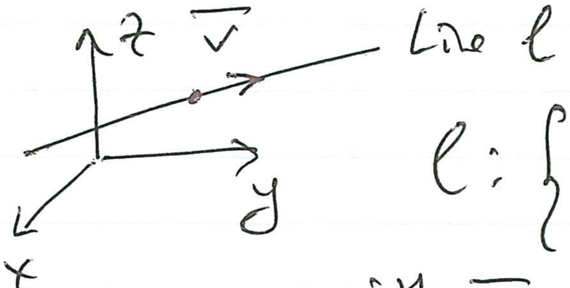
Oct. 15, 2019

Sect 12.8 Directional Derivatives.



$$\ell : \begin{cases} x = x_0 + v_x s \\ y = y_0 + v_y s \\ z = z_0 + v_z s, \quad s \in R. \end{cases}$$

with $\bar{v} = (v_x, v_y, v_z)$ s.t. $|\bar{v}| = 1$.

$$\bar{r}(s) = (x_0 + v_x s,\ y_0 + v_y s,\ z_0 + v_z s)$$

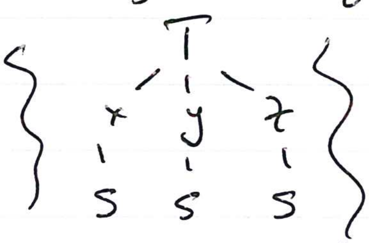
$$\frac{\partial \bar{r}}{\partial s}(s) = (v_x, v_y, v_z) = \bar{v}, \quad \left|\frac{\partial \bar{r}}{\partial s}\right| = |\bar{v}| = 1.$$

Suppose $T = T(x, y, z)$ is defined on R^3.

Q-n: what is the rate of change of T along the line ℓ?

$$f(s) = T(x_0 + v_x s,\ y_0 + v_y s,\ z_0 + \cancel{v_z}\, s), \quad s \in R.$$

$$f'(s) = \frac{\partial T}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial T}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial T}{\partial z} \cdot \frac{\partial z}{\partial s} =$$



$$= \frac{\partial T}{\partial x} \cdot v_x + \frac{\partial T}{\partial y} \cdot v_y + \frac{\partial T}{\partial z} \cdot v_z$$

$$= \underbrace{\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right)}_{\nabla T} \cdot \underbrace{(v_x, v_y, v_z)}_{\bar{v}}$$

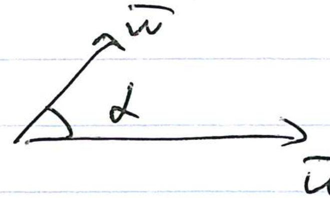
$$\therefore f'(s) = \underbrace{\nabla T \cdot \bar{v}}_{D_{\bar{v}} T}$$

Def-n: The directional derivative of a f-n $F(x,y,z)$ in the direction $\bar{v} = (v_x, v_y, v_z)$ (where $|\bar{v}| = 1$) at the point (x_0, y_0, z_0) is

$$D_{\bar{v}} F(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0) \cdot \bar{v}$$

Remark: i) $D_{\bar{v}} F$ is the component of ∇F in the direction $\bar{v}$.

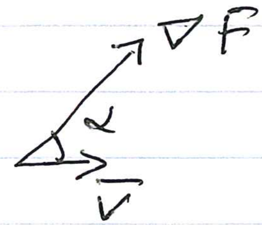
Note: If we have $\bar{u}, \bar{w} \in \mathbb{R}^3$, then $\bar{u} \cdot \bar{w} = |\bar{u}| \cdot |\bar{w}| \cdot \cos\alpha$, where α is



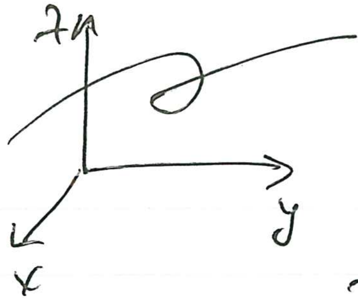
Hence,

$$D_{\bar{v}} F = |\nabla F| \cdot \underbrace{|\bar{v}|}_{=1} \cos\alpha$$

$$= |\nabla F| \cos\alpha$$



Remark: 2) The gradient ∇F of a f-n $F = F(x,y,z)$ defines the direction in which the function increases most rapidly, and the maximum rate of change is $|\nabla F|$.



$\vec{r}(s) = (x(s), y(s), z(s))$.

$T = T(x, y, z)$

$f(s) = T(x(s), y(s), z(s))$

$$f'(s) = \frac{\partial T}{\partial x} \cdot x'(s) + \frac{\partial T}{\partial y} y'(s) + \frac{\partial T}{\partial z} \cdot z'(s)$$

$$= \nabla T \cdot \vec{r}\,'(s) =$$

$\left\{ \begin{array}{l} \text{If } s = \text{arc-length parameter,} \\ \text{then } |\vec{r}\,'(s)| = 1 \end{array} \right\}$.

$$= \nabla T \cdot \hat{T}$$

Rate of change of T with respect to distance travelled along the curve is

$\nabla T \cdot \hat{T}$, where

$\hat{T}$ is the unit tangent to this curve.

$$\left\{ \begin{array}{l} \vec{r} = \vec{r}(t) \text{ curve} \\ \hat{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \end{array} \right\}$$

#2, p.843 Find $D_{\vec{v}} f$ if

1) $f(x,y,z) = x^2y + xz$

2) $P(-1, 1, -1)$

3) $\vec{v}$ is the direction of vector that joins $(3,3,1)$ ~~and~~ to $(3,1,-1)$.

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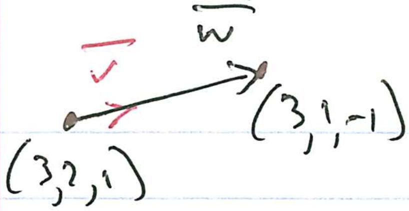
$\overline{w} = (3,1,-1) - (3,2,1) = (0,-1,-2)$



$\overline{v} = \dfrac{\overline{w}}{|\overline{w}|} = \dfrac{(0,-1,-2)}{\sqrt{5}}$.

$\nabla f = (2xy + z,\ x^2,\ x)$

$\nabla f(-1,1,-1) = (-3,1,-1)$

$\therefore\ D_{\overline{v}} f(-1,1,-1) = (-3,1,-1) \cdot \dfrac{(0,-1,-2)}{\sqrt{5}} =$

$= \dfrac{-1+2}{\sqrt{5}} = \dfrac{1}{\sqrt{5}}$.

#6 Same q-n if

1) $f(x,y) = \sin(x+y)$
2) at $(2,-2)$
3) along the line $3x+4y=-2$ in the direction of decreasing y.

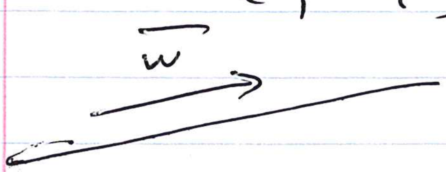
$3x+4y=-2 \iff y = -\frac{3}{4}x - \frac{1}{2}$

~~[illegible]~~ $\begin{cases} x = t \\ y = -\frac{3}{4}t - \frac{1}{2} \end{cases}$, $t \in \mathbb{R}$

$\overline{r}(t) = (1,-\frac{3}{4})t + (0,-\frac{1}{2})$, $t \in \mathbb{R}$.

As $t \uparrow$, $y \downarrow$ $\therefore$ Orientation is correct.

$\overline{w} = (1,-\frac{3}{4})$



$\overline{v} = \dfrac{\overline{w}}{|\overline{w}|} = \dfrac{(1,-\frac{3}{4})}{\sqrt{1+\frac{9}{16}}} =$

$= \dfrac{(1,-\frac{3}{4})}{\frac{5}{4}} = \dfrac{(4,-3)}{5}$.

$$\left\{ \underline{\text{Method 2:}} \quad \bar{w} \parallel \bar{w}_1 = (4,-3) \quad \therefore \bar{v} = \frac{\bar{w}_1}{|\bar{w}_1|} = \frac{(4,-3)}{\sqrt{16+9}} = \frac{(4,-3)}{5} \right\}$$

$$D_{\bar{v}} f = \nabla f(2,-2) \cdot \bar{v}$$

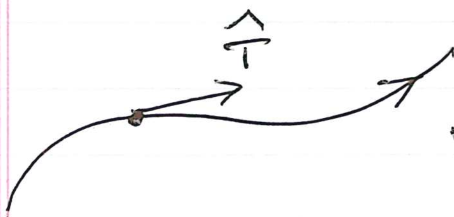
$$\nabla f = (\cos(x+y), \cos(x+y))$$

$$\nabla f(2,-2) = (\cos 0, \cos 0) = (1,1)$$

$$\therefore D_{\bar{v}} f(2,-2) = (1,1) \cdot \frac{(4,-3)}{5} = \frac{1}{5}.$$

<u>#11, p. 893</u> Find the rate of change of f with respect to distance travelled along the curve:

$f(x,y,z) = xy + z^2$ at $(1,0,-2)$ along the curve $y = x^2 - 1$, $z = -2x$ in the direction of increasing X.



The rate of change is

$$D_{\bar{v}} f(1,0,-2) = \nabla f(1,0,-2) \cdot \hat{T}(1,0,-2)$$

$$\nabla f = (y, x, 2z)$$

$$\nabla f(1,0,-2) = (0, 1, -4)$$

we need to find $\hat{T}(1,0,-2)$!

Method 1: Find a parametrization for this curve Q2

$$\begin{cases} y = x^2 - 1 \\ z = -2x \end{cases} \iff \begin{cases} x = t \\ y = t^2 - 1 \\ z = -2t \end{cases}, \quad t \in \mathbb{R}.$$

As $t \uparrow$, $x \uparrow$, i.e., the curve is directed in the direction of increasing x.

$\bar{r}(t) = (t, t^2 - 1, -2t)$

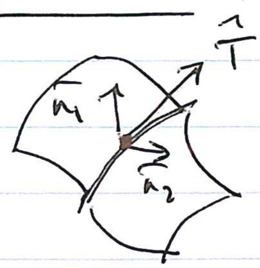
$\bar{r}'(t) = (1, 2t, -2)$

$(1, 0, -2)$ corresponds to $t = 1$.

and $\bar{r}'(1) = (1, 2, -2)$

$$\therefore \hat{T}(1) = \frac{\bar{r}'(1)}{|\bar{r}'(1)|} = \frac{(1,2,-2)}{\sqrt{1+4+4}} = \frac{(1,2,-2)}{3}.$$

Method 2:



$\hat{T} \parallel \bar{n}_1 \times \bar{n}_2$

Surface 1: $y - x^2 + 1 = 0$

$F(x,y,z) = 0$, where

$F(x,y,z) = y - x^2 + 1$.

~~Hence~~ $\nabla F = (-2x, 1, 0)$

At $(1, 0, -2)$, we have

$\bar{n}_1 = \nabla F(1, 0, -2) = (-2, 1, 0)$

Surface 2: $2x + z = 0$, i.e., $G(x,y,z) = 0$ where $G(x,y,z) = 2x + z$.

$\nabla G = (2, 0, 1)$

$\bar{n}_2 = \nabla G(1, 0, -2) = (2, 0, 1)$

$$\overline{n_1} \times \overline{n_2} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = (1, 2, -2)$$

$$\frac{\overline{n_1} \times \overline{n_2}}{|\overline{n_1} \times \overline{n_2}|} = \frac{(1,2,-2)}{3}$$

positive ∴ At this point, x is ↑ (i.e, orientation is correct)

$$\therefore \overline{v} = \frac{(1,2,-2)}{3}.$$

$$\therefore D_{\bar{v}} f(1,0,-2) = (0,1,-4) \cdot \frac{(1,2,-2)}{3} = \frac{10}{3}.$$

#21, p. 844 In what direction (if any) is the rate of change of $f(x,y,z) = xy + z$ at $(0,1,-2)$ equal to (a) 0, (b) 1, and (c) -20?

$$D_{\bar{v}} f(0,1,-2) = \nabla f(0,1,-2) \cdot \overline{v}$$

$$\nabla f(x,y,z) = (y, x, 1)$$

$$\nabla f(0,1,-2) = (1,0,1)$$

$$\therefore D_{\bar{v}} f(0,1,-2) = (1,0,1) \cdot \overline{v} \quad (|\overline{v}| = 1)$$

(a) $D_{\bar{v}} f(0,1,-2) = 0$

$$(1,0,1) \cdot \overline{v} = 0, \quad |\overline{v}| = 1$$

$\overline{v}$ is any unit vector which is $\perp (1,0,1)$.

Find all these $\bar{v}$ explicitly.

If $\bar{v}=(a,b,c)$, then

$(1,0,1)\cdot\bar{v}=0 \iff a+c=0$

and $|\bar{v}|=1 \Rightarrow a^2+b^2+c^2=1$

$$\therefore \begin{cases} a+c=0 \\ a^2+b^2+c^2=1 \end{cases} \iff \begin{cases} c=-a \\ 2a^2+b^2=1 \end{cases} \iff$$

$$\iff \begin{cases} c=-a \\ b^2=1-2a^2 \end{cases} \iff \begin{cases} c=-a \\ b=\pm\sqrt{1-2a^2}, \end{cases}$$

$\bar{v}=(a,\pm\sqrt{1-2a^2},-a)$, $a\in\mathbb{R}$ s.t. $1-2a^2\geq 0$.

$$\Updownarrow$$
$$a^2\leq\frac{1}{2}$$
$$\Updownarrow$$
$$|a|\leq\frac{1}{\sqrt{2}}$$
$$\Updownarrow$$
$$a\in\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right].$$

(b) $D_{\bar{v}}f(0,1,-2)=1$

$\underbrace{(1,0,1)\cdot\bar{v}=1}$ and $|\bar{v}|=1$

$$\Updownarrow$$
$$|(1,0,1)|\cdot\underbrace{|\bar{v}|}_{=1}\cos\alpha=1$$
$$\Updownarrow$$
$$\sqrt{2}\cos\alpha=1$$
$$\Updownarrow$$
$$\cos\alpha=\frac{1}{\sqrt{2}}$$
$$\Updownarrow$$
$$\alpha=\frac{\pi}{4}$$

$\therefore$ $\bar{v}$ is any unit vector that makes an angle $\frac{\pi}{4}$ with $(1,0,1)$

Find all $\bar{v}$ explicitly:

Let $\bar{v} = (a, b, c)$. Then $a^2 + b^2 + c^2 = 1$

$(1,0,1) \cdot \bar{v} = 1 \iff a + c = 1 \iff c = 1 - a.$

$$\begin{cases} c = 1 - a \\ a^2 + b^2 + (1-a)^2 = 1 \end{cases} \iff \begin{cases} c = 1 - a \\ 2a^2 - 2a + b^2 = 0 \end{cases}$$

$$\iff \begin{cases} c = 1 - a \\ b = \pm\sqrt{2a(1-a)} \end{cases}$$

$\therefore \bar{v} = \left(a, \pm\sqrt{2a(1-a)}, 1-a\right)$, where

a is s.t. $a(1-a) \geq 0$, i.e., $0 \leq a \leq 1$.

(c) $D_{\bar{v}} f(0,1,-2) = -20$

$\underline{(1,0,1) \cdot \bar{v} = -20}$, $|\bar{v}| = 1$

$\parallel$

$|(1,0,1)| \cdot |\bar{v}| \cos\alpha = -20$

$\parallel$

$\sqrt{2} \cos\alpha = -20$

$\parallel$

$\cos\alpha = -\frac{20}{\sqrt{2}} < -1 \quad \therefore$ Such α d.n.e.

$\therefore$ Such $\bar{v}$ d.n.e.