October 24, 2019

Sect. 12.11 Absolute maxima / minima.

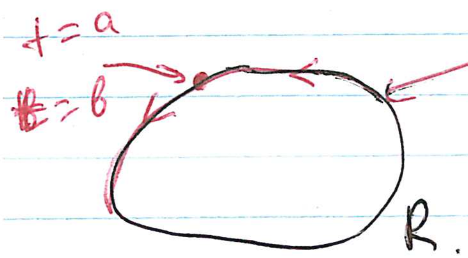
Def-n: The absolute (global) maximum of $f = f(x,y)$ on a region R is $f(x_0, y_0)$ if $(x_0, y_0) \in R$ and $f(x,y) \le f(x_0, y_0)$ for all $(x,y) \in R$.

— " ——— minimum ——— " ——— $f(x,y) \ge f(x_0, y_0)$ — " —

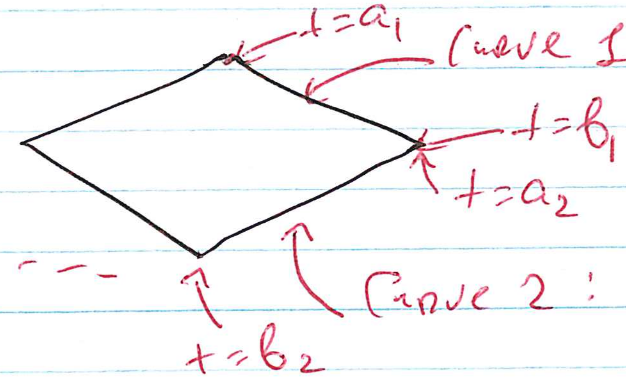
Theorem: Let R be a bounded region in $\mathbb{R}^2$ that contains its boundary (i.e., R is a ~~...~~ bounded closed set), and let $f = f(x,y)$ be continuous on R. Then f attains its abs. max and abs. min values.

How to find these abs max/min values for such a f-n?

1) Find all critical pts inside R
2) Find all critical pts. of the univariate f-n which is the restriction of f to the boundary of R (the endpoints)



Boundary = curve $\bar{r}(t) = (x(t), y(t))$, $a \le t \le b$.



Curve 1: $\bar{r}_1(t) = (x_1(t), y_1(t))$, $a_1 \le t \le b_1$

Curve 2: $\bar{r}_2(t)$, $a_2 \le t \le b_2$

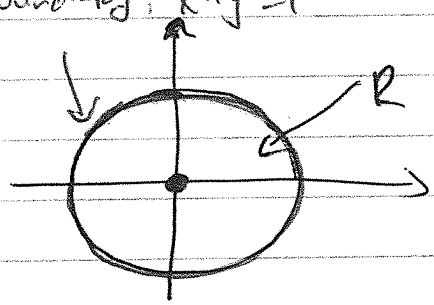
Consider $g(t) = f(\bar{r}(t)) = f(x(t), y(t))$, $a \le t \le b$.

Find all critical pts. of g on $[a, b]$
+ consider g at the endpoints a and b.

3) Find the values of f at all of these points.
The largest of these values = abs. max value of f on R
The smallest — " — = abs. min value of f on R.

<u>#1, p. 867</u> Find abs. max/min of $f(x,y) = x^2 + y^3$ on $R = \{(x,y) \mid x^2 + y^2 \le 1\}$

Boundary: $x^2+y^2=1$



$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 3y^2$$

$$\begin{cases} 2x = 0 \\ 3y^2 = 0 \end{cases} \iff \boxed{(x,y) = (0,0)} \text{ critical point}$$

Boundary: curve $x^2+y^2=1$

<u>Method 1:</u> $\begin{cases} x = \cos t \\ y = \sin t, \quad 0 \le t \le 2\pi \end{cases}$

f restricted to this curve:

$g(t) = f(\cos t, \sin t) = \cos^2 t + \sin^3 t$, $\underline{0 \le t \le 2\pi}$

$g'(t) = 2\cos t(-\sin t) + 3\sin^2 t \cos t =$

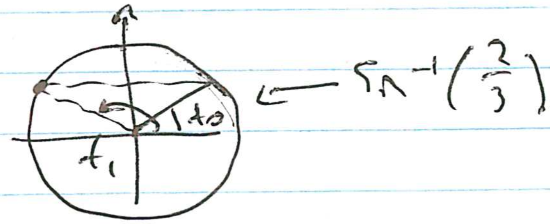
$= \sin t \cos t(-2 + 3\sin t)$

$g'(t) = 0 \iff \sin t = 0$, $\cos t = 0$ or $\sin t = \frac{2}{3}$

$\sin t = 0$ (on $[0, 2\pi]$) if $t = 0, \pi, 2\pi$

$\cos t = 0$ (on $[0, 2\pi]$) if $t = \frac{\pi}{2}, \frac{3\pi}{2}$

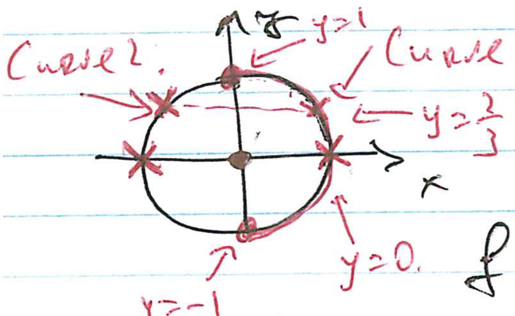
$\sin t = \frac{2}{3}$ if $t_0 = \sin^{-1}\left(\frac{2}{3}\right)$ or



$t_1 = \pi - \sin^{-1}\left(\frac{2}{3}\right)$

etc.

<u>Method 2:</u> Curve $x^2 + y^2 = 1 \Leftrightarrow x^2 = 1 - y^2$



Curve 1 $\Leftrightarrow x = \sqrt{1-y^2}$, $-1 \le y \le 1$

or

$x = -\sqrt{1-y^2}$, $-1 \le y \le 1$.

f restricted to either of these curves:

$$g(y) = f(x,y) = x^2 + y^3 = 1 - y^2 + y^3, \quad -1 \le y \le 1.$$

$x = \pm\sqrt{1-y^2}$

$$g'(y) = -2y + 3y^2 = y(-2+3y)$$

$g'(y) = 0$ iff $y = 0$ or $y = \frac{2}{3}$

All the points "of interest":

$(0,0)$, $(1,0)$, $(-1,0)$, $\left(\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$, $\left(-\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$,

$(0,-1)$ and $(0,1)$.

$\left\{ \begin{array}{l} \text{if } y = \frac{2}{3} \\ x^2 = 1 - \frac{4}{9} = \frac{5}{9} \end{array} \right\}$

$f(x,y) = x^2 + y^3$

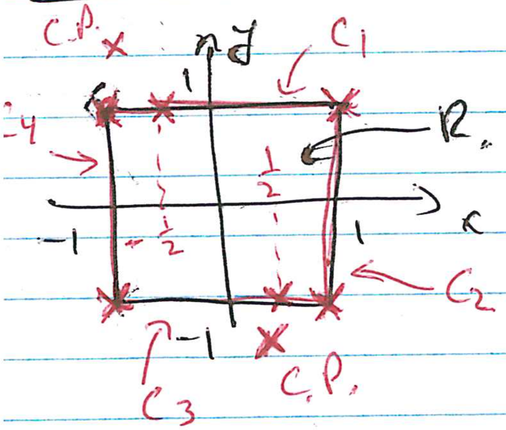
$f(0,0) = 0$, $f(\pm 1, 0) = \boxed{1}$, $f\left(\pm \frac{\sqrt{5}}{3}, \frac{2}{3}\right) = \frac{5}{9} + \frac{8}{27} = \frac{15+8}{27} = \frac{23}{27}$

$f(0,-1) = -1$, $f(0,1) = \boxed{1}$

The abs. max. value of f on $x^2+y^2 \le 1$ is 1 which is attained at $(1,0)$, $(-1,0)$ and $(0,1)$.

The abs. min. value of f on $x^2+y^2 \le 1$ is -1 attained at $(0,-1)$.

#4, p. 867 $f(x,y) = x^2 y + x y^2 + y$ on $R: -1 \le x \le 1$, $-1 \le y \le 1$.



1. $\frac{\partial f}{\partial x} = 2xy + y^2$

$\frac{\partial f}{\partial y} = x^2 + 2xy + 1$

$$\begin{cases} 2xy + y^2 = 0 \\ x^2 + 2xy + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} y(2x+y) = 0 \\ \text{—"—} \end{cases} \Leftrightarrow$$

{If $y = 0$: $x^2 + 1 = 0$ NO sol-n}

$$\Leftrightarrow \begin{cases} y = -2x \\ x^2 + 2x(-2x) + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} y = -2x \\ -3x^2 + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} x = \pm \frac{1}{\sqrt{3}} \\ y = -2x \end{cases}$$

Note: $\frac{2}{\sqrt{3}} > 1$ and $-\frac{2}{\sqrt{3}} < -1$.

$\therefore$ Critical points $\left(\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ are NOT inside R.

$\therefore$ No critical pts inside R.

11C

$\underline{\text{ii}}$. C_1 : $\begin{cases} y=1 \\ x=t, & -1 \le t \le 1 \end{cases}$

$g_1(t) = f(t,1) = t^2 + t + 1, \quad -1 \le t \le 1.$

$g_1'(t) = 2t+1, \quad g_1'(t)=0 \Leftrightarrow t = -\frac{1}{2}.$

$\therefore$ $\underline{\left(-\frac{1}{2}, 1\right) \text{ is a pt. of "interest" as well}}$

as the endpoints : $\underline{(-1,1),\ (1,1)}$.

C_2 : $\begin{cases} x=1 \\ y=t, & -1 \le t \le 1. \end{cases}$

$g_2(t) = f(1,t) = t + t^2 + t = t^2 + 2t$

$g_2'(t) = 2t+2, \quad g_2'(t) = 0 \Leftrightarrow t = -1.$

Pts of interest : $(1,-1), (1,1)$.

C_3 : $\begin{cases} y=-1 \\ x=t, & -1 \le t \le 1. \end{cases}$

$g_3(t) = f(t,-1) = -t^2 + t - 1, \quad -1 \le t \le 1.$

$g_3'(t) = -2t+1, \quad g_3'(t) = 0 \Leftrightarrow t = \frac{1}{2}$

Pts of interest : $\left(\frac{1}{2}, -1\right), (1,-1), (-1,-1)$

III

C_4: $\begin{cases} x = -1 \\ y = t, \ -1 \le t \le 1. \end{cases}$

$g_4(t) = f(-1, t) = t - t^2 + t = 2t - t^2$

$g_4'(t) = 2 - 2t$, $g_4'(t) = 0 \iff t = 1.$

$\therefore$ pts of interest: $(-1, 1)$, $(-1, -1)$

$\overline{\text{III}}$. All pts of interest:

$(-1,1), (1,1), (1,-1), (-1,-1), (-\frac{1}{2}, 1), (\frac{1}{2}, -1).$

$f(-1,1) = 1 - 1 + 1 = 1$

$f(1,1) = 1 + 1 + 1 = \boxed{3}$

$f(1,-1) = -1 + 1 - 1 = -1$

$f(-1,-1) = -1 - 1 - 1 = \boxed{-3}$

$f(-\frac{1}{2}, 1) = \frac{1}{4} - \frac{1}{2} + 1 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

$f(\frac{1}{2}, -1) = -\frac{1}{4} + \frac{1}{2} - 1 = \frac{1}{4} - 1 = -\frac{3}{4}.$

$\therefore$ Abs. max value of f on R: 3 attained at $(1,1)$.

Abs. min value of f on R: -3 attained at $(-1,-1)$.

#10, p. 867 Find the point on the plane

$x+y-2z=6$ closest to the origin.

if $P(x,y,z)$, then $[\text{dist}(P,O)]^2 = x^2+y^2+z^2$

$x+y-2z=6 \iff y = 2z-x+6$

$[\text{dist}(P,O)]^2 = x^2+(2z-x+6)^2+z^2$, i.e.,

we want to find the abs. minimum of

$f(x,z) = x^2+(2z-x+6)^2+z^2$ on

$R = \{(x,z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$.

Since the abs. min is attained (clear from the statement), it will have to be at one of the critical points.

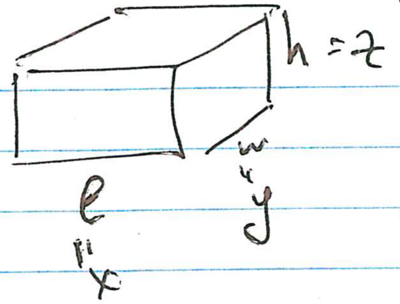
$$\frac{\partial f}{\partial x} = 2x + 2(2z-x+6)(-1) = 2(2x-2z-6) = 4(x-z-3)$$

$$\frac{\partial f}{\partial z} = 2(2z-x+6)\cdot 2 + 2z = 2(4z-2x+12+z) = 2(5z-2x+12) = \cancel{4(3z-x+6)}$$

Solve $\begin{cases} x-z-3=0 \\ 5z-2x+12=0 \end{cases}$ which will give us the critical pt. where abs. min value is attained.

(HW)

#15, p. 867.



Post office demands:

$l + 2(w+h) \le 250$ (cm).

Find dimensions s.t.

Volume = $lwh \longrightarrow$ maximized.

we have: $x + 2(y+z) = 250$ clear.

$V = xyz$ and s.t. $x \ge 0, y \ge 0, z \ge 0$.

$x = 250 - 2(y+z)$

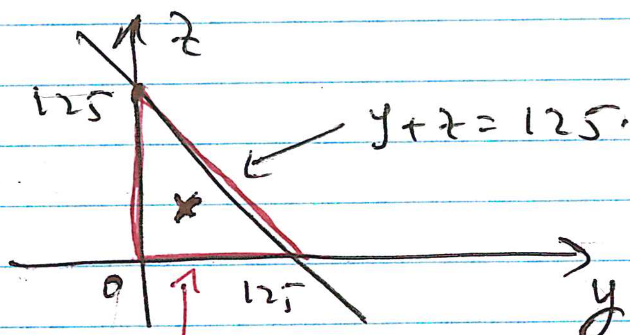
$f(y,z) = [250 - 2(y+z)]\,yz = 2(125-y-z)yz$

$y \ge 0, z \ge 0$ and $125 - y - z \ge 0$.

$\therefore$ we need to find abs. max value of

$f(y,z) = 2yz(125-y-z)$ on

$R = \{(y,z) \mid y \ge 0,\ z \ge 0,\ y + z \le 125\}$.



Find critical pts. inside R.

$$\frac{1}{2}\frac{\partial f}{\partial y} = z(125-y-z) + yz(-1) = z(125 - 2y - z)$$

$$\frac{1}{2}\frac{\partial f}{\partial z} = y[125 - y - z - z] = y[125 - y - 2z].$$

$$\frac{\partial f}{\partial y}=0 \quad \frac{\partial f}{\partial z}=0$$

$$\begin{cases} 2y+z=125 \\ y+2z=125 \end{cases}$$

(note $y\neq 0$, $z\neq 0$ since we are looking for C.P. inside R)

2 (2nd eq-n) − 1st eq-n :

$3z=125 \iff z=\frac{125}{3}$.

$y=125-2\cdot\frac{125}{3}=\frac{125}{3}$.

$\therefore f\left(\frac{125}{3},\frac{125}{3}\right)=2\cdot\left(\frac{125}{3}\right)^2\cdot\frac{125}{3}=$

$=2\cdot\left(\frac{125}{3}\right)^3 \leftarrow$ abs. max. value.

$\boxed{\text{\underline{HW}: \#24, p. 867.}}$