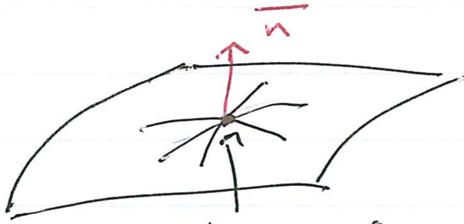
Oct. 17, 2019

Sect. 12.9 Tangent Lines and Tangent Planes.



$F(x,y,z)=0$

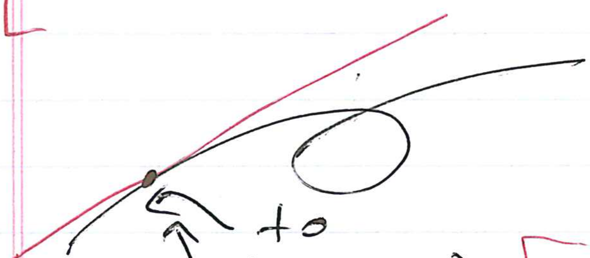
$\bar{n} = \nabla F(x_0, y_0, z_0)$

$= (F_x, F_y, F_z)$

(x_0, y_0, z_0)

Tangent plane: $F_x(x_0,y_0,z_0)(x-x_0) +$

$+ F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0)=0$



t_0

(x_0, y_0, z_0)

$\bar{r}(t) = (x(t), y(t), z(t)).$

$\bar{r}'(t_0) \parallel$ line

$$\begin{cases} x = x_0 + x'(t_0)t \\ y = y_0 + y'(t_0)t \\ z = z_0 + z'(t_0)t, \quad t \in \mathbb{R}. \end{cases}$$

or $(x,y,z) = (x_0, y_0, z_0) + \bar{r}'(t_0)t, \quad t \in \mathbb{R}.$

Tangent line

$$\frac{x-x_0}{x'(t_0)} = \frac{y-y_0}{y'(t_0)} = \frac{z-z_0}{z'(t_0)}.$$

(!) If ~~t~~ a curve is given as the curve of intersection of two surfaces $F(x,y,z)=0$ and $G(x,y,z)=0$, then $\nabla F \times \nabla G$ is $\parallel$ to this curve at each point on the curve.

Hence, if (x_0, y_0, z_0) is a pt. on this curve, i.e., $\begin{cases} F(x_0,y_0,z_0)=0 \\ G(x_0,y_0,z_0)=0 \end{cases}$, then the eq-n of the tangent line to this curve at (x_0,y_0,z_0) is

$$(x,y,z) = (x_0,y_0,z_0) + \left[\nabla F(x_0,y_0,z_0) \times \nabla G(x_0,y_0,z_0)\right] t$$

$t \in \mathbb{R}$.

#1, p. 850 Find eq-n for the tangent line.

$y = x^2$, $z = 0$ at $(-2,4,0)$.

Method 1: $\begin{cases} F(x,y,z) = y - x^2 \\ G(x,y,z) = z \end{cases}$

Curve: $F=0$, $G=0$

$$\bar{n}_1 = \nabla F(-2,4,0) = (-2x, 1, 0)\Big|_{(-2,4,0)} =$$
$$= (4,1,0)$$

$$\bar{n}_2 = \nabla G(-2,4,0) = (0,0,1)$$

$$\bar{n}_1 \times \bar{n}_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1,-4,0)$$

$$(x,y,z) = (-2,4,0) + t(1,-4,0), \quad t \in \mathbb{R}$$

or

$$\begin{cases} x = -2+t \\ y = 4-4t \\ z = 0 \end{cases}, \quad t \in \mathbb{R}.$$

Method 2: Curve $\begin{cases} x = t \\ y = t^2 \\ z = 0, & t \in \mathbb{R}. \end{cases}$

$\bar{r}(t) = (t, t^2, 0)$

$(-2, 4, 0)$ corresponds to $t_0 = -2$.

$\bar{r}'(t_0) \parallel$ tangent line.

$\bar{r}'(t) = (1, 2t, 0)$

$\bar{r}'(-2) = (1, -4, 0)$

$\therefore (x, y, z) = (-2, 4, 0) + (1, -4, 0)t, \; t \in \mathbb{R}.$

#13, p. 850 Curve $\begin{cases} x^2 + y^2 + z^2 = 4 \\ z^2 = x^2 + y^2 \end{cases}$ at $(1, 1, -\sqrt{2})$

$F(x, y, z) = x^2 + y^2 + z^2 - 4$

$G(x, y, z) = x^2 + y^2 - z^2$

$\underbrace{F = 0, \; G = 0}_{\text{curve.}}$

$\nabla F = (2x, 2y, 2z) = 2(x, y, z) \parallel (x, y, z)$

$\bar{n}_1 = (x, y, z)\Big|_{(1,1,-\sqrt{2})} = (1, 1, -\sqrt{2}) \; \Big\} \parallel \nabla F(1, 1, -\sqrt{2})$

$\nabla G = (2x, 2y, -2z) = 2(x, y, -z) \parallel (x, y, -z)$

$\bar{n}_2 = (x, y, -z)\Big|_{(1,1,-\sqrt{2})} = (1, 1, \sqrt{2}).$

$$\bar{u} = \bar{n}_1 \times \bar{n}_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & -\sqrt{2} \\ 1 & 1 & \sqrt{2} \end{vmatrix} = (2\sqrt{2}, -2\sqrt{2}, 0).$$

$$= 2\sqrt{2}(1, -1, 0) \parallel \boxed{(1, -1, 0) = \bar{u}_1}$$

Tangent line: $(x,y,z) = (1,1,-\sqrt{2}) + t(1,-1,0)$, $t \in \mathbb{R}$.

#23, p.850 Eq-n of tangent plane to

$$\underbrace{x^2y + y^2z + z^2x + 3}_{F(x,y,z)} = 0 \quad \text{at } (2,-1,-1).$$

$$\bar{n} = \nabla F(2,-1,-1) = \left.(2xy + z^2,\ x^2 + 2yz,\ y^2 + 2xz)\right|_{(2,-1,-1)}$$

$$= (-4+1,\ 4+2,\ 1-4)$$

$$= (-3,6,-3) = -3(1,-2,1)$$

$\therefore$ $(1,-2,1) \perp$ tangent plane.

$\therefore$ Eq-n of tan. plane $(x-2) - 2(y+1) + (z+1) = 0$

#27, p.850 Show that the curve

$$\bar{r}(t) = \left(\frac{2}{3}(t^3+2),\ 2t^2,\ 3t-2\right)$$

intersects the surface $x^2 + 2y^2 + 3z^2 = 15$ at right angles at $(2,2,1)$.

1) Find all t s.t. $\bar{r}(t) = (2,2,1)$

$$\begin{cases} \frac{2}{3}(t^3+2) = 2 \\ 2t^2 = 2 \\ 3t-2 = 1 \end{cases} \iff \begin{cases} \frac{2}{3}\cdot 3 = 2 \\ 2\cdot 1 = 2 \\ t = 1 \end{cases} \iff \boxed{t=1}$$

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$\bar{r}'(1)$ is $\parallel$ curve (and, hence, to its tangent) at $(2,2,1)$

$\therefore$ we need to show

$$\bar{r}'(1) \perp \text{surface at } (2,2,1)$$

$$\bar{r}'(t) = (2t^2, 4t, 3)$$

$$\underline{\bar{r}'(1) = (2, 4, 3)}$$

Let $F(x,y,z) = x^2 + 2y^2 + 3z^2 - 15$

(then $F = 0$ defines our surface).

$\nabla F(2,2,1) \perp$ surface

$$\nabla F(x,y,z) = (2x, 4y, 6z) = 2(x, 2y, 3z)$$

$$\parallel (x, 2y, 3z)$$

$\therefore \nabla F(2,2,1)$ is $\parallel (2,4,3)$

Hence, we need to show that

$$\underline{(2,4,3) \text{ is } \parallel (2,4,3)} \quad \text{(obvious)}$$

$$\bar{r}'(1) \parallel \nabla F(2,2,1)$$

#28, p. 850 Verify that the curve

$x^2 - y^2 + z^2 = 1, \quad xy + xz = 2$

is tangent to the surface $xyz - x^2 - 6y + 6 = 0$

at $(1,1,1)$.

If $\overline{u}$ is $\parallel$ curve at $(1,1,1)$, and $\overline{n} \perp$ surface at $(1,1,1)$, we need to show that $\overline{u} \perp \overline{n}$.

$G(x,y,z) = x^2 - y^2 + z^2 - 1$ Curve: $G=0, H=0$

$H(x,y,z) = xy + xz - 2$

$F(x,y,z) = xyz - x^2 - 6y + 6$ Surface: $F=0$.

$\overline{u} \parallel \nabla G(1,1,1) + \nabla H(1,1,1)$

$\overline{n} \parallel \nabla F(1,1,1)$

$\nabla G = (2x, -2y, 2z) \parallel (x, -y, z)$

$\nabla H = (y+z, x, x)$

$\overline{n_1} = (x, -y, z)\Big|_{(1,1,1)} = (1, -1, 1)$

$\overline{n_2} = (y+z, x, x)\Big|_{(1,1,1)} = (2, 1, 1)$

$$\overline{n_1} + \overline{n_2} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = (-2, 1, 3)$$

$\therefore \boxed{\overline{u} = (-2, 1, 3)}$

$\nabla F = (yz-2x, xz-6, xy)$

$\nabla F(1,1,1) = \boxed{(-1,-5,1) = \overline{n}}$

$\overline{u} \perp \overline{n} \iff \overline{u}\cdot\overline{n} = 0$

$\overline{u}\cdot\overline{n} = (-2,1,3)\cdot(-1,-5,1) =$

$= 2-5+3 = 0$

$\therefore$ Curve is tangent to surface at $(1,1,1)$.

#30, p. 850 Find the indicated derivative:

$f(x,y,z) = 2x^2+y^2z^2$ at $(3,1,0)$ with respect to distance along the curve $x+y+z=4$, $x-y+z=2$ in the direction of increasing x.

$D_{\overline{v}} f(3,1,0)$ where $\overline{v}$ is the unit tangent to the curve oriented so that $x\uparrow$.

$\overline{r} = \overline{r}(t) = (x(t), y(t), z(t))$

$\overline{r}'(t) = (x'(t), y'(t), z'(t))$

$x\uparrow \Rightarrow x' > 0 \quad \therefore$ If $\hat{T}_1 = (a,b,c)$

$\hat{T}_2 = (-a,-b,-c)$,

then we need to pick the vector whose 1st component is >0.

$F(x,y,z) = x+y+z-4$

$G(x,y,z) = x-y+z-2$

$\underbrace{F=0,\ G=0}_{\text{curve}}$

$(\nabla F \times \nabla G)\big|_{(3,1,0)} \parallel$ curve

$\nabla F = (1,1,1)$

$\nabla G = (1,-1,1)$

$$\nabla F \times \nabla G = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = (2,0,-2) = 2(1,0,-1) \parallel (1,0,-1).$$

$(1,0,-1) \parallel$ curve at $(3,1,0)$

Unit tangents are: $\underbrace{\frac{(1,0,-1)}{\sqrt{2}}}_{\hat{T}}$ and $\frac{(-1,0,1)}{\sqrt{2}}$

$$\therefore D_{\bar{v}} f(3,1,0) = D_{\hat{T}} f(3,1,0) =$$

$$= \nabla f(3,1,0) \cdot \hat{T}$$

$\nabla f = (4x, 2yz^2, 2y^2z)$

$\nabla f(3,1,0) = (12, 0, 0)$

$$\therefore D_{\bar{v}} f(3,1,0) = (12,0,0) \cdot \frac{(1,0,-1)}{\sqrt{2}} = \frac{12}{\sqrt{2}} = 6\sqrt{2}.$$

Sect. 12.10: Relative (local) maxima and minima.

Note: relative $\equiv$ local.

Definition: A f-n $f(x,y)$ is said to have a relative (local) maximum $f(x_0,y_0)$ at a point (x_0,y_0) if there is a circle in the xy-plane centered at (x_0,y_0) s.t. for all points (x,y) inside this circle

$$f(x,y) \le f(x_0,y_0)$$

Def-n: — " — relative (local) minimum

— " — $f(x,y) \ge f(x_0,y_0)$.

Def-n: A point in the domain of a f-n $f(x,y)$ is said to be a critical point of $f(x,y)$ if

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = 0 \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = 0$$

or if one (or both) of these partial derivatives d.n.e. at (x_0,y_0).

Def-n: If a critical point of $f(x,y)$ at which $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ yields neither a relative max or relative min, then it is called a saddle point.

Theorem: Suppose that (x_0, y_0) is a critical pt. of $f(x,y)$ at which $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both 0. Suppose that $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ exist near (x_0, y_0) and are continuous at (x_0, y_0). Define

$$A = f_{xx}(x_0, y_0)$$

$$B = f_{xy}(x_0, y_0) \quad \left(= f_{yx}(x_0, y_0) \right)$$

$$C = f_{yy}(x_0, y_0)$$

(i) If $A > 0$ and $B^2 - AC < 0$, then $f(x,y)$ has a local minimum at (x_0, y_0).

(ii) If $A < 0$ and $B^2 - AC < 0$, then $f(x,y)$ has a local maximum at (x_0, y_0).

(iii) If $B^2 - AC > 0$, then $f(x,y)$ has a saddle point at (x_0, y_0)

(iv) If $B^2 - AC = 0$, no conclusion can be made.