattering based on the orthogonal wavelet transformation [7] is extended in the present paper to the problem of scattering by a conducting BOR.

Implementing a thresholding procedure with appropriate threshold values (as shown in the next Section), the matrix \( [Z'] \) becomes a sparse matrix. In order to accelerate the iterative solution algorithm, the elements of this sparse matrix are arranged using the “row-indexed sparse storage mode” [6], the solution time thus becoming practically proportional to the matrix sparsity. The GMRES iterative algorithm described in [11], [12] is implemented to obtain the solution of the sparse matrix equation since it converges faster than other iterative algorithms and also due to the fact that the orthogonality of the wavelets employed provides an efficient way to select the termination criterion. Since the Euclidean norm \( ||[W]|| = 1 \) for the orthogonal wavelets, the residual error \( r \) for (7) becomes the residual error \( r' \) for (12) before thresholding. Recognizing this, a solution of (7) with a required accuracy \( \varepsilon \) is sought for by using the GMRES algorithm in two steps. First, the sparse matrix equation obtained after thresholding (12) with a threshold value \( \delta \) is solved by imposing the same accuracy \( \varepsilon \), the residual error \( r_\delta \) of this matrix equation being thus reduced to \( r_\varepsilon \). Then, the error \( r \) for (7) is checked and the iterative procedure is terminated if \( r \leq \varepsilon \); otherwise the GMRES algorithm is restarted with the new termination criterion \( |r_\delta - r_\varepsilon + \Delta| \leq \varepsilon_\delta \), \( \Delta = |r - r_\delta| \) being.
determined in the previous step and $\varepsilon_\delta$ being an accuracy value chosen to be less than $\varepsilon$. It should be remarked that $\Delta$ evaluates quantitatively the influence of thresholding on the residual error $r$. This error is usually small since a very small value of $\delta$ is used in practice. Thus, in the second step the GMRES iterative process converges rapidly, yielding the solution of (7), which therefore is found practically by performing only the one full matrix–vector multiplication necessary to evaluate $r$. This procedure was tested and found to be more efficient than the conjugate gradient iterative method used in [6], where a smaller value of the residual error is required for the termination criterion to ensure the same accuracy $\varepsilon$ for the solution of the original moment-method equation.

### IV. Numerical Results

Based on the procedure described in the preceding Sections, a computer program has been written to analyze the electromagnetic wave scattering by an arbitrary conducting BOR. The functions $f_k(t)$ in (5) and $u_k(t)$ in (6) were chosen to be triangular pulse functions, and the elements of the impedance matrix in (9) were evaluated by applying the integration procedure described in [2]. Daubechies’ wavelets with 8 vanishing moments have been used to perform the wavelet matrix transforms in (13) [6], [7]. A solution of the resulting sparse matrix equations is sought for by using the GMRES iterative process described in the previous Section, with $\varepsilon = 0.01$, as in [6], and $\varepsilon_\delta = 0.0001$. Numerical experiments were performed on a Pentium 120 personal computer. In what follows, the sparsity $S$ of a matrix is defined as its percentage content of nonzero elements.

The first example considered is that of a conducting sphere with a radius $\beta a$ illuminated by an axially incident plane wave as shown in Fig. 1. To illustrate the efficiency of the presented method, experiments were conducted for different large electrical dimensions, namely $\beta a = 25.6$, 51.2, and 102.4, while the number of basis functions per wavelength was kept 20, as in [4]. Figs. 2 and 3 show the computed $E$- and $H$-plane radar cross section (RCS) for $\beta a = 51.2$ obtained by the proposed method in comparison with that by the conventional MoM employing an LU decomposition algorithm. A sparse matrix with $S = 11.28\%$, obtained from $[Z^\prime]$ with a small threshold value of $\delta = 9 \times 10^{-6}$, has a structure as illustrated in Fig. 4. Table I shows the efficiency of the method presented as compared to the MoM with the LU solver. A highly sparse matrix is produced with selected small values of $\delta$ in all cases and the matrix sparsity increases as the matrix size increases, the number of wavelets involved also increasing correspondingly. A 45% to 80% reduction in CPU time can be observed, i.e., the method presented is 1.9 to 5 times faster, including the time taken to perform the FWT. It should be remarked that the computational efficiency increases substantially with the matrix size, which

<table>
<thead>
<tr>
<th>$\beta a$</th>
<th>Matrix Size</th>
<th>Threshold Value $\delta$</th>
<th>Sparsity $S$ (%)</th>
<th>CPU Time in Seconds</th>
<th>Reduction in CPU Time (%)</th>
</tr>
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<tbody>
<tr>
<td>25.6</td>
<td>512x512</td>
<td>$3.5 \times 10^{-6}$</td>
<td>15.37</td>
<td>22.31</td>
<td>24.98</td>
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<tr>
<td>51.2</td>
<td>1024x1024</td>
<td>$9 \times 10^{-6}$</td>
<td>11.28</td>
<td>91.83</td>
<td>154.47</td>
</tr>
<tr>
<td>102.4</td>
<td>2048x2048</td>
<td>$6 \times 10^{-6}$</td>
<td>7.15</td>
<td>595.94</td>
<td>690.19</td>
</tr>
</tbody>
</table>

Fig. 2. $E$-plane RCS for a conducting sphere with $\beta a = 51.2$.

Fig. 3. $H$-plane RCS for a conducting sphere with $\beta a = 51.2$.

Fig. 4. Structure of the sparse matrix with $S = 11.28\%$ obtained after thresholding $[Z^\prime]$ in (13) with $\delta = 9 \times 10^{-6}$.

Table 1: Relative to Spheres of Large Sizes
A second example considered is that of a conducting cone with spherical caps, \( \beta a = 12, \beta b = 6, \) and \( \beta h = 48, \) for \( N = 512 \) basis functions along \( C \) and a sparsity \( S = 7.64\% \).

Fig. 5. \( E \)-plane RCS for a conducting cone with spherical caps, \( \beta a = 12, \beta b = 6, \) and \( \beta h = 48, \) for \( N = 512 \) basis functions along \( C \) and a sparsity \( S = 7.64\% \).

Fig. 6. \( H \)-plane RCS for a conducting cone with spherical caps, \( \beta a = 12, \beta b = 6, \) and \( \beta h = 48, \) for \( N = 512 \) basis functions along \( C \) and a sparsity \( S = 7.64\% \).

recommends the method presented for an efficient solution of electrically large problems.

A second example considered is that of a conducting cone with spherical caps illuminated by an axially incident plane wave as shown in Fig. 5. The efficiency of the proposed procedure is examined for fixed dimensions of the body, \( \beta a = 12, \beta b = 6, \) and \( \beta h = 48, \) but with the total number \( N \) of basis functions considered along the generator curve of the body taken to be 256 and then increased to 512. A more dense sampling usually results in moment-method matrices with a higher condition number. The computed \( E \)- and \( H \)-plane RCS are plotted in Figs. 5 and 6, respectively, and the results obtained by the MoM with the LU solver are also given as reference. As shown in Table II, a 48.92\% and a 78.29\% reduction in CPU time is obtained for \( N = 256 \) (matrix size \( 512 \times 512 \)) and for \( N = 512 \) (matrix size \( 1024 \times 1024 \)), respectively, in comparison with the MoM employing the LU solver, i.e., the proposed procedure is, respectively, 1.9 and 4.9 times faster. As expected, the reduction in computation time is greater than that for the corresponding cases in the first example.

V. Conclusions

A fast solution technique based on the FWT and employing the GMRES iterative algorithm for the problem of wave scattering by a conducting BOR has been investigated. The orthogonal wavelet transformation scheme in [7] was effectively adapted to obtain highly sparse moment-method matrices, whose elements are then stored using a row-indexing storage mode. A new approach has been proposed, where a solution of the transformed moment-method equations is obtained by applying the GMRES iterative algorithm for a given accuracy, taking into account the influence of thresholding and by implementing an effective termination criterion that exploits the orthogonal property of the wavelets. The computational efficiency of this method is illustrated by comparing it with the conventional MoM using an LU solver for different numbers of basis functions per wavelength. Various numerical results generated show that the method presented is highly efficient for the analysis of wave scattering by conducting bodies of revolution, being especially recommended for electrically large problems.

REFERENCES