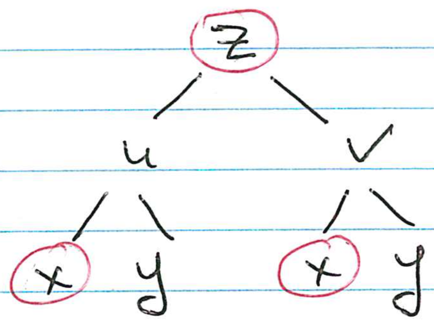
Oct. 8, 2019.

Sect. 12.6 Chain Rule

Univariate case: $f = f(u)$, $u = u(x)$, $x = x(t)$, $t = t(z)$

$$\frac{df}{dz} = \left\{ \begin{array}{c} f \rightarrow u \\ \downarrow \\ x \\ \downarrow \\ t \\ \downarrow \\ z \end{array} \right\} = \frac{df}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dt} \cdot \frac{dt}{dz}$$

Multivariate Case: $z = z(u,v)$, $u = u(x,y)$, $v = v(x,y)$.



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

Theorem: Let $u = g(x,y)$ and $v = h(x,y)$ be continuous and have first partial derivatives w.r.t. x at (x_0, y_0), and let $z = f(u,v)$ have first partial derivatives inside a circle centered at the pt. $(u_0, v_0) = (g(x_0,y_0), h(x_0,y_0))$

Then, at (x_0, y_0),

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}.$$

#1, p. 830 $\quad z = f(x,t)$, $x = g(t)$, $z = \dfrac{xt^2}{x+t}$,

$x = e^{3t}$.

Q-n: $\dfrac{dz}{dt}$ - ?

$$\begin{tikzcd} & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[d, dash] & & t \\ t & & \end{tikzcd}$$

z is a f-n of t only, so

$$\frac{\partial z}{\partial t} = \frac{dz}{dt}$$

Notation

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial t}$$

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problem with our notation.

<u>Notation:</u>

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \left.\frac{\partial z}{\partial t}\right)_x$$

partial der. wrt t assuming that x is fixed.

We could write this as:

$$\frac{\partial z}{\partial t} = \left.\frac{\partial z}{\partial x}\right)_t \cdot \frac{\partial x}{\partial t} + \left.\frac{\partial z}{\partial \cancel{t}}\right)_x$$

can use but do not have to.

$$z = \frac{xt^2}{\cancel{x}+t}, \quad x = e^{3t}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \overset{\frac{dx}{dt}}{\frac{\partial x}{\partial t}} + \left.\frac{\partial z}{\partial t}\right)_x$$

$$\frac{\partial z}{\partial x} = t^2 \frac{\partial}{\partial x}\left[\frac{x}{x+t}\right] = t^2 \cdot \frac{1\cdot(x+t) - x\cdot 1}{(x+t)^2} =$$

$$= t^2 \cdot \frac{t}{(x+t)^2} = \frac{t^3}{(x+t)^2}$$

$$\frac{dx}{dt} = \left\{ = \frac{\partial x}{\partial t} \right\} = 3e^{3t}$$

$$\left.\frac{\partial z}{\partial t}\right)_x = x \frac{\partial}{\partial t}\left[\frac{t^2}{x+t}\right]\Bigg)_x =$$

$$= x \cdot \frac{2t(x+t) - t^2}{(x+t)^2} = x \cdot \frac{2xt + t^2}{(x+t)^2}.$$

$$\therefore \frac{dz}{dt} = \left\{ = \frac{\partial z}{\partial t} \right\} = \frac{t^3}{(x+t)^2} \cdot 3 \cdot e^{3t} + x \cdot \frac{2xt+t^2}{(x+t)^2}.$$

<u>Homogeneous F-s:</u>

$$\underline{f(tx, ty, tz) = t^n f(x,y,z)} \qquad (*)$$

Homogeneous f-n of degree n

<u>Ex:</u> $f(x,y,z) = x^3 + y^3 + z^3$ ← homog. f-n of deg. 3

$f(tx, ty, tz) = t^3x^3 + t^3y^3 + t^3z^3 = t^3 f(x,y,z)$

Differentiate $(*)$ wrt t:

$\{ f = f(u,v,w),\ u = tx,\ v = ty,\ w = tz \}$.

$$\frac{\partial f}{\partial u} \cdot x + \frac{\partial f}{\partial v} \cdot y + \frac{\partial f}{\partial w} \cdot z = n t^{n-1} f(x,y,z).$$

$t=1$: $\quad \dfrac{\partial f}{\partial u} x + \dfrac{\partial f}{\partial v} y + \dfrac{\partial f}{\partial w} \cdot z = n f(x,y,z).$

$\left\{ \text{If } f = f(x,y,z), \text{ then } \dfrac{\partial f}{\partial u} = \dfrac{\partial f}{\partial x}, \text{ etc.} \right\}$

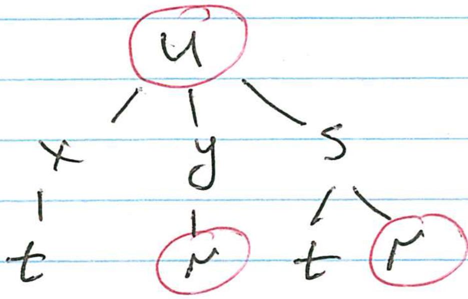
$\therefore$ <u>Euler's Thm:</u> $\quad \boxed{x \dfrac{\partial f}{\partial x} + y \dfrac{\partial f}{\partial y} + z \dfrac{\partial f}{\partial z} = n f(x,y,z)}$

#5, p.830 $\left.\frac{\partial u}{\partial r}\right)_t$ if $u = f(x, y, s)$

$x = g(t)$

$y = h(r)$

$s = k(r, t)$



$\frac{dy}{dr}$

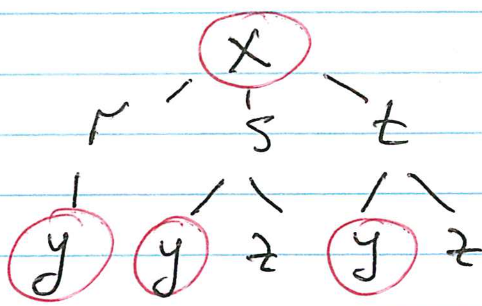
$$\left.\frac{\partial u}{\partial r}\right)_t = \left.\frac{\partial u}{\partial y}\right)_{x,s} \cdot \frac{\partial y}{\partial r} + \left.\frac{\partial u}{\partial s}\right)_{x,y} \cdot \left.\frac{\partial s}{\partial r}\right)_t$$

#8, p.831 $\left.\frac{\partial x}{\partial y}\right)_z$ if $x = f(r, s, t)$

$r = g(y)$

$s = h(y, z)$

$t = k(y, z)$



$\frac{dr}{dy}$

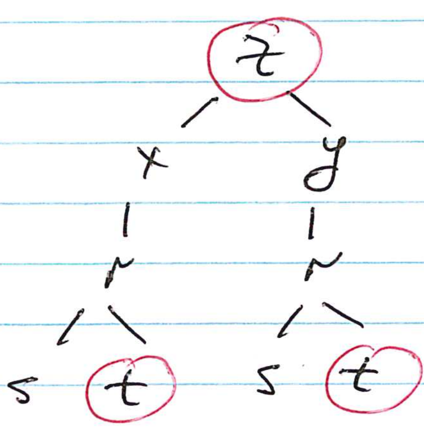
$$\left.\frac{\partial x}{\partial y}\right)_z = \frac{\partial x}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial x}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial x}{\partial t} \cdot \frac{\partial t}{\partial y}.$$

#9, p.831 $\left.\frac{\partial z}{\partial t}\right)_s$ if $z = e^{x+y}$

$x = 2r + s$

$y = 2r - s$

$r = t \ln(s^2 + t^2)$



$$\left.\frac{\partial z}{\partial t}\right)_s = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} \cdot \frac{\partial r}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r} \cdot \frac{\partial r}{\partial t}$$

$$\left.\frac{\partial z}{\partial t}\right)_s = e^{x+y}\cdot 2\cdot\left[\ln(s^2+t^2) + t\cdot\frac{1}{s^2+t^2}\cdot 2t\right]$$

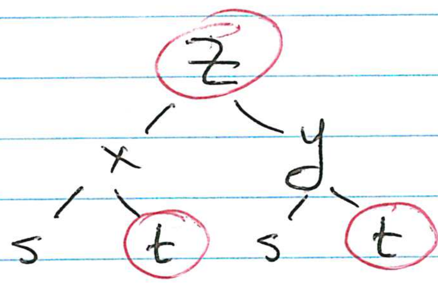
$$+\, e^{x+y}\cdot 2\cdot\left[\ln(s^2+t^2) + t\cdot\frac{1}{s^2+t^2}\cdot 2t\right]$$

$$= 4e^{x+y}\left[\ln(s^2+t^2) + \frac{2t^2}{s^2+t^2}\right].$$

#11, p. 831. $\left.\frac{\partial^2 z}{\partial t^2}\right)_s$ if $z = x^2y^2 + xe^y$

$x = s + t^2$

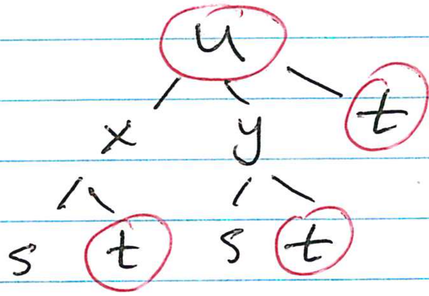
$y = s - t^2$.



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial t} = (2xy^2 + e^y)\cdot 2t + (2yx^2 + xe^y)(-2t)$$

$$= 4xty^2 + 2te^y - 4ytx^2 - 2xte^y = u.$$



$u = u(x, y, t)$

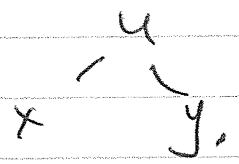
$$\left.\frac{\partial^2 z}{\partial t^2}\right)_s = \left.\frac{\partial u}{\partial t}\right)_s = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} + \left.\frac{\partial u}{\partial t}\right)_{x,y}$$

$$\left.\frac{\partial^2 z}{\partial t^2}\right)_s = (4ty^2 - 8ytx - 2te^y)\cdot 2t +$$

$$+ (8xty + 2te^y - 4tx^2 - 2xte^y)(-2t)$$

$$+ (4xy^2 + 2e^y - 4yx^2 - 2xe^y).$$

#27, p. 832 If $f(s)$ is differentiable, show that $u(x,y) = f(4x-3y) + 5(y-x)$ satisfies $3\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} = 5$.



$u(x,y) = z$

$$\frac{\partial u}{\partial x} = f'(4x-3y)\cdot\frac{\partial}{\partial x}(4x-3y) + \frac{\partial}{\partial x}\left[5(y-x)\right]$$

$$= f'(4x-3y)\cdot 4 - 5$$

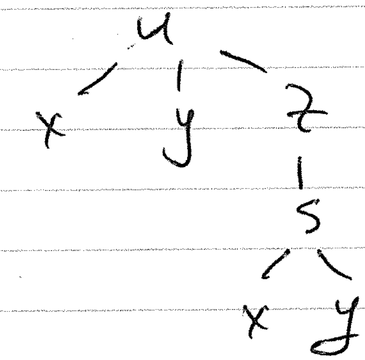
$$\frac{\partial u}{\partial y} = f'(4x-3y)\cdot\frac{\partial}{\partial y}(4x-3y) + \frac{\partial}{\partial y}\left[5(y-x)\right]$$

$$= f'(4x-3y)\cdot(-3) + 5$$

$$\therefore\ 3\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} = 3\left[f'(4x-3y)\cdot 4 - 5\right] + 4\left[f'(4x-3y)\cdot(-3)+5\right] =$$

$$= -15 + 20 = \underline{5} \quad \therefore\ 3\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} = 5.$$

$$\left\{\begin{array}{l} u = z + 5(y-x) \\ z = f(s) \\ s = 4x - 3y \end{array}\right.$$



HW: #25, 26, 28-30 on p. 832.

Sect. 12.7 Implicit Differentiation.

#5, p. 838 $\quad x^2 \sin z - y e^z = 2x \quad (*)$

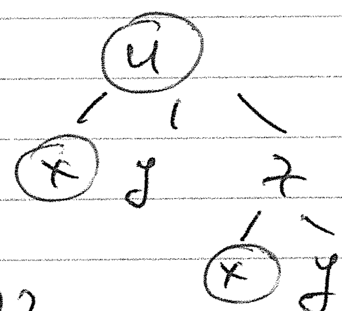
Defines $z = z(x,y)$ implicitly.

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Differentiate $(*)$ wrt x and wrt y thinking about z as a f-n of x and y.

$$\frac{\partial}{\partial x}\left[x^2 \sin z - y e^z\right] = \frac{\partial}{\partial x}\left[2x\right]$$

$$u = x^2 \sin z - y e^z$$



$$\frac{\partial u}{\partial x} = \left.\frac{\partial u}{\partial x}\right)_{y,z} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$2x \sin z + x^2 \cdot \cos z \cdot \frac{\partial z}{\partial x} - y \cdot e^z \cdot \frac{\partial z}{\partial x} = 2$$

$$\frac{\partial z}{\partial x}\left(x^2 \cos z - y e^z\right) = 2 - 2x \sin z$$

$$\frac{\partial z}{\partial x} = \frac{2 - 2x \sin z}{x^2 \cos z - y e^z}$$

$$\frac{\partial}{\partial y}\left[x^2 \sin z - y e^z\right] = \frac{\partial}{\partial y}\left[2x\right].$$

$$x^2 \cos z \cdot \frac{\partial z}{\partial y} - e^z - y \cdot e^z \cdot \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{e^z}{x^2 \cos z - y e^z}.$$

Ex: $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \longrightarrow$ defines $u = u(x,y)$, $v = v(x,y)$.

$\frac{\partial u}{\partial x}$ - ? Diff. both eq-s wrt x:

$$\begin{cases} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x} = 0 \\ \frac{\partial G}{\partial x} + \frac{\partial G}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial G}{\partial v} \cdot \frac{\partial v}{\partial x} = 0 \end{cases}$$

Notation: $\frac{\partial F}{\partial x} = F_x$, $\frac{\partial F}{\partial u} = F_u$, $\frac{\partial F}{\partial y} = F_y$, etc.

$$\begin{cases} F_u \cdot \boxed{\frac{\partial u}{\partial x}} + F_v \cdot \boxed{\frac{\partial v}{\partial x}} = -F_x \\ G_u \cdot \boxed{\frac{\partial u}{\partial x}} + G_v \cdot \boxed{\frac{\partial v}{\partial x}} = -G_x \end{cases}$$

Solve for $\frac{\partial u}{\partial x}$ using Cramer's Rule:

$$\frac{\partial u}{\partial x} = \frac{\det \begin{bmatrix} -F_x & F_v \\ -G_x & G_v \end{bmatrix}}{\det \begin{bmatrix} F_u & F_v \\ G_u & G_v \end{bmatrix}}$$