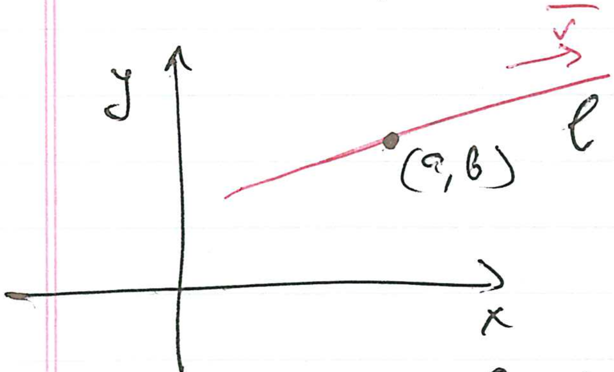
Oct. 22, 2019

Let $f = f(x,y)$ be s.t. (a,b) is a critical pt.



$\bar{v} = (t_1, t_2)$ where $\bar{v} \neq \bar{0}$ (i.e., $t_1^2 + t_2^2 \neq 0$)

Line ℓ: $\begin{cases} x = a + t_1 u \\ y = b + t_2 u, \ u \in \mathbb{R}. \end{cases}$

Restrict $f = f(x,y)$ to this line ℓ.

$$g(u) = f(a + t_1 u, b + t_2 u), \quad u \in \mathbb{R}.$$

If $u = 0$, then $g(0) = f(a,b)$

Note: $g : \mathbb{R} \to \mathbb{R}$ is defined near $u = 0$.

$$g'(u) = f_x(a + t_1 u, b + t_2 u) \cdot t_1 + \\ + f_y(a + t_1 u, b + t_2 u) \cdot t_2$$

$$g'(0) = \underbrace{f_x(a,b)}_{=0} \cdot t_1 + \underbrace{f_y(a,b)}_{=0} \cdot t_2 = 0$$

((a,b) is a critical pt. for f: $f_x(a,b) = 0$ and $f_y(a,b) = 0$).

$$g''(u) = f_{xx}(a + t_1 u, b + t_2 u)\, t_1^2 + \\ + f_{xy}(\dots)\, t_1 t_2 + \\ + f_{yx}(\dots)\, t_1 t_2 + f_{yy}(\dots)\, t_2^2 =$$

$= \{$ If f_{xy}, f_{yx} are "nice" then $f_{xy} = f_{yx}$ $\}$.

$= f_{xx}(a+t_1 u, b+t_2 u)\, t_1^2 +$
$+ 2 f_{xy}(\ldots)\, t_1 t_2 + f_{yy}(\ldots)\, t_2^2$

$$\therefore\ g''(0) = \underbrace{f_{xx}(a,b)}_{A}\, t_1^2 + 2 \underbrace{f_{xy}(a,b)}_{B}\, t_1 t_2 + \underbrace{f_{yy}(a,b)}_{C}\, t_2^2$$

i.e., $g''(0) = A t_1^2 + 2B t_1 t_2 + C t_2^2$

Q-n: For what A, B and C $g''(0)$ is >0 or <0 for all $(t_1, t_2) \neq (0,0)$.

1) If $t_2 = 0$, $g''(0) = A t_1^2 \begin{cases} >0 \text{ if } A>0 \\ <0 \text{ if } A<0 \end{cases}$

(note: $t_1 \neq 0$)

2) If $t_2 \neq 0$, then

$$g''(0) = \underbrace{t_2^2}_{>0} \left[A\left(\frac{t_1}{t_2}\right)^2 + 2B \frac{t_1}{t_2} + C \right]$$

Let $\frac{t_1}{t_2} = w$. Then $g''(0) = \underbrace{\text{const}}_{>0} \left[A w^2 + 2Bw + C \right]$

$Aw^2 + 2Bw + C = 0 \iff$ Discriminant

$(2B)^2 - 4AC = 4(B^2 - AC)$

$Aw^2 + 2Bw + C > 0$ if $B^2 - AC < 0$ and $A > 0$

— " — < 0 if $B^2 - AC < 0$ and $A < 0$

changes sign if $B^2 - AC > 0$

$\therefore$ If $B^2 - AC < 0$ and $A > 0$, then
$g''(0) > 0$ $\therefore$ (a,b) is a pt. of local minimum.

If $B^2 - AC < 0$ and $A < 0$, then
$g''(0) < 0$ $\therefore$ (a,b) — " — local maximum.

If $B^2 - AC > 0$, then $g''(0)$ is positive for some (t_1, t_2) and negative for some other (t_1, t_2), i.e., (a,b) is a saddle pt.

If $B^2 - AC = 0$, then the test is inconclusive.

__Remarks:__ 1) $\det \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$ ← Hessian of f

$$\overset{\shortparallel}{\det} \begin{bmatrix} A & B \\ B & C \end{bmatrix} = AC - B^2$$

2) why $B^2 - AC = 0 \Rightarrow$ the test is inconclusive?

__Ex:__ $f(x,y) = y^2$

$f_x = 0$, $f_y = 2y$ $\therefore$ $(0,0)$ is a crit. pt.

$\underset{\overset{\shortparallel}{A}}{f_{xx} = 0}$, $\underset{\overset{\shortparallel}{B}}{f_{xy} = 0}$, $\underset{\overset{\shortparallel}{C}}{f_{yy} = 2}$

$B^2 - AC = 0 - 0 \cdot 2 = 0$

If (x,y) is close to $(0,0)$ (actually, for any (x,y)) $f(x,y) \geq f(0,0) = 0$ $\therefore$ $(0,0)$ is a pt. of local minimum

$g(x,y) = -y^2$

$g_x = 0,\ g_y = -2y$ $\therefore (0,0)$ is a crit. pt.

$\underset{\text{"}A}{g_{xx} = 0},\ \underset{\text{"}B}{g_{xy} = 0},\ \underset{\text{"}C}{g_{yy} = -2}$

$B^2 - AC = 0$, but $(0,0)$ is a pt. of local maximum,

since $g(x,y) \le g(0,0) = 0$.

$h(x,y) = y^3$

$h_x = 0,\ h_y = 3y^2$ $\therefore (0,0)$ is a crit. pt.

$\underset{\text{"}A}{h_{xx} = 0},\ \underset{\text{"}B}{h_{xy} = 0},\ \underset{\text{"}C}{h_{yy} = 6y}$

$B^2 - AC = 0$ and now $(0,0)$ is a saddle pt., since $h(x,y) = y^3 > 0$ if $y > 0$ and < 0 if $y < 0$.

#3, p. 859 Find all critical pts and classify.

$f(x,y) = x^3 - 3x + y^2 + 2y$

$f_x = 3x^2 - 3$

$f_y = 2y + 2$

Critical pts:

$$\begin{cases} 3x^2 - 3 = 0 \\ 2y + 2 = 0 \end{cases} \iff \begin{cases} x^2 = 1 \\ y = -1 \end{cases} \iff \begin{cases} x = \pm 1 \\ y = -1 \end{cases}$$

$\therefore$ Critical pts are: $(1,-1)$ and $(-1,-1)$.

Critical pt	A	B	C	B^2-AC	Nature
$(1,-1)$	$\boxed{6}^{>0}$	0	2	$\boxed{-12}^{<0}$	Local minimum
$(-1,-1)$	-6	0	2	$\boxed{12}^{>0}$	Saddle pt.

$f_{xx} = 6x \leftarrow A$

$f_{xy} = 0 \leftarrow B$

$f_{yy} = 2 \leftarrow C$

<u>#6, p. 859</u> $\quad f(x,y) = x\sin y$

$f_x = \sin y$

$f_y = x\cos y$

$$\begin{cases} \sin y = 0 \\ x\cos y = 0 \end{cases} \Leftrightarrow \begin{cases} y = \pi k, \ k \in \mathbb{Z} \\ x = 0 \end{cases}$$

Critical points: $(0, \pi k)$, $k \in \mathbb{Z}$

$\{ \ \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\} \ \}$

$f_{xx} = 0 \leftarrow A$

$f_{xy} = \cos y \leftarrow B$

$f_{yy} = -x\sin y \leftarrow C$

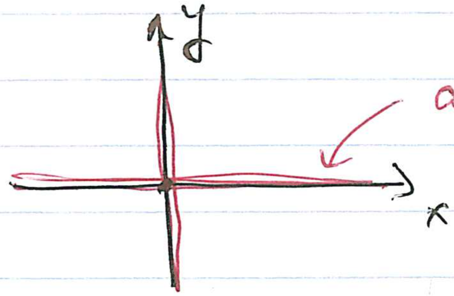
Critical pt	A	B	C	B^2-AC	Nature
$(0, 2\pi n), n \in \mathbb{Z}$	0	1	0	$\boxed{1}^{>0}$	Saddle pt.
$(0, \pi(2n+1)),$ $n \in \mathbb{Z}$	0	-1	0	$\boxed{1}^{>0}$	Saddle pt.

#10, p. 859. $f(x,y) = x^4 y^3$

$$f_x = 4x^3y^3 \qquad f_y = 3x^4y^2$$

$$\begin{cases} 4x^3y^3 = 0 \\ 3x^4y^2 = 0 \end{cases} \Rightarrow x = 0 \text{ or } y = 0$$

Critical points: $(0, b)$, $b \in \mathbb{R}$
$(a, 0)$, $a \in \mathbb{R}$



any pt. of the x- and y-axes is a critical point.

$$f_{xx} = 12x^2y^3 \qquad f_{xy} = 12x^3y^2 \qquad f_{yy} = 6x^4y$$

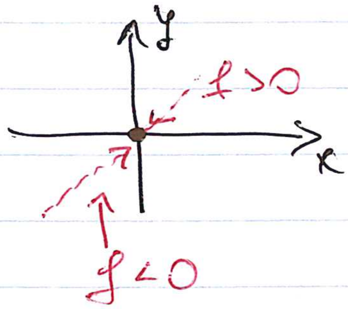
$$B^2 - AC = \left(12x^3y^2\right)^2 - 12x^2y^3 \cdot 6x^4y$$
$= 0$ at all critical points

$\therefore$ "Second derivative test" fails at all critical points.

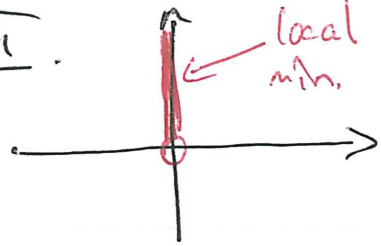
I. C.P. $(0,0)$

$$f(x,y) \begin{cases} > 0 \text{ if } y > 0, x \neq 0 \\ < 0 \text{ if } y < 0, x \neq 0 \end{cases}$$

and $f(0,0) = 0$ $\therefore$ $(0,0)$ is a saddle point.



II.



C.P. $(0,b)$, $b>0$

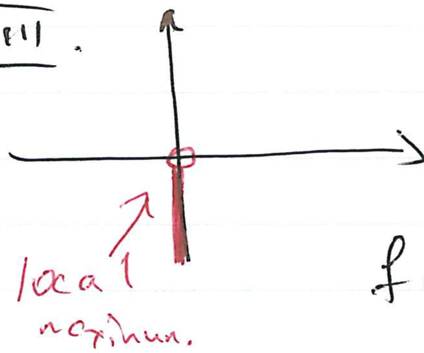
$$f(0,b) = 0 \cdot b^3 = 0$$

$$f(x,y) = \underbrace{x^4}_{\geq 0}\, \underbrace{y^3}_{\sim b^3 > 0} \text{ when } (x,y) \sim (0,b).$$

$\therefore$ $f(x,y) \geq \underset{f(0,b)}{\overset{\shortparallel}{0}}$ for all (x,y) "close" to $(0,b)$.

$\therefore$ $(0,b)$ is a pt. of local minimum.

III.



C.P. $(0,b)$, $b<0$

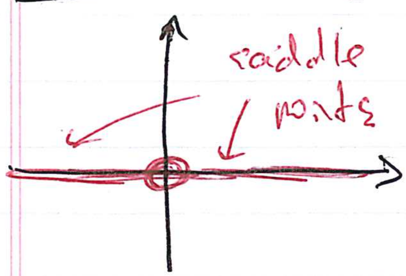
$$f(0,b) = 0 \cdot b^3 = 0$$

$$f(x,y) = \underbrace{x^4}_{\geq 0}\, \underbrace{y^3}_{\sim b^3 < 0} \leq 0 = f(0,b)$$

when (x,y) is close to $(0,b)$.

$\therefore$ $(0,b)$ is a pt. of local maximum.

IV. C.P. $(a,0)$, $a \neq 0$



$$f(a,0) = a^4 \cdot 0^3 = 0$$

$$f(x,y) = \underbrace{x^4}_{a^4>0}\, \underbrace{y^3}_{\substack{>0 \text{ if } y>0 \\ <0 \text{ if } y<0}} \text{ when } (x,y) \sim (a,0)$$

$\therefore$ $f(x,y) > 0$ when $(x,y) \sim (a,0)$ and $y>0$

$f(x,y) < 0$ when $(x,y) \sim (a,0)$ and $y<0$

$\therefore$ $(a,0)$, $a \neq 0$ are saddle points

#12, p.859 $f(x,y) = |x| + y^2$

$$f_x = \frac{\partial}{\partial x}(|x|) \longleftarrow \text{d.n.e. if } x = 0$$

HW: #13, p.859

$\therefore$ Some of critical points:

$$(0, b),\ b \in \mathbb{R}.$$

$x > 0$: $f(x,y) = x + y^2$

$f_x = 1$

$f_y = 2y$

$$\begin{cases} 1 = 0 \\ 2y = 0 \end{cases} \Rightarrow \text{no sols}$$

$x < 0$: $f(x,y) = -x + y^2$

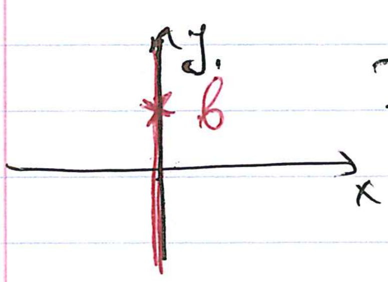
$f_x = -1$

$f_y = 2y$

$$\begin{cases} -1 = 0 \\ 2y = 0 \end{cases} \Rightarrow \text{no sols.}$$

$\therefore$ There are no other critical points.

$\therefore$ The set of all critical pts: $\{(0,b) \mid b \in \mathbb{R}\}$



I. C.P.: $(0,b)$, $b > 0$

$f(0,b) = |0| + b^2 = b^2$

$f(x,y) = |x| + y^2$

we want to know if $f(x,y) \overset{?}{\gtrless} f(0,b)$ when (x,y) is close to $(0,b)$

$$\underbrace{|x|}_{\geq 0} + \underbrace{y^2}_{} \quad ? \quad b^2$$

$y > b$, $y^2 > b^2$

$y < b$, $y^2 < b^2$

More precisely,

$$f(0,y) = y^2 \quad \begin{array}{l} > b^2 \text{ if } y > b \\ < b^2 \text{ if } y < b \end{array}$$

$\therefore$ $(0,b)$, $b>0$ is a saddle point.

II. $(0,b)$, $b<0$ $\leftarrow$ same thing happens (check)

$\therefore$ $(0,b)$, $b<0$ is a saddle point.

III. $(0,0)$ $\qquad f(0,0) = |0| + 0^2 = 0$

$$f(x,y) = \underbrace{|x|}_{\geq 0} + \underbrace{y^2}_{\geq 0} \geq 0 = f(0,0)$$

$\therefore$ $(0,0)$ is a pt. of local minimum.

Note:

Def-n 1. Saddle point $\Leftrightarrow$ a critical pt. which is NOT a pt. of local max or min.

Def-n 2. Saddle point $\Leftrightarrow$ a critical pt. s.t. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ and not a point of local max or min.

If (a,b) is s.t. $\frac{\partial f}{\partial x}(a,b)$ OR $\frac{\partial f}{\partial y}(a,b)$ d.n.e. and it is NOT a pt. of local max/min, then it IS a saddle point under Def. 1 but NOT a saddle point under Def. 2.

#17, p. 860 Find all critical points

113

$f(x,y,z) = xyz + x^2yz - y$

$f_x = yz + 2xyz$

$f_y = xz + x^2z - 1$

$f_z = xy + x^2y$

Solve $\begin{cases} yz + 2xyz = 0 \\ xz + x^2z - 1 = 0 \\ xy + x^2y = 0 \end{cases} \Leftrightarrow \begin{cases} yz(1+2x) = 0 \\ xz + x^2z = 1 \\ xy(1+x) = 0 \end{cases}$

$\underline{y=0}$ $\quad xz + x^2z = 1 \Leftrightarrow z = \frac{1}{x+x^2}$

$\therefore \boxed{\left(x, 0, \frac{1}{x+x^2}\right) \text{ is a critical pt}}.$

$\underline{z=0}$ $\quad$ 2nd eq-n: $0 = 1$ NO soln.

$x = -\frac{1}{2}$ $\quad$ 3d eq-n: $y \cdot \left(-\frac{1}{2}\right) \cdot \frac{1}{2} = 0 \Rightarrow y = 0$

2nd eq-n: $-\frac{1}{2}z + \frac{1}{4}z = 1 \Leftrightarrow -\frac{z}{4} = 1$

$\Leftrightarrow z = -4.$

$\therefore \boxed{\left(-\frac{1}{2}, 0, -4\right) \text{ is another crit. pt.}}$