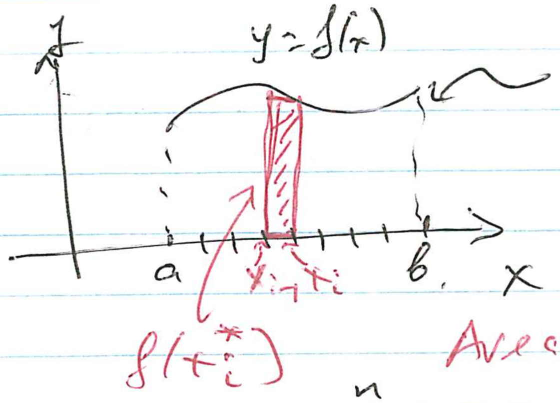
Oct. 29, 2019

Sect. 13.1 Double integrals.

$\mathbb{R}^1$



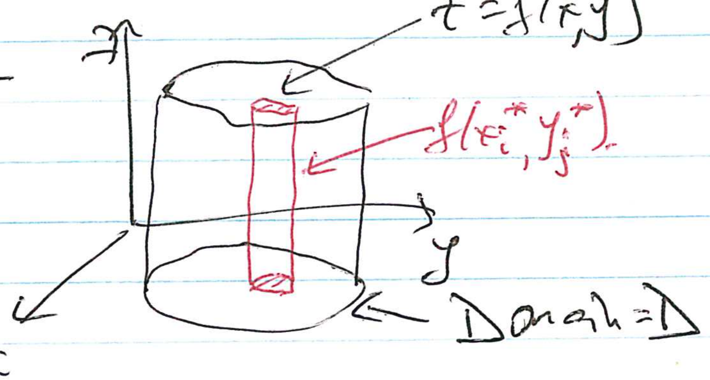
Area under $y = f(x)$, $a \le x \le b$.

$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$

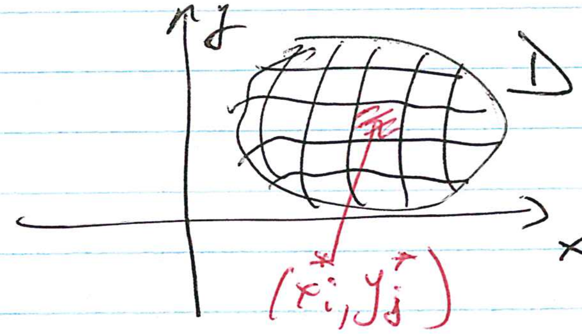
Area of red rectangle $= f(x_i^*) \underbrace{(x_i - x_{i-1})}_{\Delta x_i}$

$$\text{Area} = \lim_{\substack{\max \|\Delta x_i\| \to 0 \\ 1 \le i \le n}} \sum_{i=1}^{n} f(x_i^*) \Delta x_i = \int_a^b f(x)\,dx.$$

$\mathbb{R}^2$



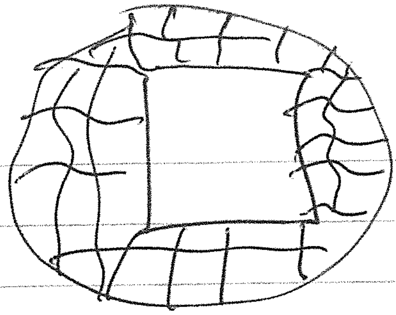
Volume of solid under the graph of f-n $z = f(x,y)$, i.e., under $z = f(x,y)$, $(x,y) \in D$.



Volume $= f(x_i^*, y_j^*) \underbrace{A_{ij}}_{\text{area of this piece.}}$

$$\text{Volume} = \lim_{\|A\| \to 0} \sum_{i,j} f(x_i^*, y_j^*) A_{ij}, \text{ where}$$

$$\|A\| = \max_{i,j} A_{ij}.$$



NOT allowed.

Q-n: Does this limit always exist?

A: NO

Ex: $f(x,y) = \begin{cases} 1 \text{ if } x \text{ } \cancel{\text{or}} \text{ and } y \text{ } \cancel{\text{is}} \text{ are rational} \\ 0 \text{ otherwise.} \end{cases}$

1) $\sum_{i,j} f(x_i^*, y_j^*) A_{ij} = \sum_{i,j} 1 \cdot A_{ij}$ = area of D

if x_i^* and y_j^* are rational.

2) $\sum_{i,j} f(x_i^*, y_j^*) A_{ij} = \sum_{i,j} 0 \cdot A_{ij} = 0$

if at least one of x_i^*, y_j^* is irrational.

$\therefore$ The limit depends on the choice of (x_i^*, y_j^*)

Theorem: Let C be a closed, piecewise-smooth curve that encloses region R with finite area, and let $f = f(x,y)$ be continuous on R.

Then $\lim_{\|A\| \to 0} \sum_{i,j} f(x_i^*, y_j^*) A_{ij}$ exists for any choice of (x_i^*, y_j^*) and does not depend on this choice, and is called the double integral of f over R.

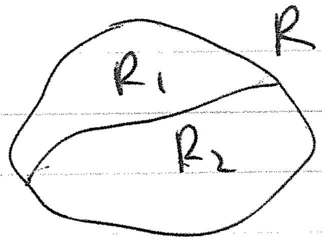
Notation: $\iint_R f(x,y)\, dA$.

Properties:

1) $$\iint_R c f(x,y)\,dA = c \iint_R f(x,y)\,dA$$, for any constant c

2) $$\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

3) If $R = R_1 \cup R_2$ and R_1 and R_2 do not have any interior points in common, then



$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

4) If $f(x,y) = 1$ for all $(x,y) \in R$, then

$$\iint_R f(x,y)\,dA = \iint_R 1\,dA = \text{area of } R.$$

Double Iterated Integrals.

Ex: #1, p. 900

$$\int_{-1}^{2} \underbrace{\int_{y}^{y+2} (x^2 - xy)\,dx}\,dy$$

$$\left\{ \int_a^b \underbrace{\left[\int_{h(y)}^{g(y)} f(x,y)\,dx \right]}_{F(y)} dy \right\}$$

$$\int_y^{y+2} (x^2 - xy)\,dx = \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 y\right)\Bigg|_{x=y}^{x=y+2} =$$

$$= \left[\frac{1}{3}(y+2)^3 - \frac{1}{2}(y+2)^2 y\right] - \left[\frac{1}{3}y^3 - \frac{1}{2}y^3\right]$$

$$= (y+2)^2\left[\frac{1}{3}(y+2) - \frac{1}{2}y\right] + \frac{1}{6}y^3 =$$

$$= (y+2)^2\left[\frac{2}{3} - \frac{1}{6}y\right] + \frac{1}{6}y^3 = F(y).$$

$$\therefore \int_{-1}^{2} \underbrace{\int_y^{y+2} (x^2 - xy)\,dx}_{\text{"} F(y)} \,dy = \int_{-1}^{2} F(y)\,dy = \ldots$$

#5, p. 900

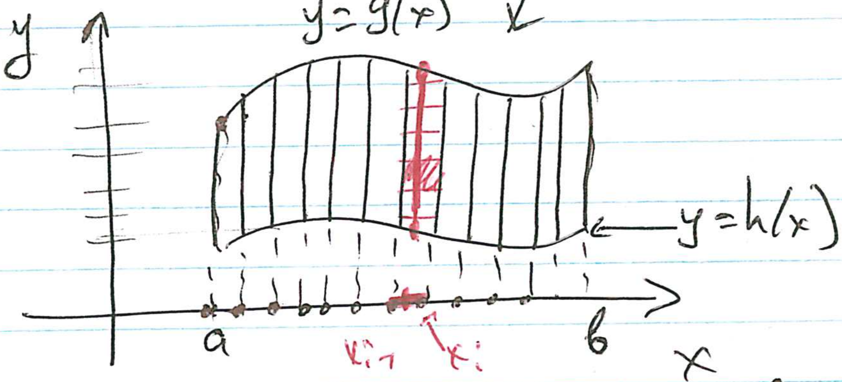
$$\int_3^5 \underbrace{\int_0^{\pi/2} x \sin y \,dy}\,dx = $$

$$\int_0^{\pi/2} x \sin y\,dy = -x\cos y\Bigg|_{y=0}^{y=\frac{\pi}{2}} = -x\cos\frac{\pi}{2} + x\cos 0 = +x$$

$$= \int_3^5 (+x)\,dx = \frac{1}{2}x^2\Bigg|_3^5 = \frac{1}{2}5^2 - \frac{1}{2}3^2 = \frac{16}{2} = 8.$$

Sect. 13.2 Evaluation of Double Integrals by double iterated integrals.

$$\iint_D f(x,y)\,dA =$$



Area $= \Delta x_i \Delta y_j$

$\boxed{}\ \Delta y_j$

Δx_i

$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b.$

$$= \lim_{\substack{\max\|\Delta x_i\|\to 0 \\ \max\|\Delta y_j\|\to 0}} \sum_{i,j} f(x_i^*, y_j^*)\,\Delta x_i\,\Delta y_j =$$

$$= \lim_{\max\|x_i\|\to 0} \sum_i \Bigg[\underbrace{\lim_{\max\|\Delta y_j\|\to 0} \sum_j f(x_i^*, y_j^*)\,\Delta y_j}_{\int_{h(x_i^*)}^{g(x_i^*)} f(x_i^*, y)\,dy} \Bigg] \Delta x_i =$$

$\cancel{\left(\int f(x, y_j^*)\,dx\right.}$

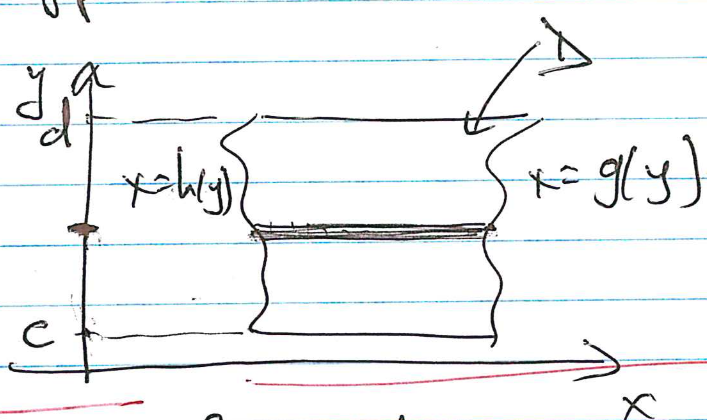
$$= \lim_{\substack{\max\|x_i\|\to 0 \\ i}} \sum_i \underbrace{\int_{h(x_i^*)}^{g(x_i^*)} f(x_i^*, y)\,dy}_{F(x_i^*)}\ \Delta x_i =$$

$$= \int_a^b \left[\int_{h(x)}^{g(x)} f(x,y)\, dy \right] dx.$$

$$\therefore \iint_D f(x,y)\, dA = \int_a^b \int_{h(x)}^{g(x)} f(x,y)\, dy\, dx,$$

where $D = \{(x,y) \mid h(x) \le y \le g(x),\ a \le x \le b\}$.

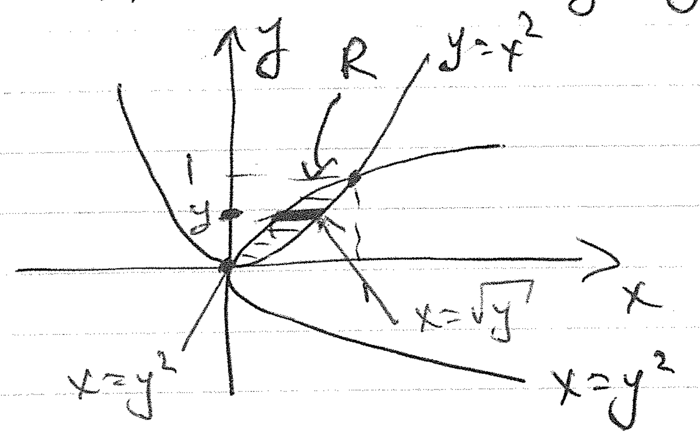
Suppose now that D is s.t.



$D = \{(x,y) \mid h(y) \le x \le g(y),\ c \le y \le d\}$,

$$\iint_D f(x,y)\, dA = \int_c^d \left[\int_{h(y)}^{g(y)} f(x,y)\, dx \right] dy$$

#1, p. 905 Evaluate $\iint_R (x^2+y^2)\,dA$, where R is bounded by $y=x^2$, $x=y^2$.



$$\begin{cases} y=x^2 \\ x=y^2 \end{cases} \iff \begin{cases} y=x^2 \\ x=x^4 \end{cases} \iff \begin{cases} x=0 \text{ or } x=1 \\ y=x^2 \end{cases}$$

$\therefore$ $(0,0)$ and $(1,1)$ are pts of intersection.

$$\iint_R (x^2+y^2)\,dA = \int_0^1 \int_{y^2}^{\sqrt{y}} (x^2+y^2)\,dx\,dy =$$

$$= \int_0^1 \left\{ \left(\frac{1}{3}x^3 + xy^2\right)\Big|_{x=y^2}^{x=\sqrt{y}} \right\} dy =$$

$$= \int_0^1 \left[\left(\frac{1}{3}y\sqrt{y} + y^2\sqrt{y}\right) - \left(\frac{1}{3}y^6 + y^4\right)\right] dy$$

$$= \int_0^1 \left[\frac{1}{3}y^{\frac{3}{2}} + y^{\frac{5}{2}} - \frac{1}{3}y^6 - y^4\right] dy =$$

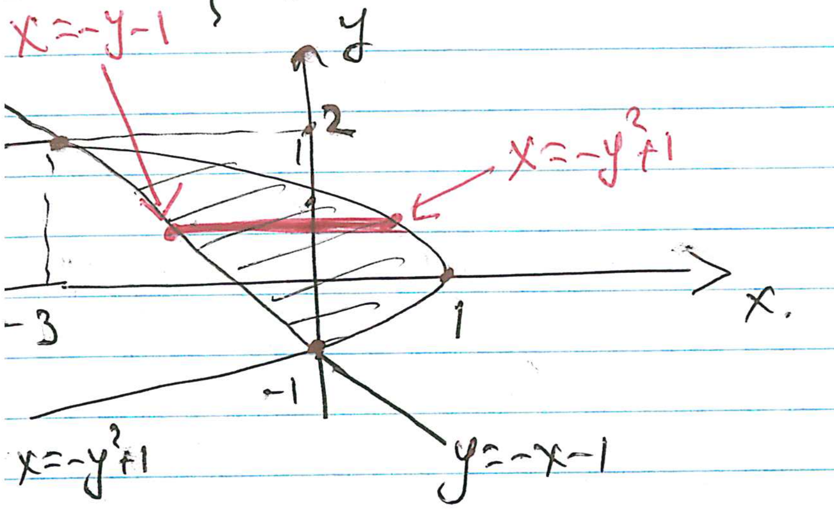
$$= \left(\frac{1}{3}\cdot\frac{2}{5}y^{\frac{5}{2}} + \frac{2}{7}y^{\frac{7}{2}} - \frac{1}{3}\cdot\frac{1}{7}y^7 - \frac{1}{5}y^5\right)\Big|_0^1 =$$

$$= \frac{2}{15} + \frac{2}{7} - \frac{1}{21} - \frac{1}{5} = \ldots$$

#4, p. 905 $\iint_R xy^2\,dA$

R is bounded by $x+y+1=0$ and $x+y^2=1$.

$\{ y=-x-1 \text{ and } x=-y^2+1 \}$.



$$\begin{cases} x=-y^2+1 \\ x=-y-1 \end{cases}$$

$-y-1=-y^2+1 \iff$

$\iff y^2-y-2=0.$

$\iff y=\frac{1\pm\sqrt{1+8}}{2}=\frac{1\pm 3}{2}$

$\iff y=-1$ or $y=2.$

$$\iint_R xy^2\,dA = \int_{-1}^{2}\int_{-y-1}^{-y^2+1} xy^2 \; dx \; dy =$$

$$=\int_{-1}^{2}\left[\left(\frac{1}{2}x^2y^2\right)\Big|_{x=-y-1}^{x=-y^2+1}\right]dy =$$

$$=\int_{-1}^{2}\frac{1}{2}y^2\left[(-y^2+1)^2-(-y-1)^2\right]dy =$$

$$=\frac{1}{2}\int_{-1}^{2}y^2\left[y^4-2y^2+1-(y^2+2y+1)\right]dy =$$

$$=\frac{1}{2}\int_{-1}^{2}y^2(y^4-3y^2-2y)\,dy=\frac{1}{2}\int_{-1}^{2}(y^6-3y^4-2y^3)\,dy$$

$$=\frac{1}{2}\left(\frac{1}{7}y^7-\frac{3}{5}y^5-\frac{1}{2}y^4\right)\Big|_{-1}^{2}=\ldots$$