Oct. 10, 2019

**Def-n:** The Jacobian determinant (= Jacobian) of f-s $F, G, H$ with respect to variables $u, v$ and $w$ is

$$\frac{\partial(F,G,H)}{\partial(u,v,w)} = \det\begin{bmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{bmatrix} = \det\begin{bmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} & \frac{\partial F}{\partial w} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} & \frac{\partial G}{\partial w} \\ \frac{\partial H}{\partial u} & \frac{\partial H}{\partial v} & \frac{\partial H}{\partial w} \end{bmatrix}.$$

~~Ex~~: Note:

$$\frac{\partial(F,G)}{\partial(x,y)} = \det\begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}, \text{ etc.}$$

**Notation:**

$$\det\begin{bmatrix} a_{11} \dots a_{1n} \\ \dots \\ a_{n1} \dots a_{nn} \end{bmatrix} = \begin{vmatrix} a_{11} \dots a_{1n} \\ \dots \\ a_{n1} \dots a_{nn} \end{vmatrix}.$$

**Ex (considered last time)**

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \longrightarrow \text{defines } \begin{matrix} u=u(x,y) \\ v=v(x,y). \end{matrix}$$

Diff. wrt $x$:

$$\begin{cases} F_x + F_u \cdot \frac{\partial u}{\partial x} + F_v \cdot \frac{\partial v}{\partial x} = 0 \\ G_x + G_u \cdot \frac{\partial u}{\partial x} + G_v \cdot \frac{\partial v}{\partial x} = 0 \end{cases}$$

$\Updownarrow$

$$\underbrace{\begin{bmatrix} F_u & F_v \\ G_u & G_v \end{bmatrix}}_{\frac{\partial(F,G)}{\partial(u,v)}} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{bmatrix} = -\begin{bmatrix} F_x \\ G_x \end{bmatrix}$$

Using Cramer's Rule:

$$\frac{\partial u}{\partial x} = \frac{\det\begin{bmatrix} -F_x & F_v \\ -G_x & G_v \end{bmatrix}}{\frac{\partial(F,G)}{\partial(u,v)}} = - \frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}$$

$$\frac{\partial v}{\partial x} = - \frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}}$$

Similarly, $\frac{\partial u}{\partial y} = - \frac{\frac{\partial(F,G)}{\partial(y,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}$, $\frac{\partial v}{\partial y} = - \frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(u,v)}}$

Ex: $\begin{cases} F(x,y,u,v,w)=0 \\ G(x,y,u,v,w)=0 \\ H(x,y,u,v,w)=0 \end{cases}$ $\rightarrow$ defines $u$, $v$ and $w$ as f-s of $x$ and $y$.

Then $$\frac{\partial u}{\partial x} = - \frac{\frac{\partial(F,G,H)}{\partial(x,v,w)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}}$$

why?

Diff. wrt $x$:

$$\begin{cases} F_x + F_u \cdot \frac{\partial u}{\partial x} + F_v \frac{\partial v}{\partial x} + F_w \frac{\partial w}{\partial x} = 0 \\ G_x + \dots \\ H_x + \dots \end{cases}$$

#9, p. 838 $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ if

$$\begin{cases} x^2 - y^2 + u^2 + 2v^2 = 1 \\ x^2 + y^2 = 2 + u^2 + v^2 \end{cases}$$

$\{u, v$ are f-s of $x$ and $y\}$

Let $F(x,y,u,v) = x^2 - y^2 + u^2 + 2v^2 - 1$

$G(x,y,u,v) = x^2 + y^2 - u^2 - v^2 - 2$

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}$$

$$\frac{\partial(F,G)}{\partial(u,v)} = \det \begin{bmatrix} F_u & F_v \\ G_u & G_v \end{bmatrix}$$

$$\frac{\partial(F,G)}{\partial(x,v)} = \det \begin{bmatrix} F_x & F_v \\ G_x & G_v \end{bmatrix}$$

| | |
|---|---|
| $F_u = 2u$ | $G_u = -2u$ |
| $F_v = 4v$ | $G_v = -2v$ |
| $F_x = 2x$ | $G_x = 2x$ |

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} 2x & 4v \\ 2x & -2v \end{vmatrix}}{\begin{vmatrix} 2u & 4v \\ -2u & -2v \end{vmatrix}} = -\frac{4\begin{vmatrix} x & 2v \\ x & -v \end{vmatrix}}{(-4)\begin{vmatrix} u & 2v \\ u & v \end{vmatrix}} =$$

$$= \frac{-3xv}{-uv} = \frac{3x}{u}.$$

$$\frac{\partial v}{\partial y} = - \frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(u,v)}} = - \frac{\begin{vmatrix} 2u & -2y \\ -2u & 2y \end{vmatrix}}{\begin{vmatrix} 2u & 4v \\ -2u & -2v \end{vmatrix}} =$$

$$\left\{ \begin{array}{l} F_y = -2y \\ G_y = 2y \end{array} \right\}$$

$$= - \frac{4uy - 4uy}{\cdots} = 0$$

#10, p. 838 $\frac{\partial x}{\partial t} = ?$ if $\sin(x+t) - \sin(x-t) = z$

defines $x$ as a f-n of $t$ and $z$

Method 1 (using the formula).

$$F(t, z, x) = 0$$

$$\frac{\partial x}{\partial t} = - \frac{\frac{\partial F}{\partial t}}{\frac{\partial F}{\partial x}} =$$

$$F(t, z, x) = \sin(x+t) - \sin(x-t) - z$$

$$= - \frac{\cos(x+t) + \cos(x-t)}{\cos(x+t) - \cos(x-t)}$$

Method 2 (w/o using the general formula).

Diff. $\sin(x+t) - \sin(x-t) = z$ w.r.t. $t$ thinking about $x$ as a f-n of $t$ and $z$.

$$\cos(x+t)\cdot\left[\frac{\partial x}{\partial t}+1\right]-\cos(x-t)\cdot\left[\frac{\partial x}{\partial t}-1\right]=0$$ 83

$$\stackrel{\Downarrow}{}\frac{\partial x}{\partial t}\left[\cos(x+t)-\cos(x-t)\right]=-\cos(x+t)-\cos(x-t$$

$$\stackrel{\Downarrow}{}\frac{\partial x}{\partial t}=\frac{-\cos(x+t)-\cos(x-t)}{\cos(x+t)-\cos(x-t)}.$$

#11, p. 838 $\quad \left.\frac{\partial\varphi}{\partial x}\right)_{y,z}$ if $\begin{cases} x=r\sin\varphi\cos\theta \\ y=r\sin\varphi\sin\theta \\ z=r\cos\varphi \end{cases}$

define $r,\varphi,\theta$ as f-s of $x, y$ and $z$.

$$F(x,y,z,r,\varphi,\theta)=r\sin\varphi\cos\theta-x$$
$$G(\text{---}''\text{---})=r\sin\varphi\sin\theta-y$$
$$H(\text{---}''\text{---})=r\cos\varphi-z.$$

$$\frac{\partial\varphi}{\partial x}=-\frac{\dfrac{\partial(F,G,H)}{\partial(r,x,\theta)}}{\dfrac{\partial(F,G,H)}{\partial(r,\varphi,\theta)}}$$

$$\frac{\partial(F,G,H)}{\partial(r,\varphi,\theta)}=\det\begin{bmatrix}\sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi & 0\end{bmatrix}$$

$$=r^2\det\begin{bmatrix}\sin\varphi\cos\theta & \cos\varphi\cos\theta & -\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \cos\varphi\sin\theta & \sin\varphi\cos\theta \\ \cos\varphi & -\sin\varphi & 0\end{bmatrix}$$

$$= r^2\left[\cos\varphi \begin{vmatrix} \cos\varphi\cos\theta & -\sin\varphi\sin\theta \\ \cos\varphi\sin\theta & \sin\varphi\cos\theta \end{vmatrix} + \right.$$

$$\left. + \sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta \end{vmatrix}\right] =$$

$$= r^2\left[\cos^2\varphi\sin\varphi \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} + \right.$$

$$\left. + \sin^3\varphi \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}\right] =$$

$$= r^2 \sin\varphi\left(\cos^2\varphi + \sin^2\varphi\right) = r^2\sin\varphi$$

$$\boxed{\frac{\partial(F,G,H)}{\partial(r,\varphi,\theta)} = r^2\sin\varphi.}$$

$$\frac{\partial(F,G,H)}{\partial(r,x,\theta)} = \det\begin{bmatrix} \sin\varphi\cos\theta & -1 & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & 0 & r\sin\varphi\cos\theta \\ \cos\varphi & 0 & 0 \end{bmatrix}$$

$$= \begin{vmatrix} \sin\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & 0 \end{vmatrix} = -r\sin\varphi\cos\varphi\cos\theta$$

$$\therefore \frac{\partial\varphi}{\partial x} = + \frac{r\cancel{\sin\varphi}\cos\varphi\cos\theta}{r^2\cancel{\sin\varphi}} = \frac{\cos\varphi\cos\theta}{r}.$$

HW: #14, p. 838

#18 $\begin{cases} x = e^u \cos v \\ y = e^u \sin v \end{cases}$ ← u, v are f-s of x and y

If $z = u^3 v + \sin(uv)$, find $\left.\frac{\partial z}{\partial y}\right)_x$

$$\frac{\partial z}{\partial y} = 3u^2 v \frac{\partial u}{\partial y} + u^3 \frac{\partial v}{\partial y} + \cos(uv)\left[u \frac{\partial v}{\partial y} + v \frac{\partial u}{\partial y}\right]$$

HW: #29 p. 839.

#19, p. 839. If $z^3 - xz - y = 0$ defines $z$ as a f-n of $x$ and $y$, show

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{3z^2 + x}{(3z^2 - x)^3}.$$

$\frac{\partial z}{\partial y}$ - ? Diff. w.r.t. $y$:

$$3z^2 \frac{\partial z}{\partial y} - x \frac{\partial z}{\partial y} - 1 = 0$$

$$\frac{\partial z}{\partial y} = \frac{1}{3z^2 - x}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{1}{3z^2 - x}\right] =$$

$$= -\frac{1}{(3z^2 - x)^2} \cdot \left[6z \frac{\partial z}{\partial x} - 1\right]$$

Now, find $\frac{\partial z}{\partial x}$.

Diff. $z^3 - xz - y = 0$ w.r.t. $x$:

$$3z^2 \frac{\partial z}{\partial x} - z - x\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{z}{3z^2 - x}$$

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{(3z^2-x)^2}\left[6z \cdot \frac{z}{3z^2-x} - 1\right] =$$

$$= -\frac{1}{(3z^2-x)^3}\left(6z^2 - (3z^2-x)\right) =$$

$$= -\frac{1}{(3z^2-x)^3}\cdot(3z^2+x)$$

<u>Remark:</u> In #20,

$$\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$$ define $u$ and $v$ as f-s of $x$ and $y$

$$\frac{\partial^2 u}{\partial x^2} = \underline{\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right]}$$

In general, you will need $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$.