Oct. 31, 2019

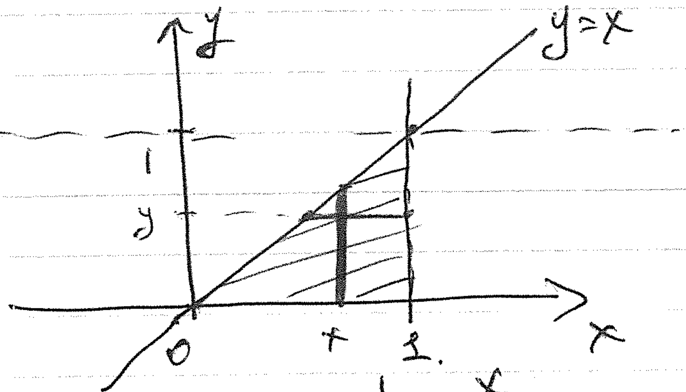
#14, p. 905 $\displaystyle\int_0^1 \int_y^1 \sin(x^2)\,dx\,dy = I$

$\left\{ \displaystyle\int_y^1 \sin(x^2)\,dx \longrightarrow \int \sin(x^2)\,dx \leftarrow \text{cannot evaluate} \right\}$

$\therefore$ reverse the order

$D = \{(x,y) \mid 0 \le y \le 1,\ y \le x \le 1\} =$

$= \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x\}$



$$\therefore\ I = \int_0^1 \int_0^x \sin(x^2)\,dy\,dx =$$

$$= \int_0^1 \sin(x^2)\cdot\left[y \Big|_{y=0}^{y=x} \right] dx = \int_0^1 \underbrace{\sin(x^2)\left[x-0\right]}_{x\sin(x^2)}dx$$

$$= -\frac{1}{2}\cos(x^2)\Big|_0^1 = -\frac{1}{2}\left[\cos 1 - \cos 0\right] = \frac{1}{2}(1-\cos 1)$$

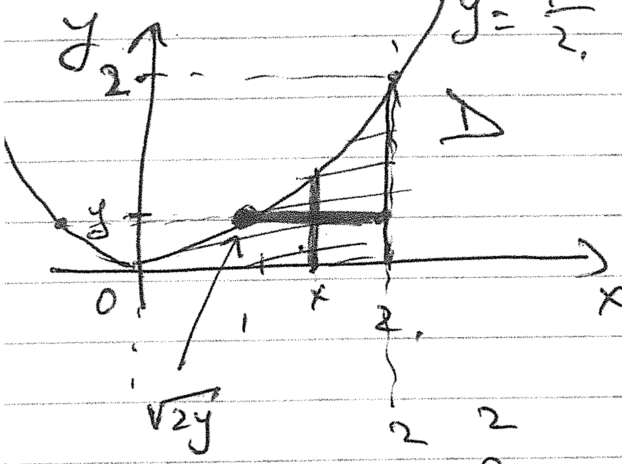
#17, p. 905 $\displaystyle\int_0^2 \int_0^{x^2/2} \frac{x}{\sqrt{1+x^2+y^2}}\,dy\,dx = I$

$\left\{ \displaystyle\int \frac{1}{\sqrt{a^2+y^2}}\,dy \right.$ — can be evaluated (in terms of $\tan^{-1}$)

However, it will be easier to change the order of integration $\left.\right\}$

$$D=\left\{(x,y)\ \middle|\ 0\le x\le 2,\ 0\le y\le \frac{x^2}{2}\right\} =$$

$$= \left\{(x,y)\ \middle|\ 0\le y\le 2,\ \sqrt{2y}\le x\le 2\right\}$$



$$\therefore\ I=\int_0^2\int_{\sqrt{2y}}^{2}\frac{x}{\sqrt{1+x^2+y^2}}\,dx\,dy=$$

$$=\int_0^2\left[\left(1+x^2+y^2\right)^{\frac{1}{2}}\Big|_{x=\sqrt{2y}}^{x=2}\right]dy=$$

$$=\int_0^2\left[\left(5+y^2\right)^{\frac{1}{2}}-\left(1+2y+y^2\right)^{\frac{1}{2}}\right]dy=$$

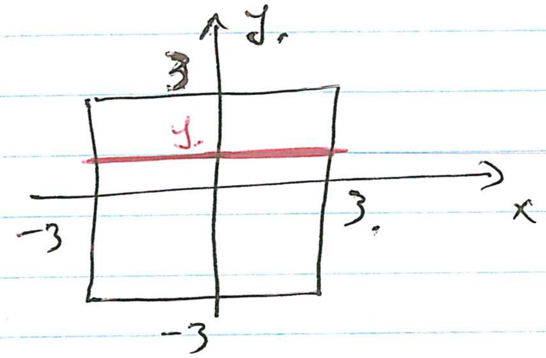
$$=\int_0^2\left[\sqrt{y^2+5}-\sqrt{(y+1)^2}\right]dy=$$

$$=\int_0^2\left[\sqrt{y^2+5}-|y+1|\right]dy=$$

$$=\int_0^2\left(\sqrt{y^2+5}-y-1\right)dy=\ \ldots\ \text{HW.}$$

$$\left\{\int\sqrt{y^2+1}\,dy=\left\{\begin{array}{l} y=\dfrac{e^u-e^{-u}}{2}\\ y^2+1=\left(\dfrac{e^u+e^{-u}}{2}\right)^2\end{array}\right\}=\int\frac{e^u+e^{-u}}{2}\cdot\frac{e^u+e^{-u}}{2}\,du=\text{etc.}\right\}$$

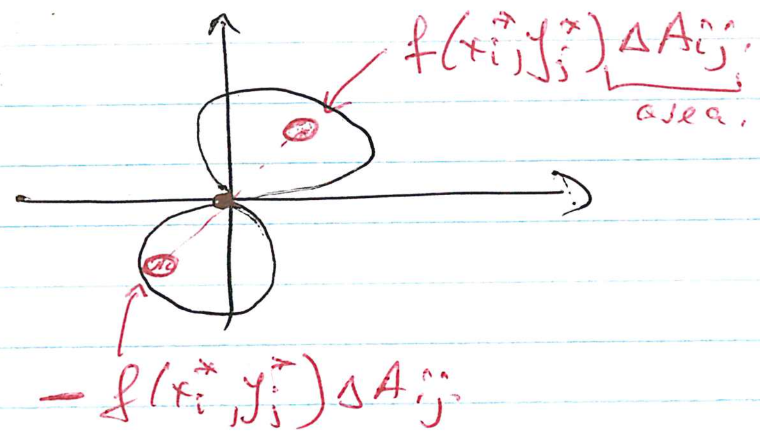
#23, p. 905 $\displaystyle\iint_R (x+y)\,dA$, where R is the square with vertices $(\pm 3, 0)$ and $(0, \pm 3)$.



Idea: 1) If f is s.t. $f(-x,y) = -f(x,y)$, $-a \le x \le a$

Then $\displaystyle\int_{-a}^{a} f(x,y)\,dx = 0$

2) If $f = f(x,y)$ is s.t. $f(-x,-y) = -f(x,y)$ and ~~$|\Delta_x f(x,y)|$ if~~ D is s.t. if $(x,y) \in D$, then $(-x,-y) \in D$ (i.e., D is symmetric w.r.t. the origin), then $\displaystyle\iint_D f(x,y)\,dA = 0$.



Back to the problem:

1) $f(x,y) = x+y$, so $f(-x,-y) = -x-y = -(x+y) = -f(x,y)$

2) If $(x,y) \in R$, then $(-x,-y) \in R$.

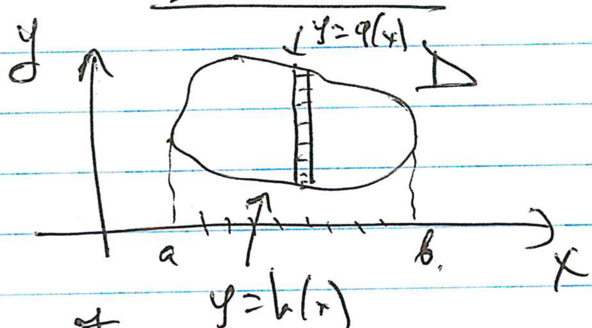
$\therefore \displaystyle\iint_R (x+y)\,dA = 0$.

Method 2: $\iint_R (x+y)\, dA = \iint_R x\, dA + \iint_R y\, dA$,

and ~~apply~~ use observation 1 above, i.e.,

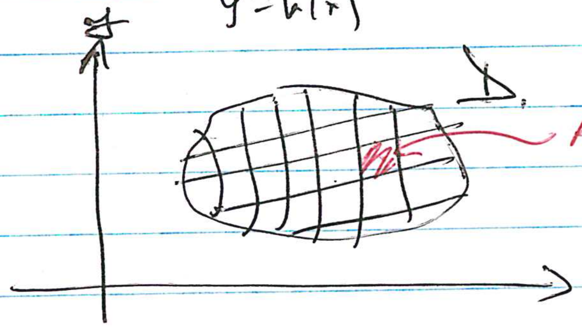
$f(x,y) = x$ and $f(-x,y) = -x = -f(x,y)$, for each fixed y. Also, for each y, the region R restricted to this y is symmetric.

Sect. 13.3 Areas and Volumes of Solids of Revolution.



Area of D – ?

$$\int_a^b [g(x) - h(x)]\, dx \quad (*).$$



A_{ij}.

Area of $D \approx \sum_{i,j} A_{ij}$

$$\boxed{\therefore \text{Area of } D = \iint_D 1\, dA}$$

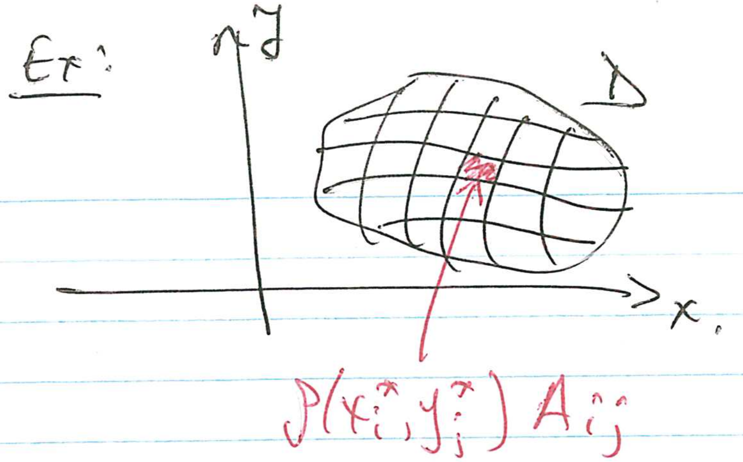
If $D = \{(x,y) \mid a \le x \le b,\ h(x) \le y \le g(x)\}$, then

$$\text{Area of } D = \int_a^b \int_{h(x)}^{g(x)} 1 \quad dy \quad dx =$$

$$= \int_a^b \left[y \Big|_{y=h(x)}^{y=g(x)} \right] dx = \int_a^b [g(x) - h(x)]\, dx \text{ which}$$

is the above formula $(*)$.

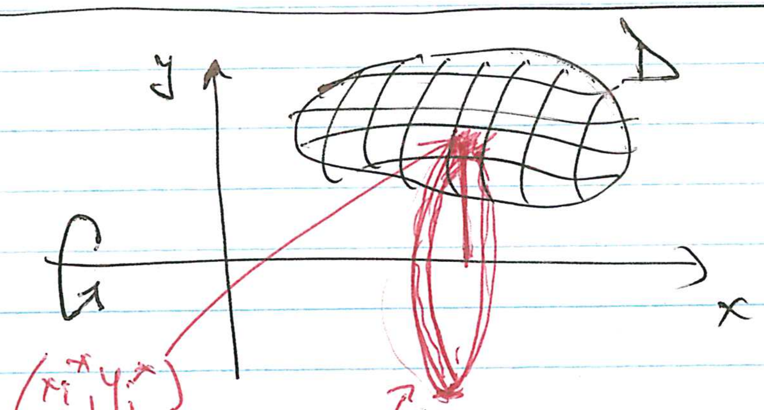
Ex:



Density at (x,y) is $\rho(x,y)$.

Mass of this plate =

$$= \iint_D \rho(x,y)\, dA.$$



Suppose now that the region D is revolved around the x-axis. Find the volume of the resulting solid.

Volume?

$$V_{ij} = 2\pi R A_{ij} = 2\pi y_j^* A_{ij}$$

$$\boxed{V = \iint_D 2\pi y\, dA \quad \text{(around the } x\text{-axis)}}$$

If $D = \{(x,y) \mid a \le x \le b,\ h(x) \le y \le g(x)\}$,

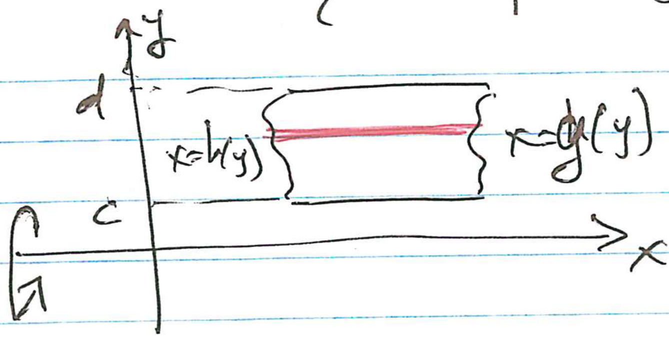
then
$$V = \iint_D 2\pi y\, dA = 2\pi \int_a^b \int_{h(x)}^{g(x)} y\, dy\, dx =$$

$$= 2\pi \int_a^b \left[\frac{1}{2} y^2 \Big|_{y=h(x)}^{g(x)}\right] dx = 2\pi \int_a^b \frac{1}{2}\left[[g(x)]^2 - [h(x)]^2\right] dx$$

$$= \int_a^b \pi\left[(g(x))^2 - (h(x))^2\right] dx$$

formula for the "washer method".

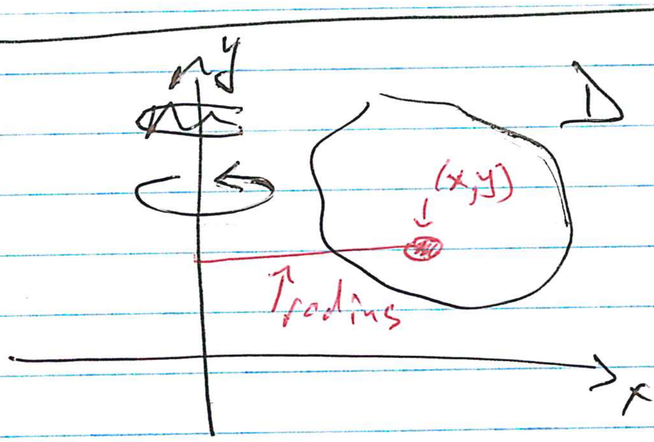
If $D = \{(x,y) \mid c \le y \le d,\ h(y) \le x \le g(y)\}$, then



$$V = \iint_D 2\pi y \, dA = \int_c^d \int_{h(y)}^{g(y)} 2\pi y \, dx \, dy =$$

$$= \int_c^d 2\pi y \left(g(y) - h(y) \right) dy$$

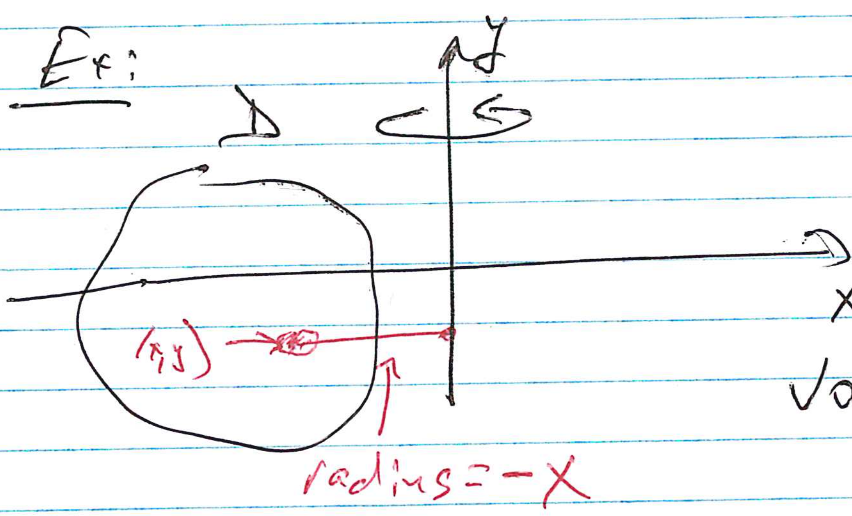
formula for the "shell method".



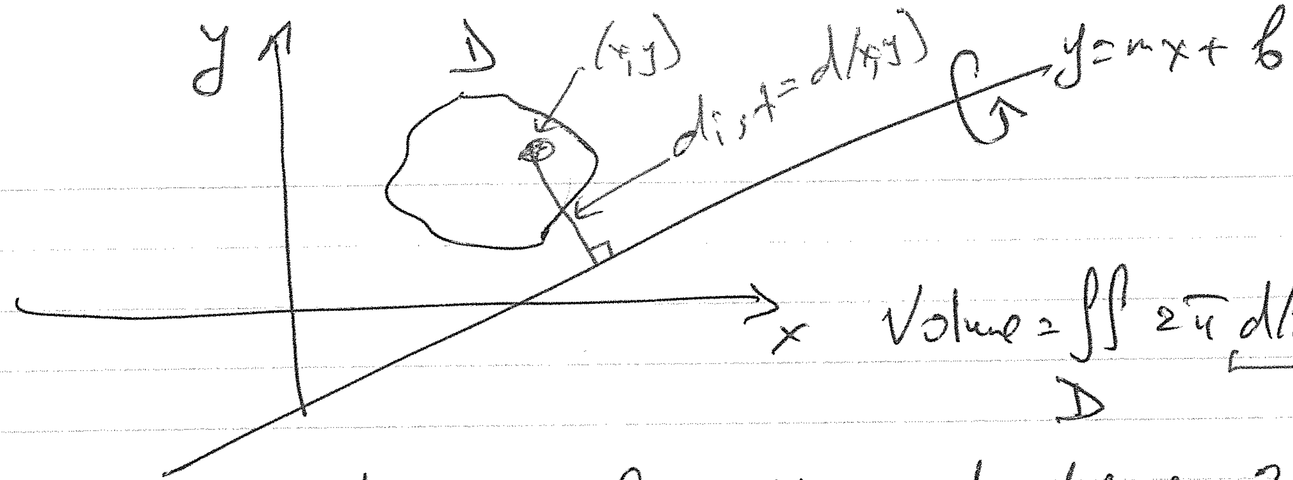
$$\boxed{V = \iint_D 2\pi x \, dA}$$

(revolution around the y-axis).

__Ex:__



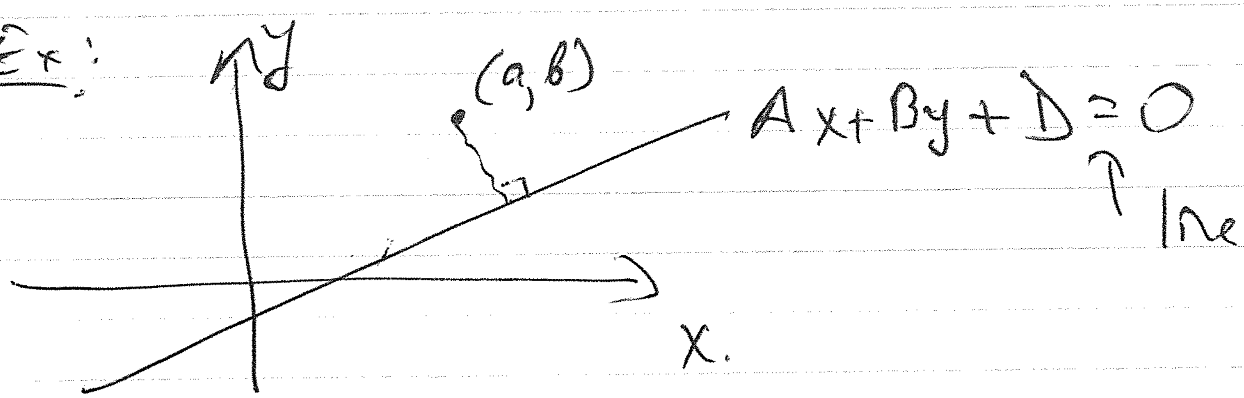
$$\text{Volume} = \iint_D (-2\pi x) \, dA$$



$$\text{Volume} = \iint_D 2\pi\, \underbrace{d(x,y)}\, dA$$

Qn: how to find this $d = d(x,y)$?

Ex:



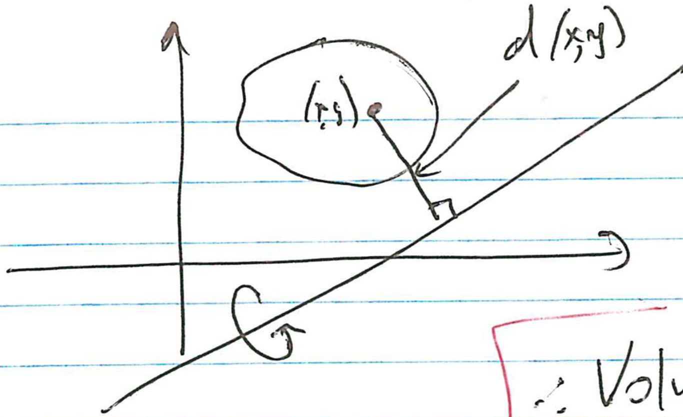
Distance from (a,b) to the line $=$

$$= \left\{ \begin{array}{l} \mathbb{R}^3 : (a,b,c) \text{ and plane } Ax+By+Cz+D=0 \\ \text{Distance from this pt. to the plane is} \\ \dfrac{|Aa+Bb+Cc+D|}{\sqrt{A^2+B^2+C^2}} \end{array} \right\}$$

$=$ distance from $(a,b,0)$ to the plane $Ax+By+0\cdot z+D =$

$$= \frac{|Aa+Bb+D|}{\sqrt{A^2+B^2}}$$

Back to our problem:



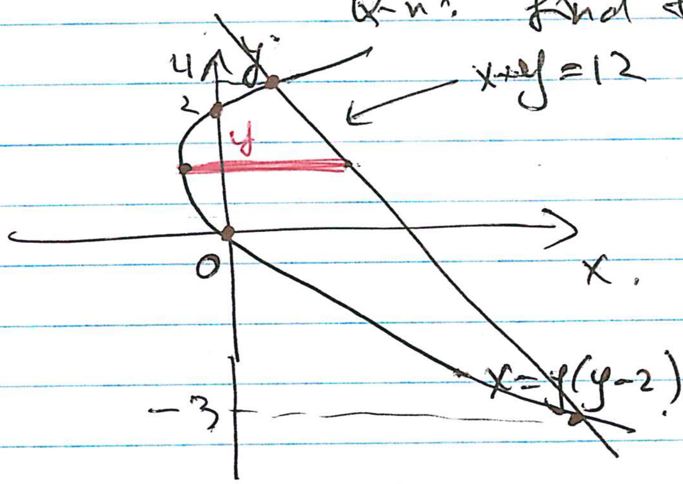
$$y = mx + b \iff y - mx - b = 0$$

$$d(x,y) = \frac{|y - mx - b|}{\sqrt{1+m^2}}.$$

$$\boxed{\therefore \text{Volume} = \iint_D \frac{|y-mx-b|}{\sqrt{1+m^2}}\, dA.}$$

<u>#8, p. 910</u> Region bounded by $x = y(y-2)$, $x + y = 12$

Q-n: find the area.



$$\text{Area} = \iint_D 1\, dA$$

Find pts of intersection:

$$\begin{cases} x = y(y-2) \\ x = 12 - y \end{cases}$$

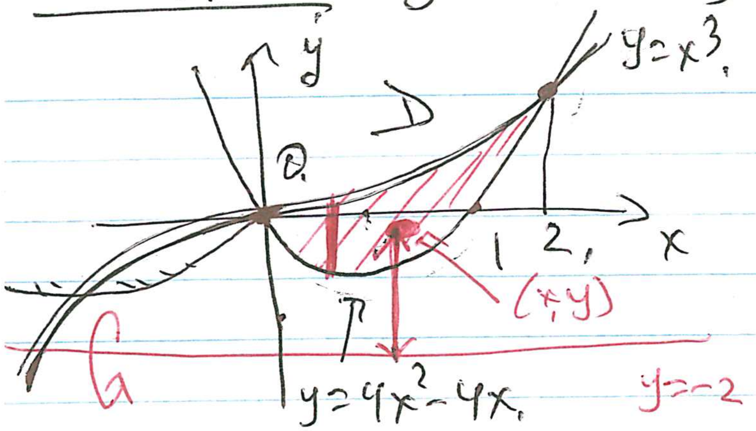
$$\therefore 12 - y = y^2 - 2y \iff y^2 - y - 12 = 0 \iff y = \frac{1 \pm \sqrt{1+48}}{2} = \frac{1 \pm 7}{2}$$

$\therefore y = -3$ or $y = 4$.

$$\text{Area} = \int_{-3}^{4} \int_{y(y-2)}^{12-y} 1 \; dx \; dy =$$

$$= \int_{-3}^{4} \left[12 - y - y(y-2)\right] dy = \ldots (\text{HW}).$$

#17, p. 911 $\quad y = 4x^2 - 4x,\ y = x^3$ about $y = -2$.



$y = 4x^2 - 4x = 4x(x-1)$

$$\begin{cases} y = 4x^2 - 4x \\ y = x^3 \end{cases}$$

$\therefore\ x^3 = 4x^2 - 4x \iff x = 0$ ok.

$x^2 = 4x - 4 \iff x^2 - 4x + 4 = 0$

$\iff (x-2)^2 = 0 \Rightarrow x = 2$.

Note: the vertex of parabola $y = 4x^2 - 4x$ is at

$x = \frac{1}{2}$: $y = 4 \cdot \frac{1}{4} - 4 \cdot \frac{1}{2} = 1 - 2 = -1 > -2$.

Radius $= y - (-2) = y + 2$

$\therefore$ Volume $= \iint\limits_D 2\pi(y+2)\,dA =$

$$= \int_0^2 \int_{4x^2-4x}^{x^3} 2\pi(y+2)\,dy\,dx =$$

$$= \int_0^2 2\pi \left(\frac{(y+2)^2}{2} \Bigg|_{y=4x^2-4x}^{y=x^3} \right) dx =$$

$$= \int_0^2 2\pi \cdot \frac{1}{2} \left[(x^3+2)^2 - (4x^2-4x+2)^2 \right] dx = \ldots \text{ HW.}$$

HW: #37, 38 p. 911