A Single-Source Surface Integral Formulation for Eddy-Current Problems

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Abstract—A novel integral equation satisfied by a single unknown surface current density is formulated for the two-dimensional analysis of the quasi-stationary field in systems of parallel conductors. This is an alternative to the coupled boundary integral equations formulated in terms of two unknowns, the magnetic vector potential and its normal derivative over each conductor surface. The accuracy of the results computed by the proposed solution method is demonstrated by comparison with results from exact analytical solutions. In order to demonstrate the increased efficiency of the proposed method, various test results are also compared with those obtained from the existent boundary integral equations. Significant reductions in the computation time have been achieved.

Index Terms—Eddy currents, quasi-stationary fields, surface integral equations.

I. INTRODUCTION

VARIOUS FINITE difference and finite element techniques have been developed for computing eddy currents in homogeneous and nonhomogeneous solid conductors [1]. In these techniques, the entire conducting region is discretized and the respective nodal unknowns have to be determined throughout the region. Eddy currents in homogeneous conductors can also be analyzed by using coupled boundary integral equations (BIEs) formulated in terms of two unknowns, for instance, the magnetic vector potential and its normal derivative distributed over the surface of the conducting bodies [2], [3].

In this paper, a formulation of a single-source surface integral equation (SSSIE) is presented for the analysis of eddy currents induced in solid conductors. The vector potential in the region inside the conductor is expressed in terms of a single unknown surface density of electric current distributed over the surface of the conducting bodies [2], [3].

In this work, the proposed method, numerical results have been generated for single and multiple conductors and are compared with those obtained from the existent boundary integral equations. Significant reductions in the computation time have been achieved.

The magnetic vector potential is chosen to have only a component parallel to the conductor, independent of $z$. Inside the conducting region $D$, it satisfies a nonhomogeneous Helmholtz equation

$$ (\nabla^2 + k^2)A(r) = \mu \sigma \frac{\partial V}{\partial z}, \quad r \in D $$

where $V$ is the classical electric scalar potential, with $\partial V/\partial z =$ const inside the conductor, $k^2 \equiv -j \omega \mu \sigma$, $j \equiv \sqrt{-1}$, $\omega$ is the angular frequency, and $r$ is the position vector of the observation point. Equation (1) can be written in the form

$$ (\nabla^2 + k^2)A^c(r) = 0, \quad r \in D $$

with $A^c = A + C_0$ and the constant $C_0 \equiv -(j/\omega)\partial V/\partial z$ to be determined. In the free-space region $D_e$, the quasi-stationary magnetic vector potential satisfies the Laplace equation.

The following continuity conditions across the interface $S$ between the conducting and the nonconducting regions are to be imposed:

$$ A(r) = A^c(r), \quad r \in S $$

where $A$ is the vector potential in $D$, $\mu_0$ is the permeability of free space, and $\partial/\partial n$ denotes the normal derivative.

In order to construct a single-source surface integral equation, we first assume to have everywhere the same conducting material as in the region $D$ and that the actual $A^c$ in $D$ is produced by a single layer of electric current parallel to the vector potential, of density $J_s$, distributed over the conductor surface, i.e.

$$ A^c(r) = AJ_s, \quad r \in DUS $$

where the integral operator $A$ acts as

$$ AJ_s = -\frac{j \mu}{4} \int_C J_s(r')H_0^{(2)}(kR)dl' $$

with $C$ being the conductor cross-sectional contour, $H_0^{(2)}$ the Hankel function of second kind and zero order, $R = |r - r'|$, and $r'$ the position vector of the source point. The tangential component of the actual magnetic field intensity on the surface $S$ just inside the conductor can be written in the form

$$ H_s(r) = -\frac{1}{\mu} \frac{\partial A}{\partial n} = \frac{1}{2} J_s + \nabla J_s, \quad r \in S $$

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where the integral operator \( \mathcal{H} \) is defined from

\[
\mathcal{H} J_s = \frac{j}{4} \int_C J_s(r') \frac{\partial}{\partial n} H_0^{(2)}(kR) dl'
\]  

(8)

with the integral taken in principal value.

On the other hand, the potential \( A_e \) in the region \( D_e \), outside the conductor, can be represented by applying the Green theorem. Assuming that the vector potential vanishes at infinity

\[
A_e(r) = A_0(r) - \frac{1}{2\pi} \left[ \int_C \frac{\partial A_e(r')}{\partial n'} \ln \left( \frac{1}{R} \right) dl' \right] + \int_C A_e(r') \frac{\partial}{\partial n'} \left( \ln \left( \frac{1}{R} \right) \right) dl', \quad r \in D_e
\]  

(9)

where \( A_0 \) is the vector potential which corresponds to the external field \( B_0 \).

Taking into account (3), (4), and (7), the actual potential on the surface \( S \) just outside the conductor is obtained from (9) in the form

\[
A_e(r) = A_0(r) + A_0^0 H_t + \left( \frac{1}{2} I + A_0^m \right) A, \quad r \in S
\]  

(10)

where \( I \) is the identity operator and with the operators \( A_0^0 \) and \( A_0^m \) acting as

\[
A_0^0 H_t = \frac{\mu_0}{2\pi} \int_C H_t(r') \ln \left( \frac{1}{R} \right) dl'
\]  

(11)

\[
A_0^m A = \frac{1}{2\pi} \int_C A(r') \frac{\partial}{\partial n'} \left( \ln \left( \frac{1}{R} \right) \right) dl'
\]  

(12)

the integral in (12) being evaluated in principal value.

Imposing in (10) the continuity condition in (3) and substituting \( H_t \) from (7) and \( A = A^c - C_0 \) from (5) yields a single-source surface integral equation in \( J_s \)

\[
\left[ A_0^0 \left( -\frac{1}{2} I + H_t \right) + \left( \frac{1}{2} I + A_0^m \right) A \right] J_s + C_0 = -A_0,
\]  

\[ r \in C. \]  

(13)

To specify the value \( I_c \) of the total current carried by the solid conductor, we apply Ampère’s theorem along the contour \( C \) which yields an additional equation, i.e., [with (7)]

\[
\int_C \left( -\frac{1}{2} J_s + \mathcal{H} J_s \right) dl = I_c.
\]  

(14)

It should be noted that, in the form given above, the expressions in (10) and (13) have been derived for observation points on the conductor boundary where its curvature is finite.

Once the unknown current density \( J_s \) and the constant \( C_0 \) are determined from (13) and (14), the magnetic vector potential in \( D \) and \( D_e \) is obtained, respectively, from (5) and (9), i.e.

\[
A(r) = A J_s - C_0, \quad r \in D
\]  

(15)

\[
A_e(r) = A_0 + \left[ A_0^0 \left( -\frac{1}{2} I + H_t \right) + A_0^m A \right] J_s,
\]  

\[ r \in D_e. \]  

(16)

The current density inside the conductor is calculated from

\[
J = -j\omega \sigma (A + C_0), \text{ i.e. } J = -j\omega \sigma A J_s.
\]  

(17)

For a system of \( n \) parallel homogeneous conductors of arbitrary cross sections the integral equation (13) has the same form, but with the integrals in the operators \( A_0^0 \) and \( A_0^m \) [see (11) and (12)] performed over the union \( C = C_1 U C_2 U \ldots U C_n \) of the contours of the conductors; when the integration points in \( A_0^0 \) and \( A_0^m \) are on the contour \( C_i \) of the cylinder \( i \), of conductivity \( \sigma_i \) and permeability \( \mu_i \), the operators \( A \) and \( \mathcal{H} \) in (6) and (8) are taken with the integrals performed over \( C_i \), with \( \mu = \mu_i \) and \( k^2 = k_i^2 \equiv -j\omega \mu_i \sigma_i \). For each \( r \in C_i \subset C \), the unknown constant \( C_0 \) has a specific value \( C_{0_i} \). The given currents in the \( n \) conductors are fixed through \( n \) additional equations obtained from (14) written for each contour \( C_i \) with the corresponding current \( I_{c_i} \).

The electric and magnetic field intensities inside the conductors are calculated from (17) and (15) as

\[
E = -j\omega A J_s, \quad r \in D
\]  

(18)

\[
H = \frac{j k}{4} \int C J_s(r') H_0'(kR) \frac{a_z \times R}{R} dl', \quad r \in D
\]  

(19)

where \( a_z \) is the unit vector along the positive direction of the \( z \) axis. For the region outside the conductors \( H \) is calculated by taking the curl in (16). We remark that the fields \( J, H, \) and \( E \) in (18) are all expressed in terms of the single surface current \( J_s \) distributed over the conductor boundary.

III. NUMERICAL RESULTS

The single-source surface integral equation has been implemented numerically for various structures of cylindrical conductors by employing a point-matching method of moments.

The first example considered is that of a circular cylinder of conductivity \( \sigma = 5.8 \times 10^7 \) S/m (copper) and permeability \( \mu = \mu_0 \), immersed in a uniform magnetic field of flux density \( B_0 \) with a time-harmonic variation. In Fig. 1, the magnitude of the induced current density \( J \) normalized to \( B_0/(\mu_0 r_c) \) is plotted versus the ratio \( r/r_c \) for various depths of penetration \( \delta \), where \( r_c \) is the cylinder radius and \( r \) is the distance from the cylinder center along the direction perpendicular to the direction of the external field.

![Fig. 1. Normalized current density induced in a circular cylindrical conductor by a uniform magnetic field for various skin depths. —: SSIE. x: exact analytical solution.](image-url)
In a second example, we consider a cylinder of conductivity \( \sigma = 5.8 \times 10^7 \text{ S/m} \), excited by a parallel wire carrying a current \( I_0 \) and located very close to the conductor surface in order to produce a highly nonuniform field, as shown in Fig. 2. The current density is computed at points along a radial direction from the conductor center to the current wire. There is a good agreement between the results obtained from the SSSIE and the analytical results even when the skin effect is pronounced. As in the first example, the cylinder cross-sectional contour was discretized into a number of about 60 segments, with a constant surface current density \( J_s \) over each segment.

The numerical experiment in the first example has also been performed, as shown in Figs. 3 and 4, for a magnetic material conductor and a small depth of penetration, for which results generated by a hybrid integro-differential finite element (IDFE) technique are also available [4]. On a PC Intel Celeron 500 MHz, the SSSIE results converge to two significant digits in a CPU time of 7.3 s using 55 contour segments, while for the same accuracy, the BIE requires 15.9 s using 80 contour segments. This decrease in CPU time by a factor of more than 2 is mainly achieved due to the reduction in the required contour discretization when employing the SSSIE method.

In Fig. 5, we present computed results for the current distribution in a nonmagnetic hollow cylinder of conductivity \( \sigma = 3.6 \times 10^7 \text{ S/m} \) in the presence of a uniform magnetic field of flux density \( B_0 \). The same number of 80 segments was used on the inner and on the outer contours in order to achieve a two significant digit accuracy in both the SSSIE and the BIE methods. The SSSIE method yields approximately a 1.5 times reduction in CPU time as compared to the BIE method.

Results for a system of two parallel circular cylindrical conductors of radii \( 5\delta \), conductivity \( \sigma = 5.8 \times 10^7 \text{ S/m} \) and permeability \( \mu = \mu_0 \), carrying equal and opposite currents, are given in Fig. 6. For the same accuracy, the surface discretization required by the SSSIE solution (40 segments per cylinder) is reduced to half compared to that required by the BIE solution. The CPU time when using the SSSIE method (23.5 s) is more than 7 times smaller than that corresponding to the BIE method (170 s). When one employs the same surface discretization, e.g., a number of 40 segments on each contour, the CPU time required by the SSSIE is half of that required by the BIE.
A single-source surface integral equation method has been developed for the analysis of eddy currents in cylindrical conductors, all the field quantities of interest being determined in terms of only one surface current distributed over each conductor boundary. Its accuracy was tested for various conductor configurations for which exact analytical solutions are available. A large range of frequencies was considered, including the case of strong skin effect. The computational efficiency with respect to existent coupled surface integral equation formulations has also been demonstrated, substantial reductions in CPU time being achieved, especially for multiply-connected conductors and for multi-conductor systems.

REFERENCES


